Universal recursive preference structures*

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Abstract

Give a set $\mathcal{X}$ of "outcomes" and a set $\mathcal{T}$ of "types", a *recursive preference structure* (RPS) is a function that assigns a continuous partial order over $\mathcal{T}\times\mathcal{X}$ to every element of $\mathcal{T}$. This describes an agent who has preferences not only over the outcomes in $\mathcal{X}$, but also over *her own preferences* (as encoded by the types). We prove the existence of a *universal* RPS —one into which any other RPS can be mapped in a unique way. Formally, this universal RPS is a *terminal coalgebra* of a suitably defined endofunctor on the category of compact Hausdorff spaces.

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> A taste is almost *defined* as a preference about which you do not argue —de gustibus non est disputandum. A taste about which you argue, with others *or yourself*, ceases ipso facto being a taste —it turns into a *value*. —Albert Hirschman

What if you could choose your own preferences? How would you do this? Does this even make sense? Is the rational selection of one's own preferences even possible? These questions have inspired a rich philosophical literature concerning personal autonomy, moral responsibility, and transformative experiences (see e.g. Frankfurt 1971; Paul 2014; Pettigrew 2019). But they are also relevant to normative decision theory (Pettigrew, 2015). Recently, Pivato (2023a) proposed a decision-theoretic framework to examine these issues. Its focus is normative rather than descriptive; it does not ask how people *actually* choose their preferences, but rather, how an ideally rational agent *could* choose her preferences. What is the decision problem faced by such an agent? What is the feasible set, and what is the appropriate notion of rational choice?

One of the models introduced in Pivato (2023a) is the *recursive preference structure*. Given a set $\mathcal{X}$ of possible outcomes (e.g. consumption bundles), a recursive preference structure over $\mathcal{X}$ consists of a *type space* $\mathcal{T}$ together with a function $\phi:\mathcal{T}\longrightarrow\mathcal{P}$, where

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$\mathcal{P}$ is a set of preference orders over $\mathcal{T} \times \mathcal{X}$. In other words: ϕ maps each type in $\mathcal{T}$ to a preference order over *type-outcome pairs*. This represents an agent who has preferences not only over outcomes in $\mathcal{X}$, but also preferences about her own type. Since types determine preferences, this means she implicitly has *preferences over her own preferences*.

Such higher-order preferences represent a desire for *autonomy*: an agent's desire to decide what kind of person she will be, and what kind of preferences she will have. Her degree of autonomy is directly proportional to the diversity of preferences over $\mathcal{T} \times \mathcal{X}$ that are theoretically available to her via ϕ. For this reason, we might want $\mathcal{P}$ to include *incomplete* preferences, describing an agent who regards some pairs of options as *incommensurable*: neither is better than the other.

More generally, the size of the image of ϕ determines the agent's autonomy: a larger image allows more autonomy. Thus, one might say that she is *fully autonomous* if the image of ϕ contains *every* possible preference order over $\mathcal{T} \times \mathcal{X}$. In general, this is impossible, because the set of *all* possible preferences over $\mathcal{T} \times \mathcal{X}$ always has a strictly larger cardinality than $\mathcal{T}$ itself, so there can be no surjection from the latter set into the former. But if we posit topologies on $\mathcal{T}$ and $\mathcal{X}$, and restrict attention to *continuous* preferences on $\mathcal{T} \times \mathcal{X}$, then such a fully autonomous recursive preference structure is possible (Proposition 1.4).

An even greater degree of autonomy would be obtained by a recursive preference structure which not only covers every preference order on $\mathcal{T} \times \mathcal{X}$, but contains an image of *every other recursive preference structure*. Such an agent would be truly autonomous: not only could she, in principle, adopt any possible preference order on $\mathcal{T} \times \mathcal{X}$ — she could even mimic any other recursive preference structure. The main result of this paper establishes the existence of such a "universal" recursive preference structure (Theorem 2.4). As explained below, it is a *terminal coalgebra* of an endofunctor on the category of compact Hausdorff spaces. Terminal coalgebras have many important properties, and play an important role in logic and theoretical computer science (Jacobs and Rutten, 1997; Rutten, 2000). They also have interesting applications in theoretical economics (Vassilakis, 1992). Since the seminal papers of Moss and Viglizzo (2004, 2006), it has been understood that the universal type spaces of Bayesian game theory are terminal coalgebras; see Pintér (2010), Heinsalu (2014), Fukuda (2021), Guarino (2022) and Galeazzi and Marti (2023) for recent applications of this approach.[1] But not all endofunctors have terminal coalgebras. Proving their existence is nontrivial (Adámek et al., 2018). The main contribution of this paper is proving that a terminal coalgebra exists for the endofunctor that describes recursive preference structures.

The remainder of this paper is organized as follows. Section 1 introduces *recursive preference structures* and related concepts. Section 2 defines the key concept of this paper: a *universal* recursive preference structure. It contains the main result (Theorem 2.4), which states that they exist under general conditions. It also contains other results describing their structure and properties. Section 3 explains how the main concepts and results of Sections 1 and 2 can be reformulated in terms of *terminal coalgebras* of endofunctors on the

[1] Heinsalu (2014), Guarino (2022), and Galeazzi and Marti (2023) explicitly formulate their results in terms of terminal coalgebras, whereas Pintér (2010) and Fukuda (2021) do not. But they do describe universal type spaces as the terminal objects in a category of type spaces and type morphisms.

category of compact Hausdorff spaces. Section 4 briefly reviews some prior literature. The proofs of all results are in the appendices. Appendix A reviews mathematical background, Appendix B contains the proofs of results from Section 1, and Appendix C contains the proofs of results from Section 2. All the results in the paper are formulated in terms of *continuous partial orders*. Finally, Appendix D explains how all concepts and results in the paper can be reformulated in terms of the dual notion of *continuous quasiorders*.

1 Recursive preference structures

Let $\mathcal{X}$ be a set. A *strict partial order* on $\mathcal{X}$ is a binary relation $\succ$ that is *transitive* (for all $x, y, z \in \mathcal{X}$, if $x \succ y$ and $y \succ z$, then $x \succ z$) and *antisymmetric* (for all $x, y \in \mathcal{X}$, it is never the case that both $x \succ y$ and $x \prec y$). Antisymmetry implies that $\succ$ is *irreflexive* ($x \not\succ x$ for all $x \in \mathcal{X}$).

Now suppose that $\mathcal{X}$ is a topological space. A strict partial order $\succ$ is *continuous* if the set $\{(x, y) \in \mathcal{X}\times\mathcal{X};\ \ x \succ y\}$ is open in $\mathcal{X}\times\mathcal{X}$. This implies that all the upper and lower contour sets of $\succ$ are open subsets of $\mathcal{X}$, but it is a slightly stronger condition.[2]

A *local continuous strict partial order* is an ordered pair $(\mathcal{Y}, \succ)$, where $\mathcal{Y} \subseteq \mathcal{X}$ is a closed subset, and where $\succ$ is a strict partial order on $\mathcal{Y}$ which is continuous *relative to the subspace topology on* $\mathcal{Y}$. (Note that $\succ$ might *not* be continuous with respect to the ambient topology on $\mathcal{X}$ —for example, this could happen if $\mathcal{Y}$ itself is nowhere dense in $\mathcal{X}$.)

Suppose $\mathcal{X}$ is a compact Hausdorff space, and let $K(\mathcal{X})$ be the set of all nonempty closed subsets of $\mathcal{X}$. Then $K(\mathcal{X})$ itself is a compact Hausdorff space, when equipped with the Vietoris topology (see Appendix A.2). For any local continuous strict partial order $(\mathcal{Y}, \succ)$, let $[\![\mathcal{Y}, \succ]\!] := \{(x, y) \in \mathcal{Y}\times\mathcal{Y};\ \ x \not\succ y\}$; this is a closed subset of $\mathcal{Y}\times\mathcal{Y}$ (because it is the complement of an open set, because $\succ$ is continuous on $\mathcal{Y}$); hence it is a closed subset of $\mathcal{X}\times\mathcal{X}$ (because $\mathcal{Y}$ itself is closed), and hence, an element of $K(\mathcal{X}\times\mathcal{X})$. Let $P(\mathcal{X})$ be the set of all local continuous strict partial orders on $\mathcal{X}$. Then the injective function $P(\mathcal{X}) \ni (\mathcal{Y}, \succ) \mapsto [\![\mathcal{Y}, \succ]\!] \in K(\mathcal{X})$ identifies $P(\mathcal{X})$ with a subset of $K(\mathcal{X}\times\mathcal{X})$, and the subspace topology induced by the Vietoris topology pulls back to define a topology on $P(\mathcal{X})$, which we will call the *co-Vietoris topology*. In this topology, $P(\mathcal{X})$ is itself a compact Hausdorff space. If $\mathcal{X}$ is metrizable, then so is $P(\mathcal{X})$ (see Proposition A.1).

Recursive preference structures. Let $\mathcal{X}$ be a compact Hausdorff space. A *recursive preference structure* (RPS) over $\mathcal{X}$ is an ordered pair $(\mathcal{T}, \phi)$, where $\mathcal{T}$ is a compact Hausdorff space, and $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ is a continuous function. So for any type $t \in \mathcal{T}$, $\phi(t)$ is a local continuous strict partial order on $\mathcal{T} \times \mathcal{X}$; this represents the preferences of type t over type-outcome pairs.

Example 1.1. Let $\mathcal{T}$ and $\mathcal{X}$ be compact Hausdorff spaces, and endow $\mathcal{T} \times \mathcal{X}$ with the product topology. Let $v : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \longrightarrow \mathbb{R}$ be continuous. For all $t \in \mathcal{T}$, define a

[2] For any $y \in \mathcal{X}$, the upper and lower contour sets are defined $\mathcal{U}(y) = \{x \in \mathcal{X}\ ;\ x \succ y\}$ and $\mathcal{L}(y) = \{x \in \mathcal{X}\ ;\ x \prec y\}$.

continuous partial order $\succ_t$ on $\mathcal{T}\times\mathcal{X}$ by stipulating that $(t_1,x_1)\succ_t(t_2,x_2)$ if and only if $v(t;\,t_1,x_1) > v(t;\,t_2,x_2)$. (In other words: $v(t;\cdot,\cdot)$ is an ordinal utility function for $\succ_t$.) The partial order $\succ_t$ is continuous because the function $v(t,\cdot,\cdot):\mathcal{T}\times\mathcal{X}\longrightarrow\mathbb{R}$ is continuous. Let $\phi(t) := (\mathcal{T}\times\mathcal{X},\succ_t)$ for all $t\in\mathcal{T}$; this yields a function $\phi:\mathcal{T}\longrightarrow P(\mathcal{T}\times\mathcal{X})$. Under mild conditions on v, the function ϕ is continuous in the co-Vietoris topology (see Proposition B.2), and hence an RPS. ♢

Example 1.2. Let $\mathcal{T}$ and $\mathcal{X}$ be Peano continua (e.g. compact, connected subsets of $\mathbb{R}^N$), and let d be a convex metric on $\mathcal{T}\times\mathcal{X}$ that is compatible with the product topology.[3] Let $\gamma:\mathcal{T}\longrightarrow\mathcal{T}\times\mathcal{X}$ be a continuous function. For all $t\in\mathcal{T}$, we define a continuous partial order $\succ_t$ on $\mathcal{T}\times\mathcal{X}$ by stipulating that $(t_1,x_1)\succ_t(t_2,x_2)$ if and only if $d\Big(\gamma(t),(t_1,x_1)\Big) < d\Big(\gamma(t),(t_2,x_2)\Big)$. Heuristically: $\gamma(t)$ is the “ideal point” of an agent of type t, and all type-outcome pairs in $\mathcal{T}\times\mathcal{X}$ are ordered by their proximity to $\gamma(t)$. The partial order $\succ_t$ is continuous because d is continuous with respect to the product topology on $(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X})$. Define $\phi(t) := (\mathcal{T}\times\mathcal{X},\succ_t)$ for all $t\in\mathcal{T}$; this yields a function $\phi:\mathcal{T}\longrightarrow P(\mathcal{T}\times\mathcal{X})$ that is continuous in the co-Vietoris topology (see Lemma B.1), and hence an RPS. ♢

Recursive optimality. Preference-maximization is the *sine qua non* of rational choice. So what is the relevant notion of rational choice for recursive preference structures?

Let $(\mathcal{T},\phi)$ be a recursive preference structure over $\mathcal{X}$. Let $t^*\in\mathcal{T}$, let $\phi(t^*) = (\mathcal{Y}_{t^*},\succ_{t^*})$, and let $x^*\in\mathcal{X}$. The type-outcome pair (t^*,x^*) in $\mathcal{T}\times\mathcal{X}$ is *recursively optimal* if $(t^*,x^*)\in\mathcal{Y}_{t^*}$ and (t^*,x^*) is undominated according to $\succ_{t^*}$. That is: there is no $t\in\mathcal{T}$ and $x\in\mathcal{X}$ such that $(t^*,x^*)\prec_{t^*}(t,x)$.

Example 1.3. Let $\mathcal{T}$ and $\mathcal{X}$ be Peano continua, let d be a convex metric on $\mathcal{T}\times\mathcal{X}$, let $\gamma:\mathcal{T}\longrightarrow\mathcal{T}\times\mathcal{X}$ be a continuous function, and define the RPS $\phi:\mathcal{T}\longrightarrow P(\mathcal{T}\times\mathcal{X})$ as in Example 1.2. Let $\gamma_1:\mathcal{T}\longrightarrow\mathcal{T}$ be the first coordinate of γ; this is a continuous self-map of $\mathcal{T}$. Suppose γ_1 has a fixed point t^*. Thus, $\gamma(t^*) = (t^*,x^*)$ for some $x^*\in\mathcal{X}$. It is easily verified that (t^*,x^*) is recursively optimal. ♢

In general, recursively optimal type-outcome pairs are not unique. Indeed, there may even be two recursively optimal type-outcome pairs (t^*,x^*) and $(t^\dagger,x^\dagger)$ such that $(t^*,x^*)\succ_{\phi(t^*)}(t^\dagger,x^\dagger)$ while $(t^\dagger,x^\dagger)\succ_{\phi(t^\dagger)}(t^*,x^*)$.

Fully Autonomous RPS. An RPS $(\mathcal{T},\phi)$ over $\mathcal{X}$ is *fully autonomous* if the map $\phi:\mathcal{T}\longrightarrow P(\mathcal{T}\times\mathcal{X})$ is surjective. In other words, by choosing a suitable type in $\mathcal{T}$, an agent with this RPS can realize *any possible* local continuous strict partial order on $\mathcal{T}\times\mathcal{X}$.[4] For example, endow $\{0,1\}$ with the discrete topology, and let $\mathbb{K} := \{0,1\}^{\mathbb{N}}$, endowed with the Tychonoff product topology. Then $\mathbb{K}$ is a totally disconnected, compact, metrizable space, called *Cantor space* (Willard 2004, Example 17.9(c); Aliprantis and Border 2006, §3.13).

[3] See Appendix A.1 for the definitions of *Peano continuum* and *convex metric*.

[4] In epistemic game theory, the analogous concept is a *complete type space* (Brandenburger, 2003).

Proposition 1.4 *Let $\mathcal{T} = \mathbb{K}$. Then for any compact, metrizable space $\mathcal{X}$, there is a fully autonomous RPS $(\mathcal{T}, \phi)$ over $\mathcal{X}$.*

Perfectly autonomous RPS. In a recursive preference structure $(\mathcal{T}, \phi)$, the function $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ could be many-to-one. Thus, there could be many different types in $\mathcal{T}$ that all determine the *same* preference order over $\mathcal{T} \times \mathcal{X}$. So these types are "behaviourally indistinguishable". Yet ϕ-image preferences on $\mathcal{T} \times \mathcal{X}$ might still have strict preferences between these types. So preferences over $\mathcal{T} \times \mathcal{X}$ might distinguish between types based on behaviourally non-observable properties. In some contexts, this might be undesirable. Motivated by this, we will say that an RPS $(\mathcal{T}, \phi)$ is *perfectly autonomous* if the map $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ is bijective, and thus, a homeomorphism. In other words: for every possible local continuous strict partial order over $\mathcal{T} \times \mathcal{X}$, there exists a *unique* type in $\mathcal{T}$ that realizes this preference order.

Proposition 1.5 *If $\mathcal{T} = \mathcal{X} = \mathbb{K}$, then there is a perfectly autonomous RPS over $\mathcal{X}$.*

An obvious limitation of this result is that it only applies when the outcome space $\mathcal{X}$ is a Cantor space. The main result of this paper will yield a much more general class of perfectly autonomous recursive preference structures. But first we must develop the appropriate mathematical framework.

2 Universal recursive preference structures

Forward images of partial orders. Let $\mathcal{X}$ and $\mathcal{X}'$ be compact Hausdorff spaces, and let $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ be continuous. For any local continuous strict partial order $(\mathcal{Y}, \succ)$ on $\mathcal{X}$, let $\phi^{\P}(\mathcal{Y}, \succ) := (\mathcal{Y}', \succ')$, where $\mathcal{Y}' := \phi(\mathcal{Y})$, and where $\succ'$ is the binary relation on $\mathcal{Y}'$ defined as follows: for any $x', y' \in \mathcal{Y}'$, $x' \succ' y'$ if and only if $x \succ y$ for all $x \in \mathcal{Y} \cap \phi^{-1}\{x'\}$ and $y \in \mathcal{Y} \cap \phi^{-1}\{y'\}$. It can be shown that $\succ'$ is itself a continuous partial order on $\mathcal{Y}'$, so that $(\mathcal{Y}', \succ') \in P(\mathcal{X}')$. Furthermore, the function $\phi^{\P}$ is continuous with respect to the co-Vietoris topologies on $P(\mathcal{X})$ and $P(\mathcal{X}')$ (see Proposition A.2).

Morphisms of RPSs. Let $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$ be two recursive preference structures over $\mathcal{X}$. (That is: $\mathcal{T}_1$ and $\mathcal{T}_2$ are compact Hausdorff spaces, and $\phi_1 : \mathcal{T}_1 \longrightarrow P(\mathcal{T}_1 \times \mathcal{X})$ and $\phi_2 : \mathcal{T}_2 \longrightarrow P(\mathcal{T}_2 \times \mathcal{X})$ are continuous.) Let $\psi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be another continuous function. Let $I_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X}$ be the identity function, and define $\psi \times I_{\mathcal{X}} : \mathcal{T}_1 \times \mathcal{X} \longrightarrow \mathcal{T}_2 \times \mathcal{X}$ in the obvious way (i.e. $(\psi \times I_{\mathcal{X}})(t, x) = (\psi(t), x)$, for all $(t, x) \in \mathcal{T}_1 \times \mathcal{X}$.) Finally, let $\psi^{\dagger} := (\psi \times I_{\mathcal{X}})^{\P} : P(\mathcal{T}_1 \times \mathcal{X}) \longrightarrow P(\mathcal{T}_2 \times \mathcal{X})$. The function ψ is a *morphism* of recursive preference structures if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_1 & \xrightarrow{\phi_1} & P(\mathcal{T}_1 \times \mathcal{X}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi^{\dagger}} \\ \mathcal{T}_2 & \xrightarrow[\phi_2]{} & P(\mathcal{T}_2 \times \mathcal{X}) \end{array} \qquad (1)$$

Example 2.1. (a) Suppose that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\psi : \mathcal{T}_1 \hookrightarrow \mathcal{T}_2$ is the inclusion map. Let $t \in \mathcal{T}_1$, and suppose $\phi_1(t) = (\mathcal{Y}, \succ)$, for some closed subset $\mathcal{Y} \subseteq \mathcal{T}_1 \times \mathcal{X}$. Then diagram (1) says that $\phi_2(t) = (\mathcal{Y}, \succ)$ also, but with $\mathcal{Y}$ now seen as a subset of the larger space $\mathcal{T}_2 \times \mathcal{X}$. A similar interpretation applies when ψ is any injective function.

(b) An *endomorphism* is a morphism from an RPS into itself. For example, let $\mathcal{X} := \{x, y\}$ and let $\mathcal{T} := \{t, s\}$ (both with the discrete topology). Suppose $\phi(t) = (\mathcal{T} \times \mathcal{X}, \succ_t)$ where $(t, x) \succ_t (s, x) \succ_t (s, y) \succ_t (t, y)$, while $\phi(s) = (\mathcal{T} \times \mathcal{X}, \succ_s)$ where $(s, x) \succ_s (t, x) \succ_s (t, y) \succ_s (s, y)$. Let $\alpha(t) := s$ and $\alpha(s) := t$. Then α is an endomorphism of $(\mathcal{T}, \phi)$.

(c) Let $\mathcal{T}$ and $\mathcal{X}$ be compact Hausdorff spaces, let $v : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \longrightarrow \mathbb{R}$ be continuous, and use v to define an RPS $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ as in Example 1.1. Let $\mathcal{T}'$ be another compact Hausdorff space, let $\psi : \mathcal{T}' \longrightarrow \mathcal{T}$ be continuous, and define $v' := v \circ (\psi \times \psi \times I_{\mathcal{X}}) : \mathcal{T}' \times \mathcal{T}' \times \mathcal{X} \longrightarrow \mathbb{R}$. Under certain conditions, we can use v' to define an RPS $\phi' : \mathcal{T}' \longrightarrow P(\mathcal{T}' \times \mathcal{X})$ as in Example 1.1.[5] In this case, ψ is an RPS morphism from $(\mathcal{T}', \phi')$ to $(\mathcal{T}, \phi)$.

(d) Let $\mathcal{X}$ and $\mathcal{T}$ be finite sets (with the discrete topology), and let $(\mathcal{T}, \phi)$ be an RPS over $\mathcal{X}$. Let $\mathcal{T}'$ be a compact Hausdorff space, and let $\psi : \mathcal{T}' \longrightarrow \mathcal{T}$ be continuous. Given $(\mathcal{Y}, \succ)$ in $P(\mathcal{T} \times \mathcal{X})$, let $\mathcal{Y}' := (\psi \times I_{\mathcal{X}})^{-1}(\mathcal{Y}) \subseteq \mathcal{T}' \times \mathcal{X}$, and for all (t'_1, x_1) and (t'_2, x_2) in $\mathcal{Y}'$ stipulate that $(t'_1, x_1) \succ' (t'_2, x_2)$ if and only if $(\psi(t'_1), x_1) \succ (\psi(t'_2), x_2)$. It is easily verified that $(\mathcal{Y}', \succ') \in P(\mathcal{T}' \times \mathcal{X})$. So if we define $\Psi(\mathcal{Y}, \succ) := (\mathcal{Y}', \succ')$ in this way for all $(\mathcal{Y}, \succ)$ in $P(\mathcal{T} \times \mathcal{X})$, we get a function $\Psi : P(\mathcal{T} \times \mathcal{X}) \longrightarrow P(\mathcal{T}' \times \mathcal{X})$; it is continuous because the co-Vietoris topology on $P(\mathcal{T} \times \mathcal{X})$ is discrete (because $\mathcal{T}$ and $\mathcal{X}$ are discrete). Now define $\phi' := \Psi \circ \phi' \circ \psi : \mathcal{T}' \longrightarrow P(\mathcal{T}' \times \mathcal{X})$. By construction, this function is continuous, so $(\mathcal{T}', \phi')$ is another RPS over $\mathcal{X}$, and ψ is an RPS morphism from $(\mathcal{T}', \phi')$ to $(\mathcal{T}, \phi)$.[6] ◊

An RPS morphism ψ is an *isomorphism* if it is a homeomorphism, and ψ^{-1} is also an RPS morphism. In this case, $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$ are essentially "the same" recursive preference structure, with two different descriptions.

The next result explains that RPS morphisms preserve the relevant notion of optimality for recursive preferences.

Proposition 2.2 *Let $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$ be two recursive preference structures over $\mathcal{X}$, and let $\psi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be a morphism. If (t_1^*, x^*) is recursively optimal for $(\mathcal{T}_1, \phi_1)$, and $t_2^* = \psi(t_1^*)$, then (t_2^*, x^*) is recursively optimal for $(\mathcal{T}_2, \phi_2)$.*

Universal RPS. Let $\mathcal{X}$ be a compact Hausdorff space, and let $(\breve{\mathcal{T}}, \breve{\phi})$ be an RPS over $\mathcal{X}$. We shall say that $(\breve{\mathcal{T}}, \breve{\phi})$ is a *universal* RPS over $\mathcal{X}$ if, for any other RPS $(\mathcal{T}, \phi)$ over $\mathcal{X}$, there is a *unique* RPS morphism $\psi : \mathcal{T} \longrightarrow \breve{\mathcal{T}}$. Thus, $(\breve{\mathcal{T}}, \breve{\phi})$ is "universal" in the sense that any other RPS can be "represented" within it. A universal RPS is thus somewhat

[5] To be precise: ψ has *local sections* if, for all $t' \in \mathcal{T}'$, there is an open neighbourhood $\mathcal{O}$ of $t = \psi(t')$ in $\mathcal{T}$, and a continuous function $\sigma : \mathcal{O} \longrightarrow \mathcal{T}'$ such that $\sigma(t) = t'$ and $\psi \circ \sigma(o) = o$ for all $o \in \mathcal{O}$. If v satisfies the hypotheses of Proposition B.2 so as to define an RPS as in Example 1.1, and ψ has local sections, then v' also satisfies these hypotheses, so it also defines an RPS. See Proposition B.3 for details.

[6] If $\mathcal{T}$ and $\mathcal{X}$ are *not* discrete, then Ψ is generally *not* continuous, even if ψ is a nice function (e.g. a coordinate projection from a product space). That is why this example assumes $\mathcal{T}$ and $\mathcal{X}$ are finite.

analogous to a *universal type spaces* in Bayesian game theory (Mertens and Zamir, 1985; Brandenburger and Dekel, 1993). The next result says that it has some special properties: it is *perfectly autonomous*, and it is essentially unique up to isomorphism.[7]

Proposition 2.3 *Suppose that* $(\breve{\mathcal{T}}, \breve{\phi})$ *is a universal RPS over* $\mathcal{X}$. *Then:*

(a) *The function* $\phi : \breve{\mathcal{T}} \longrightarrow P(\breve{\mathcal{T}} \times \mathcal{X})$ *is a homeomorphism.*

(b) *If* $(\widehat{\mathcal{T}}, \widehat{\phi})$ *is another universal RPS over* $\mathcal{X}$, *then there is a (unique) RPS isomorphism from* $(\breve{\mathcal{T}}, \breve{\phi})$ *to* $(\widehat{\mathcal{T}}, \widehat{\phi})$.

Because of Proposition 2.3(b), we can speak of "the" universal RPS over $\mathcal{X}$, if such an object exists. But its existence is not obvious. We now come to our main result.

Theorem 2.4 *For any compact Hausdorff space* $\mathcal{X}$, *there is a universal recursive preference structure over* $\mathcal{X}$. *The type space* $\breve{\mathcal{T}}$ *of this universal RPS is a compact Hausdorff space. If* $\mathcal{X}$ *is metrizable, then* $\breve{\mathcal{T}}$ *is also metrizable.*

The type space $\breve{\mathcal{T}}$ in the universal RPS of Theorem 2.4 depends on the choice of outcome space $\mathcal{X}$, and does not admit a simple description in general. The next result sheds some light on its topological properties, and provides an explicit description in certain cases.

Proposition 2.5 *Let* $\mathcal{X}$ *be a compact Hausdorff space, and let* $(\breve{\mathcal{T}}, \breve{\phi})$ *be its universal RPS.*

(a) $\breve{\mathcal{T}}$ *contains a subspace homeomorphic to* $\mathcal{X}$.

(b) *If* $\mathcal{X}$ *is homeomorphic to* $\mathbb{K}$, *then so is* $\breve{\mathcal{T}}$.

(c) *If* $\mathcal{X}$ *is a nonsingleton finite set with the discrete topology, then* $\breve{\mathcal{T}} \cong \mathbb{K}$.

(d) *If* $\mathcal{X}$ *is a continuum, then* $\breve{\mathcal{T}}$ *is a continuum.*[8]

Part (a) of Proposition 2.5 shows that the topology of $\breve{\mathcal{T}}$ is at least as large and complicated as that of $\mathcal{X}$ itself. For example, the topological dimension of $\breve{\mathcal{T}}$ must be no less than that of $\mathcal{X}$. Part (b) is a strengthened form of Proposition 1.5. Part (c) shows that even if $\mathcal{X}$ is quite small and simple, $\breve{\mathcal{T}}$ can be quite large and complicated. Part (d) shows that, in contrast to parts (b) and (c), $\breve{\mathcal{T}}$ can be a connected space.

The unique morphism from any other RPS into the universal RPS is called the *terminal morphism*. The next two results give us some insight about this morphism.

Proposition 2.6 *Let* $\mathcal{X}$ *be a compact Hausdorff space, and let* $(\breve{\mathcal{T}}, \breve{\phi})$ *be its universal RPS. Let* $(\mathcal{T}, \phi)$ *be some other RPS over* $\mathcal{X}$, *and let* $\psi : \mathcal{T} \longrightarrow \breve{\mathcal{T}}$ *be the terminal morphism.*

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[7]In fact, these are consequences of a much more general result, which has nothing to do with recursive preference structures *per se*; see Proposition 3.3 below.

[8]See Appendix A.1 for the definition of *continuum*.

(a) *For all $t_1, t_2 \in \mathcal{T}$, if $\phi(t_1) = \phi(t_2)$, then $\psi(t_1) = \psi(t_2)$.*

Thus, if ϕ is not injective, then neither is ψ.

(b) *For all $t_1, t_2 \in \mathcal{T}$, if there is an RPS endomorphism $\alpha : \mathcal{T} \longrightarrow \mathcal{T}$ such that $\alpha(t_1) = t_2$, then $\psi(t_1) = \psi(t_2)$.*

(c) *Let $(\mathcal{T}', \phi')$ be another RPS over $\mathcal{X}$, and let $\gamma_1, \gamma_2 : \mathcal{T} \longrightarrow \mathcal{T}'$ be RPS morphisms (possibly, $\gamma_1 = \gamma_2$). For all $t_1, t_2 \in \mathcal{T}$, if $\gamma_1(t_1) = \gamma_2(t_2)$, then $\psi(t_1) = \psi(t_2)$.*

Proposition 2.6(a,b) says that the terminal morphism eliminates redundancy, by merging elements of $\mathcal{T}$ which are "the same" in some sense. Proposition 2.6(c) says that the terminal morphism is "less injective" than every other RPS morphism out of $(\mathcal{T}, \phi)$: if $\psi(t_1) \neq \psi(t_2)$, then the contrapositive of Proposition 2.6(c) implies that $\gamma(t_1) \neq \gamma(t_2)$ for *any* RPS morphism γ from $(\mathcal{T}, \phi)$ to any other RPS. This might create the concern that ψ is degenerate —perhaps even a constant function. The next result alleviates this concern.

Let $(\mathcal{T}, \phi)$ be an RPS over $\mathcal{X}$. For any $t \in \mathcal{T}$, we define a strict partial order $\ggcurly_t$ on $\mathcal{X}$ as follows. Suppose $\phi(t) = (\mathcal{Y}_t, \succ_t)$, where $\mathcal{Y}_t \subseteq \mathcal{T} \times \mathcal{X}$ and $\succ_t$ is a partial order on $\mathcal{Y}_t$. For all $x_1, x_2 \in \mathcal{X}$, stipulate that $x_1 \ggcurly_t x_2$ if $(x_1, t_1) \succ_t (x_2, t_2)$ for all $t_1, t_2 \in \mathcal{T}$ such that (x_1, t_1) and (x_2, t_2) are in $\mathcal{Y}_t$. Heuristically, $\ggcurly_t$ isolates the part of type t's preferences over outcomes that is so strong that it overrides any of her preferences between different types.

Proposition 2.7 *We continue the notation of Proposition 2.6. Let $s, t \in \mathcal{T}$. If $\ggcurly_s$ and $\ggcurly_t$ are different, then $\psi(s) \neq \psi(t)$.*

Thus, if every t in $\mathcal{T}$ induces a different order $\ggcurly_t$ on $\mathcal{X}$, then ψ is injective.

The next result says that any "similarity" between two topological spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ is reflected by a corresponding similarity between the universal RPSs over these spaces.

Proposition 2.8 *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be compact Hausdorff spaces. Let $(\breve{\mathcal{T}}_1, \breve{\phi}_1)$ and $(\breve{\mathcal{T}}_2, \breve{\phi}_2)$ be the universal RPSs over $\mathcal{X}_1$ and $\mathcal{X}_2$. Let $\xi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ be continuous. There is a unique continuous function $\tau : \breve{\mathcal{T}}_1 \longrightarrow \breve{\mathcal{T}}_2$ such that the following diagram commutes:*

$$\begin{array}{ccc} \breve{\mathcal{T}}_1 & \xrightarrow{\breve{\phi}_1} & P(\breve{\mathcal{T}}_1 \times \mathcal{X}_1) \\ \tau \downarrow & & \downarrow (\tau \times \xi)^{\P} \\ \breve{\mathcal{T}}_2 & \xrightarrow[\breve{\phi}_2]{} & P(\breve{\mathcal{T}}_2 \times \mathcal{X}_2) \end{array} \tag{2}$$

If ξ is surjective (respectively, a homeomorphism), then so is τ.

Despite the similarity between diagrams (1) and (2), the function τ in Proposition 2.8 is *not* an RPS morphism, unless $\mathcal{X}_1 = \mathcal{X}_2$ and ξ is the identity map. In particular, continuous self-maps of $\mathcal{X}$ do *not* induce RPS-endomorphisms of the universal RPS over $\mathcal{X}$. (Indeed, by the uniqueness property of the universal RPS, it *has* no nontrivial RPS-endomorphisms.) Later we shall see that Proposition 2.8 is just one part of a more general result (Proposition 3.2).

3 A categorical perspective

To prove Theorem 2.4, it is helpful to reformulate recursive preference structures in the language of category theory. A (concrete) *category* is a structure $\mathcal{C} = (\mathcal{C}^\circ, \vec{\mathcal{C}})$, where $\mathcal{C}^\circ$ is a collection of sets (each perhaps with some mathematical structure), called *objects*, while $\vec{\mathcal{C}} = \{\vec{\mathcal{C}}(A,B); \ \ A, B \in \mathcal{C}^\circ\}$ is a collection of functions (called *morphisms*), one for each pair of objects in $\mathcal{C}^\circ$. For all $A, B \in \mathcal{C}^\circ$, the elements of $\vec{\mathcal{C}}(A,B)$ are functions from A to B. Furthermore, for any $A, B, C \in \mathcal{C}^\circ$, and any morphisms $\alpha \in \vec{\mathcal{C}}(A,B)$ and $\beta \in \vec{\mathcal{C}}(B,C)$ their composition $\beta \circ \alpha$ is an element of $\vec{\mathcal{C}}(A,C)$. Finally, for all $A \in \mathcal{C}^\circ$, the identity function I_A is always an element of $\vec{\mathcal{C}}(A,A)$. For example, in the category CHS, the objects are *compact Hausdorff spaces*, and the morphisms are continuous functions between them.[9]

Isomorphisms. Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}^\circ$. A morphism $\phi \in \vec{\mathcal{C}}(A,B)$ is called an *isomorphism* if there is a morphism $\psi \in \vec{\mathcal{C}}(B,A)$ (the *inverse* of ϕ) such that $\psi \circ \phi = I_A$ and $\phi \circ \psi = I_B$. If such an isomorphism exists, then we say that A and B are *isomorphic* in the category $\mathcal{C}$. For example, in the category CHS, a function is an isomorphism if and only if it is a homeomorphism.

Endofunctors. An *endofunctor* on a category $\mathcal{C}$ consists of (i) a function $F : \mathcal{C}^\circ \longrightarrow \mathcal{C}^\circ$; and (ii) for all $A, B \in \mathcal{C}^\circ$, a function $F_{A,B} : \vec{\mathcal{C}}(A,B) \longrightarrow \vec{\mathcal{C}}[F(A), F(B)]$, which preserves morphism composition. In other words: for all $A, B, C \in \mathcal{C}^\circ$, if $A' = F(A)$, $B' = F(B)$ and $C' = F(C)$, then we have functions $F_{A,B} : \vec{\mathcal{C}}(A,B) \longrightarrow \vec{\mathcal{C}}(A', B')$, $F_{B,C} : \vec{\mathcal{C}}(B,C) \longrightarrow \vec{\mathcal{C}}(B', C')$ and $F_{A,C} : \vec{\mathcal{C}}(A,C) \longrightarrow \vec{\mathcal{C}}(A', C')$ such that, for all $\alpha \in \vec{\mathcal{C}}(A,B)$ and $\beta \in \vec{\mathcal{C}}(B,C)$, $F_{A,C}(\beta \circ \alpha) = F_{B,C}(\beta) \circ F_{A,B}(\alpha)$. (Normally, we do not write the subscripts on $F_{A,B}$, $F_{B,C}$, etc.)

Example 3.1. (a) Let $\mathcal{X}$ be a compact Hausdorff space, and consider the endofunctor $F_\mathcal{X}$ on CHS defined as follows. For any space $\mathcal{T} \in \mathsf{CHS}^\circ$ let $F_\mathcal{X}(\mathcal{T}) := \mathcal{T} \times \mathcal{X}$, equipped with the Tychonoff product topology. For any spaces $\mathcal{S}, \mathcal{T} \in \mathsf{CHS}^\circ$, and any continuous function $\phi : \mathcal{S} \longrightarrow \mathcal{T}$, let $F_\mathcal{X}(\phi) := \phi \times I_\mathcal{X} : \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{T} \times \mathcal{X}$. It is easily verified through direct computation that $F_\mathcal{X}$ is an endofunctor on CHS.

(b) For any space $\mathcal{X} \in \mathsf{CHS}^\circ$, recall that $P(\mathcal{X})$ is the compact Hausdorff space of all local continuous strict partial orders on $\mathcal{X}$, with the co-Vietoris topology. For any $\mathcal{X}, \mathcal{Y} \in \mathsf{CHS}^\circ$, and any continuous function $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$, there is a continuous function $\phi^\P : P(\mathcal{X}) \longrightarrow P(\mathcal{Y})$, as explained at the start of §2. Furthermore, if $\mathcal{Z} \in \mathsf{CHS}^\circ$ is another space, and $\psi : \mathcal{Y} \longrightarrow \mathcal{Z}$ is another continuous function, then it is easily verified that $(\psi \circ \phi)^\P = \psi^\P \circ \phi^\P$. Thus, if we define $P(\phi) := \phi^\P$ for all continuous functions ϕ between compact Hausdorff spaces, then we get an endofunctor P on CHS (Pivato, 2023b, Theorem 5.1(b)). ♢

The universal RPS endofunctor. Proposition 2.8 is actually a consequence of a more general result.

[9] For more about categories, see Adámek et al. (2009), Awodey (2010) or Leinster (2014).

Proposition 3.2 *For any compact Hausdorff space $\mathcal{X}$, let* $\mathrm{URPS}(\mathcal{X})$ *be the type space of the universal RPS over $\mathcal{X}$, from Theorem 2.4. For any continuous function $\xi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, let* $\mathrm{URPS}(\xi)$ *be the function τ from Proposition 2.8. The* URPS *is an endofunctor on the category* CHS.

Coalgebras. Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor. An F*-coalgebra* is an ordered pair (T, ϕ), where $T \in \mathcal{C}^\circ$ and $\phi \in \vec{\mathcal{C}}(T, F(T))$. In other words, ϕ is a morphism from T to $F(T)$ in the category $\mathcal{C}$.

For example, let $\mathcal{X} \in \mathsf{CHS}^\circ$ be a compact Hausdorff space, let $F_{\mathcal{X}} : \mathsf{CHS} \longrightarrow \mathsf{CHS}$ and $P : \mathsf{CHS} \longrightarrow \mathsf{CHS}$ be the endofunctors introduced in Example 3.1, and let $R_{\mathcal{X}} := P \circ F_{\mathcal{X}}$. So for any space $\mathcal{T} \in \mathsf{CHS}^\circ$, we have $R_{\mathcal{X}}(\mathcal{T}) = P(\mathcal{T} \times \mathcal{X})$. This is an endofunctor on the category CHS. An $R_{\mathcal{X}}$-coalgebra is a pair $(\mathcal{T}, \phi)$, where $\mathcal{T}$ is a compact Hausdorff space, and $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ is a continuous function; in other words, it is a *recursive preference structure* over $\mathcal{X}$.

Coalgebra morphisms. Let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor on a category $\mathcal{C}$, and let (T_1, ϕ_1) and (T_2, ϕ_2) be two F-coalgebras. For any morphism $\psi \in \vec{\mathcal{C}}(T_1, T_2)$, we get a morphism $F(\psi) \in \vec{\mathcal{C}}[F(T_1), F(T_2)]$ (because F is a functor). We say that ψ is an F-*coalgebra morphism* if the following diagram commutes:

$$\begin{array}{ccc} T_1 & \xrightarrow{\phi_1} & F(T_1) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle F(\psi)} \\ T_2 & \xrightarrow[\phi_2]{} & F(T_2) \end{array} \tag{3}$$

For example, let $R_{\mathcal{X}} := P \circ F_{\mathcal{X}}$, as in the previous paragraph. By comparing diagrams (1) and (3), we see that an RPS morphism is just a morphism of $R_{\mathcal{X}}$-coalgebras.

If (T_3, ϕ_3) is another F-coalgebra, and $\xi \in \vec{\mathcal{C}}(T_2, T_3)$, is another F-coalgebra morphism, then it is easily verified that $\xi \circ \psi$ is an F-coalgebra morphism from T_1 to T_3. The collection of all F-coalgebras itself is a category, when endowed with these morphisms.

Terminal coalgebras. Let $\mathcal{C}$ be any category. An object $T \in \mathcal{C}^\circ$ is a *terminal object* if, for any other object $A \in \mathcal{C}^\circ$, there is a *unique* morphism in $\vec{\mathcal{C}}(A, T)$ (called the *terminal morphism*). For example, in the category CHS, the terminal objects are precisely the one-point spaces.[10] It is easily verified that any two terminal objects are isomorphic. Thus, we sometimes speak of "the" terminal object in a category.

Now let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor on a category $\mathcal{C}$. A *terminal* F*-coalgebra* is a terminal object in the category of F-coalgebras. In other words, it is an F-coalgebra $(T_\dagger, \phi_\dagger)$ such that, for any other F-coalgebra (T, ϕ), there is a *unique* $\mathcal{C}$-morphism $\psi \in \vec{\mathcal{C}}(T, T_\dagger)$ which is also a F-algebra morphism.

[10]Similar statements are true in many other familiar categories, such as the category of sets and functions, or the category of vector spaces and linear maps.

For example, let $\mathcal{X} \in \mathsf{CHS}^\circ$ be a compact Hausdorff space, and recall that recursive preference structures over $\mathcal{X}$ are just coalgebras of the endofunctor $R_\mathcal{X} := P \circ F_\mathcal{X} : \mathsf{CHS} \longrightarrow \mathsf{CHS}$. An RPS $(\breve{\mathcal{T}}, \breve{\phi})$ is universal if and only if it is a terminal $R_\mathcal{X}$-coalgebra. Thus, Theorem 2.4 is simply the statement: "The endofunctor $R_\mathcal{X}$ has a terminal coalgebra." Proposition 2.3 is thus an immediate consequence of the following classic result of Lambek, which says that a terminal F-coalgebra is a "fixed point" of the endofunctor F, and is unique up to isomorphism.

Proposition 3.3 (Lambek) *Let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor on a category $\mathcal{C}$, and let (T, ϕ) be a terminal F-coalgebra. Then:*

(a) *ϕ is an isomorphism from T to $F(T)$ in the category $\mathcal{C}$.*

(b) *If (T', ϕ') is another terminal F-coalgebra, then there is a (unique) F-coalgebra isomorphism from (T, ϕ) to (T', ϕ').*

Proof: See Jacobs and Rutten (1997, Lemma 6.4), Rutten (2000, Theorem 9.1) or Awodey (2010, Lemma 10.10, p.269). □

4 Prior literature

An important literature in economics seeks to explain the formation of people's preferences as a process of rational choice (Rotemberg, 1994; Becker and Mulligan, 1997; Akerlof and Kranton, 2000; Ng and Wang, 2001; Palacios-Huerta and Santos, 2004; Welsch, 2004; Östling, 2009; Bernheim et al., 2021). Unlike this literature, the focus of the present paper is *normative*, rather than descriptive. Recursive preference structures are not intended to describe how people *actually* form their preferences. Rather, they address the question: how would an ideally rational agent choose her preferences? An RPS is a formal model of the decision problem that such an agent would face. A *universal* RPS is the version of this decision problem which offers this agent maximal autonomy.

As already noted in the introduction, the coalgebraic approach of this paper is comparable to the coalgebraic construction of universal type spaces in Bayesian game theory, which was pioneered by Moss and Viglizzo (2004, 2006) and further developed by Pintér (2010), Heinsalu (2014), Fukuda (2021), Guarino (2022). But these papers concern hierarchies of probabilistic *beliefs*, whereas we are concerned with hierarchies of *preferences*.

Epstein and Wang (1996), Di Tillio (2008), Chen (2010) and Ganguli et al. (2016) have constructed type spaces involving hierarchies of preferences. But at both a conceptual level and a technical level, these papers are very different from the present work, and essentially unrelated to it. The present paper concerns the preferences of a single agent with full information, whereas these four papers are concerned with strategic interactions between multiple agents in an environment of uncertainty. Their goal is to construct game-theoretic type spaces that allow non-expected utility models of decision-making, and in which the expected utility model can be derived axiomatically *à la* Savage, rather than being assumed *a priori*. Thus, they consider hierarchies of preferences over "Savage acts" —that is, functions mapping a space $\mathcal{S}$ of possible "states of the world" into a space $\mathcal{X}$ of possible outcomes. A complication is that the each element of $\mathcal{S}$ must provide a description not only of the (unknown) state of nature, but also of the (unknown) characteristics of the other players. In the classic Harsanyi model, each point in $\mathcal{S}$ would specify, for each player, a utility function plus a hierarchy of probabilistic beliefs. In the models of the four aforementioned papers, each element of $\mathcal{S}$ specifies, for each player, a hierarchy of preferences over increasingly sophisticated acts; at level n of the hierarchy, acts are functions which depend on all the information available at all lower levels of the hierarchy, and level-n preferences are preferences over these level-n acts.

It might be thought that the central question of the present paper could be addressed by considering a "degenerate" case of the models of these four papers, in which there is only *one* player and only *one* state of nature (so that "acts" could be identified with elements of $\mathcal{X}$). But this is not correct. In their models, an agent does not have preferences *over* the preferences of other players (much less over her *own* preferences); rather, she has preferences over *acts*, whose outcomes depend on the state of the world, which includes a description of the other players' preferences. In the "degenerate" case, there is no uncertainty at all in their models, and the hierarchy never gets off the ground. Finally, because their models involve preferences over *acts*, they use a notion of "consistency" between level-n preferences and level-$(n-1)$ preferences which is unavailable in our framework.[11]

A totally different hierarchical-preference model of strategic interactions has been proposed by Bergemann et al. (2017). In their model, each agent has a finite set of possible utility functions; her *true* utility function is some convex combination of these. So the space of possible preferences for each agent is structurally isomorphic to a probability simplex. Using this isomorphism, Bergemann et al. formally represent a hierarchy of preferences as a hierarchy of *probabilities*, analogous to the standard belief hierarchy of Bayesian game theory. Their main goal is to establish strategic distinguishability between types, which has applications in mechanism design. But like the four papers in the previous paragraph, this paper has little in common with the present work, conceptually or technically.[12]

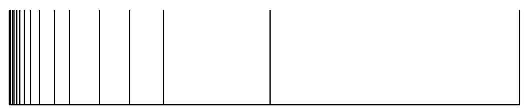
[11] There are also important technical differences. For example, all the levels in Di Tillio's hierarchy are finite sets, with no topology. Epstein and Wang's model (extended by Chen and by Ganguli et al.) *does* have a topology, but it is obtained by representing preferences with utility functions, and is very different from the Vietoris topology used in the present work. Furthermore, while Epstein and Wang and Di Tillio show that their type spaces are *complete* (i.e. there is a canonical isomorphism between the type space and the set of preferences defined with respect to that type space), they do not show that they are *universal*, in the sense of being terminal objects in a category of type spaces and type morphisms. However, Chen and Ganguli et al. do provide universality results of this kind.

[12] Rather than a hierarchy of preferences, Galeazzi and Marti (2023) construct a universal type space based on a hierarchy of *choice functions*, so as to further weaken the rationality assumptions of the model. Aside from their explicitly coalgebraic approach, their work is unrelated to the present paper.

A Mathematical preliminaries

A.1 Continua

A topological space $\mathcal{X}$ is *connected* if it cannot be written as a disjoint union of two nonempty open sets. A *continuum* is a compact, connected, metrizable topological space. A *Peano continuum* is a continuum $\mathcal{X}$ that is *locally connected*: for all $x \in \mathcal{X}$, every open neighbourhood around x contains a *connected* open neighbourhood around x. For example: any compact hypersurface in $\mathbb{R}^N$ is a Peano continuum. Meanwhile, let $\mathcal{Y} := \{0\} \cup \{\frac{1}{n};\ n \in \mathbb{N}\}$, let $\mathcal{X}_1 := \mathcal{Y} \times [0,1]$, let $\mathcal{X}_2 := \{(x,0);\ \ x \in [0,1]\}$, and let $\mathcal{X} := \mathcal{X}_1 \cup \mathcal{X}_2$ (shown below). Then $\mathcal{X}$ is a continuum, but not a Peano continuum.



A metric d on a space $\mathcal{X}$ is *convex* if, for all $x, z \in \mathcal{X}$, there exists $y \in \mathcal{X}$ (possibly not unique) such that $d(x,y) = d(y,z) = \frac{1}{2}\, d(x,z)$. This does not imply that $\mathcal{X}$ is convex in the linear algebra sense. For example: let $\mathcal{X}$ be the unit circle, and for all $x, y \in \mathcal{X}$, let $d(x,y)$ be the angular distance between x and y; then d is a convex metric. A compact topological space has a convex metric if and only if it is a Peano continuum (Bing, 1949; Moise, 1949); see also Illanes and Nadler (1999, Theorem 10.3, p.80).

A.2 Hyperspace

Let $(\mathcal{X}, d)$ be any compact metric space, and let $K(\mathcal{X})$ be the set of all nonempty closed subsets of $\mathcal{X}$. For any $\mathcal{K} \in K(\mathcal{X})$ and any $x \in \mathcal{X}$, we define

$$d(x, \mathcal{K}) \quad := \quad \inf_{k \in \mathcal{K}} d(x,k).$$

The *Hausdorff metric* on $K(\mathcal{X})$ is defined as follows: for any $\mathcal{A}, \mathcal{B} \in K(\mathcal{X})$,

$$d_H(\mathcal{A}, \mathcal{B}) \quad := \quad \max\left\{\sup_{a \in \mathcal{A}}\ d(a, \mathcal{B}),\ \sup_{b \in \mathcal{B}}\ d(b, \mathcal{A})\right\}.$$

With this metric, $K(\mathcal{X})$ is itself a compact metric space (Aliprantis and Border, 2006, Theorem 3.85(3)). It is called a *hyperspace* over $\mathcal{X}$. More generally, suppose that $\mathcal{X}$ is a locally compact Hausdorff space (not necessarily compact or metrizable). Let $\mathfrak{O}(\mathcal{X})$ be the set of all open subsets of $\mathcal{X}$. For any $\mathcal{O} \in \mathfrak{O}(\mathcal{X})$, let

$$\mathcal{O}^u \quad := \quad \{\mathcal{K} \in K(\mathcal{X})\ ;\ \emptyset \neq \mathcal{K} \subseteq \mathcal{O}\} \quad \text{and} \quad \mathcal{O}^\ell \quad := \quad \{\mathcal{K} \in K(\mathcal{X})\ ;\ \mathcal{K} \cap \mathcal{O} \neq \emptyset\}.$$

Next, for any $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_N \in \mathfrak{O}(\mathcal{X})$, let

$$\mathcal{B}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_N) \quad := \quad \mathcal{O}_0^u \cap \mathcal{O}_1^\ell \cap \cdots \cap \mathcal{O}_N^\ell. \tag{A1}$$

Let $\mathfrak{B} := \{\mathcal{B}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_N);\ \ N \in \mathbb{N} \text{ and } \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_N \in \mathfrak{O}(\mathcal{X})\}$. This is a base for a topology on $K(\mathcal{X})$, which is variously called the *Vietoris topology*, *exponential topology*, *Fell topology*, or *topology of closed convergence*.[13] Equipped with this topology, $K(\mathcal{X})$ is itself a compact Hausdorff space (Illanes and Nadler, 1999, Exercise 3.12, p.20). If $(\mathcal{X}, d)$ is a metric space, then the Vietoris topology on $K(\mathcal{X})$ is the same as the Hausdorff metric topology (Aliprantis and Border, 2006, Theorem 3.91). See Aliprantis and Border (2006, §3.17-3.18), Illanes and Nadler (1999) or Beer (1993) for more about hyperspaces.

A.3 Properties of preference spaces.

We now give precise statements of some of the claims made informally in Section 1.

Proposition A.1 *Let $\mathcal{X}$ be a compact Hausdorff space.*

(a) *$P(\mathcal{X})$ is a compact Hausdorff space in the co-Vietoris topology.*

(b) *If $\mathcal{X}$ is metrizable, then so is $P(\mathcal{X})$.*

(c) *If $\mathcal{X}$ is a continuum, then so is $P(\mathcal{X})$.*

(d) *$P(\mathcal{X})$ contains a subspace that is homeomorphic to $\mathcal{X}$.*

Proof: See Pivato 2023b, Theorems 4.1 and 4.3. □

Proposition A.2 *Let $\mathcal{X}$ and $\mathcal{X}'$ be compact Hausdorff spaces, and let $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ be continuous.*

(a) *The function $\phi^{\P} : P(\mathcal{X}) \longrightarrow P(\mathcal{X}')$ is continuous in the co-Vietoris topology.*

(b) *If ϕ is surjective, then so is $\phi^{\P}$.*

Proof: See Pivato 2023b, Theorem 4.5. □

A.4 Limit spaces

Consider a chain

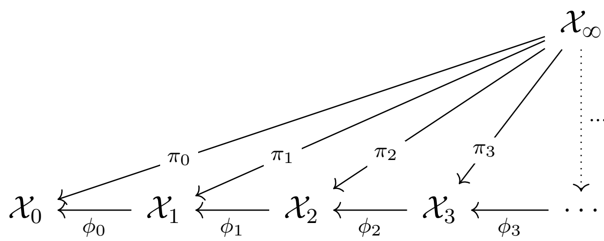
$$\mathcal{X}_0 \xleftarrow{\phi_0} \mathcal{X}_1 \xleftarrow{\phi_1} \mathcal{X}_2 \xleftarrow{\phi_2} \mathcal{X}_3 \xleftarrow{\phi_3} \cdots \tag{A2}$$

where $\mathcal{X}_0, \mathcal{X}_1, \ldots$ are topological spaces and $\phi_0, \phi_1, \ldots$ are continuous functions. The *limit* of the chain (A2) is the space

$$\mathcal{X}_\infty \quad := \quad \left\{ \mathbf{x} \in \prod_{n=0}^{\infty} \mathcal{X}_n \ ; \ \phi_n(x_{n+1}) = x_n, \ \text{for all } n \in \mathbb{N} \right\}, \tag{A3}$$

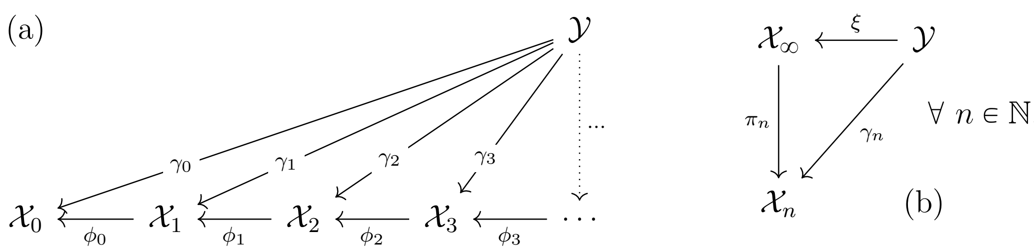
[13] For a general topological space, the Fell topology is coarser, since its base only includes sets $\mathcal{B}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_N)$ where $\mathcal{O}_0$ is *co-compact*. But if $\mathcal{X}$ is compact, then all open sets are co-compact, so Fell and Vietoris are equivalent.

equipped with the subspace topology that it inherits from the Tychonoff product topology on $\prod_{n=0}^{\infty} \mathcal{X}_n$. If $\mathcal{X}_0, \mathcal{X}_1, \ldots$ are compact Hausdorff, then $\mathcal{X}_\infty$ is also compact Hausdorff (Willard, 2004, Theorem 29.11). For all $n \in \mathbb{N}$, let $\pi_n : \mathcal{X}_\infty \longrightarrow \mathcal{X}_n$ be the coordinate projection. These maps define a *cone* over the diagram (A2). In other words, we have a commuting diagram:



(A4)

Now suppose that $\mathcal{Y}$ is some other topological space, and for all $n \in \mathbb{N}$, suppose that there are continuous functions $\gamma_n : \mathcal{Y} \longrightarrow \mathcal{X}_n$ such that we get the commuting diagram (A5)(a):



(A5)

The universal property of limits says that there is a *unique* continuous function $\xi : \mathcal{Y} \longrightarrow \mathcal{X}_\infty$ that makes the diagram obtained by combining diagrams (A4) and (A5)(a) commute, as shown in diagram (A5)(b).

A.5 Limit-preserving endofunctors

Let F be an endofunctor on the category CHS. If we apply F to the chain (A2), we get another chain

$$F(\mathcal{X}_0) \overset{F(\phi_0)}{\longleftarrow} F(\mathcal{X}_1) \overset{F(\phi_1)}{\longleftarrow} F(\mathcal{X}_2) \overset{F(\phi_2)}{\longleftarrow} F(\mathcal{X}_3) \overset{F(\phi_3)}{\longleftarrow} \cdots \tag{A6}$$

Let $\mathcal{Y}_\infty$ be the limit of the chain (A6). Meanwhile, let $\mathcal{X}_\infty$ be the limit of the original chain (A2). There is a natural continuous function from $F(\mathcal{X}_\infty)$ into $\mathcal{Y}_\infty$, as we now explain

Recall that $\mathcal{X}_\infty \subseteq \prod_{n=0}^{\infty} \mathcal{X}_n$, by defining formula (A3). For all $n \in \mathbb{N}$, let $\pi_n : \mathcal{X}_\infty \longrightarrow \mathcal{X}_n$ be the coordinate projection map. Applying F yields a continuous function $\gamma_n := F(\pi_n) : F(\mathcal{X}_\infty) \longrightarrow F(\mathcal{X}_n)$. Combining all of these together, we get a function $\Gamma : F(\mathcal{X}_\infty) \longrightarrow \prod_{n=0}^{\infty} F(\mathcal{X}_n)$ defined by setting $\Gamma(z) := (\gamma_n(z))_{n=0}^{\infty}$, for all $z \in F(\mathcal{X}_\infty)$. But we also have $\mathcal{Y}_\infty \subseteq \prod_{n=0}^{\infty} F(\mathcal{X}_n)$, by applying formula (A3) to chain (A6). It turns out that Γ actually maps $F(\mathcal{X}_\infty)$ into the subspace $\mathcal{Y}_\infty$. To see this, apply F to the cone (A4), to get a new cone

$$\begin{array}{c} F(\mathcal{X}_\infty) \\ \swarrow\ \gamma_0=F(\pi_0),\quad \swarrow\ \gamma_1=F(\pi_1),\quad \swarrow\ \gamma_2=F(\pi_2),\quad \swarrow\ \gamma_3=F(\pi_3),\quad \vdots \\ F(\mathcal{X}_0) \xleftarrow[F[\phi_0]]{} F(\mathcal{X}_1) \xleftarrow[F[\phi_1]]{} F(\mathcal{X}_2) \xleftarrow[F[\phi_2]]{} F(\mathcal{X}_3) \xleftarrow[F[\phi_3]]{} \cdots \end{array} \qquad \text{(A7)}$$

This diagram commutes. So for any $z \in F(\mathcal{X}_\infty)$, if we define $z_n := \gamma_n(z)$ for all $n \in \mathbb{N}$, then $F[\phi_n](z_{n+1}) = z_n$ for all $n \in \mathbb{N}$. Thus, the sequence $(z_n)_{n=0}^\infty$ is an element of $\mathcal{Y}_\infty$, as defined by applying formula (A3) to chain (A6). In other words: $\Gamma(z) \in \mathcal{Y}_\infty$.

We thus get a continuous function $\Gamma : F(\mathcal{X}_\infty) \longrightarrow \mathcal{Y}_\infty$. Of course, Γ is not necessarily a homeomorphism. We say that the functor F ***preserves*** ω***-limits*** on CHS if Γ is a homeomorphism from $F(\mathcal{X}_\infty)$ to $\mathcal{Y}_\infty$, for *any* chain (A2) of compact Hausdorff spaces and continuous functions.[14]

The next result says that the functor P preserves ω-limits. To be precise, suppose we apply P to the chain (A2). Then we get a new chain of compact Hausdorff spaces:

$$\mathcal{P}_0 \xleftarrow{\phi_0^\P} \mathcal{P}_1 \xleftarrow{\phi_1^\P} \mathcal{P}_2 \xleftarrow{\phi_2^\P} \mathcal{P}_3 \xleftarrow{\phi_3^\P} \cdots \qquad \text{(A8)}$$

where $\mathcal{P}_n := P(\mathcal{X}_n)$ for all $n \in \mathbb{N}$. Let $\mathcal{P}_\infty$ be the limit of (A8).

Proposition A.3 *There is a homeomorphism* $\Gamma : P(\mathcal{X}_\infty) \xrightarrow{\sim} \mathcal{P}_\infty$ *defined by setting* $\Gamma(\mathcal{Y}, \succ) := \left(\pi_n^\P(\mathcal{Y}, \succ)\right)_{n=0}^\infty$ *for all* $(\mathcal{Y}, \succ) \in P(\mathcal{X}_\infty)$.

Proof: See Pivato 2023b, Theorem 4.6. □

B Proofs from Section 1

The continuity of the function $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ in Example 1.2 is a consequence of the following slightly stronger result.

Lemma B.1 *Let* $\mathcal{Z}$ *be a Peano continuum, and let* d *be a convex metric on* $\mathcal{Z}$. *Define* δ *on* $\mathcal{Z} \times \mathcal{Z}$ *by setting* $\delta\Big((x, y),\ (x', y')\Big) := \max\Big\{d(x, x'),\ d(y, y')\Big\}$, *for all* $x, y, x', y' \in \mathcal{Z}$. *Then let* d_H *be the Hausdorff metric on* $K(\mathcal{Z} \times \mathcal{Z})$ *induced by* δ. *Via the identification* $P(\mathcal{Z}) \ni (\mathcal{Y}, \succ) \mapsto [\![\mathcal{Y}, \succ]\!] \in K(\mathcal{Z} \times \mathcal{Z})$, *this induces a metric* d_H^* *on* $P(\mathcal{Z})$.

For all $z \in \mathcal{Z}$, *define the relation* $\succ_z$ *on* $\mathcal{Z}$ *by stipulating for all* $x, y \in \mathcal{Z}$ *that* $x \succ_z y$ *if and only if* $d(x, z) < d(y, z)$. *Then* $\succ_z$ *is a continuous partial order on* $\mathcal{Z}$, *so* $(\mathcal{Z}, \succ_z) \in P(\mathcal{Z})$.

Define $\phi : \mathcal{Z} \longrightarrow P(\mathcal{Z})$ *by* $\phi(z) := (\mathcal{Z}, \succ_z)$ *for all* $z \in \mathcal{Z}$. *Then* ϕ *is Lipschitz continuous with respect to* d_H^*, *with Lipschitz factor 2.*

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[14] We have developed this idea in the category CHS. But all of this material generalizes immediately to any endofunctor defined on any category.

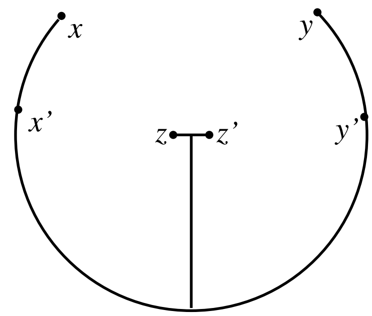


Figure 1: A counterexample showing the need for a convex metric in Lemma B.1.

Proof: For all $z \in \mathcal{Z}$, the relation $\succ_z$ is obviously a partial order. It is continuous because the function $d : \mathcal{Z}\times\mathcal{Z}\longrightarrow\mathbb{R}_+$ is continuous. It remains to show the Lipschitz continuity of ϕ.

For all $z \in \mathcal{Z}$, let $\Phi(z) := [\![\mathcal{Y}, \succ_z]\!] = \{(x,y) \in \mathcal{Z}\times\mathcal{Z};\ \ x \not\prec_z\ y\} = \{(x,y) \in \mathcal{Z}\times\mathcal{Z};$ $d(x,z) \leqslant d(y,z)\}$. Let $z, z' \in \mathcal{Z}$. We must show that $d_H\Big(\Phi(z), \Phi(z')\Big) \leqslant 2\,d(z,z')$.

To do this, let $(x,y) \in \Phi(z)$. We must show that $\inf_{(x',y')\in\Phi(z')} \delta\Big((x,y),\,(x',y')\Big) \leqslant 2\,d(z,z')$. If $d(x,z') \leqslant d(y,z')$, then $(x,y) \in \Phi(z')$, so this infimum is zero. So, assume that $d(x,z') > d(y,z')$. Now let $s := d(x,z') - d(y,z')$. Then

$$\begin{aligned} s \ &\underset{(*)}{\leqslant}\ d(x,z) + d(z,z') - d(y,z) + d(z,z') \\ &=\ d(x,z) - d(y,z) + 2\,d(z,z') \ \underset{(\dagger)}{\leqslant}\ 2\,d(z,z'), \end{aligned} \qquad \text{(B1)}$$

where $(*)$ is because $d(x,z') \leqslant\ d(x,z) + d(z,z')$ while $d(y,z') \geqslant d(y,z) - d(z',z)$, by the triangle inequality, while $(\dagger)$ is because $d(x,z) \leqslant d(y,z)$ because $(x,y) \in \Phi(z)$.

Now let $D := d(x,z')$. Then $s \leqslant D$. Since d is convex, there is an isometric embedding $\zeta : [0,D]\longrightarrow\mathcal{Z}$ with $\zeta(0) = x$ and $\zeta(1) = z'$. (Illanes and Nadler, 1999, Proposition 10.4). Let $x' := \zeta(s)$; then $d(x,x') = s$ and $d(x',z') = d(x,z') - s$, because ζ is an isometric embedding. Thus, $d(x',z') = d(y,z')$, so $(x',y) \in \Phi(z')$. But $\delta\Big((x,y),\,(x',y)\Big) = d(x,x') = s \leqslant 2\,d(z,z')$, where the last step is by inequality (B1). Thus, $\inf_{(x',y')\in\Phi(z')} \delta\Big((x,y),\,(x',y')\Big) \leqslant 2\,d(z,z')$.

This argument works for all $(x,y) \in \Phi(z)$. By a symmetric argument, we have $\inf_{(x,y)\in\Phi(z)} \delta\Big((x',y'),\,(x,y)\Big) \leqslant 2\,d(z,z')$ for all $(x',y') \in \Phi(z')$. We conclude that $d_H\Big(\Phi(z), \Phi(z')\Big) \leqslant$ $2\,d(z,z')$, as desired. □

Remark. Figure 1 shows why a convex metric is needed for the conclusion of Lemma B.1. The figure shows a Peano continuum $\mathcal{Z}$ that is a tree-like subset of the Euclidean plane $\mathbb{R}^2$. Endow $\mathcal{Z}$ with the Euclidean metric (which is *not* convex when restricted to $\mathcal{Z}$). Then along the central "crossbar" of $\mathcal{Z}$, we can find points z and z' that are arbitrarily close together, but such that $x \succ_z y$, while $x \prec_{z'} y$. Furthermore, the only points x' in $\mathcal{Z}$ that are close to x are those on the left "arc", while the only points y' close to y are those on the right "arc", as shown in the figure. Thus, for any pair (x', y') in $\mathcal{Z} \times \mathcal{Z}$ such that (x', y') is close to (x, y), we still have $x' \succ_z y'$, while $x' \prec_{z'} y'$. Thus, the infimum distance from (x, y) to $\Phi(z')$ is large, and the infimum distance from (y, x) to $\Phi(z)$ is also large. Thus, $d_H(\Phi(z), \Phi(z'))$ is large, even though z and z' are very close together. Thus, the function $\phi : \mathcal{Z} \longrightarrow P(\mathcal{Z})$ is *not* continuous in the Hausdorff metric. For any metric compatible with the topology of $\mathcal{Z}$, the corresponding Hausdorff metric on $K(\mathcal{Z} \times \mathcal{Z})$ induces the Vietoris topology (Aliprantis and Border, 2006, Theorem 3.91). So ϕ is *not* Vietoris continuous. $\diamond$

Under what conditions is the function ϕ in Example 1.1 continuous in the co-Vietoris topology? To answer this question, we need a mild technical condition. Let $(s; t, x) \in \mathcal{T} \times \mathcal{T} \times \mathcal{X}$, and let $r := v(s; t, x)$. A *locally parameterized fibre* for v at $(s; t, x)$ is a pair $(\mathcal{Q}, \varphi)$ where $\mathcal{Q} \subseteq \mathcal{T}$ is an open neighbourhood of s and $\varphi : \mathcal{Q} \longrightarrow \mathcal{T} \times \mathcal{X}$ is a continuous function such that $\varphi(s) = (t, x)$, while $v(q; \varphi(q)) = r$ for all $q \in \mathcal{Q}$.

For example, suppose that $\mathcal{T} \subset \mathbb{R}^N$ and $\mathcal{X} \subset \mathbb{R}^M$ are compact sets with nonempty interiors, and $v : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \longrightarrow \mathbb{R}$ is continuously differentiable. Write a generic element of $\mathcal{T}$ as $\mathbf{t} = (t_1, \ldots, t_N)$ and a generic element of $\mathcal{X}$ as $\mathbf{x} = (x_1, \ldots, x_M)$. Suppose that $(\mathbf{s}; \mathbf{t}, \mathbf{x})$ is in the interior of $\mathcal{T} \times \mathcal{T} \times \mathcal{X}$, and either $\frac{\partial v}{\partial t_n}(\mathbf{s}; \mathbf{t}, \mathbf{x}) \neq 0$ for some $n \in [1 \ldots N]$, or $\frac{\partial v}{\partial x_m}(\mathbf{s}; \mathbf{t}, \mathbf{x}) \neq 0$ for some $m \in [1 \ldots M]$. Then the Implicit Function Theorem says that there is a locally parameterized fibre $(\mathcal{Q}, \varphi)$ at $(\mathbf{s}; \mathbf{t}, \mathbf{x})$. More generally, if $\mathcal{T}$ and $\mathcal{X}$ are compact differentiable manifolds, and $v : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \longrightarrow \mathbb{R}$ is continuously differentiable, then a similar statement holds under a similar condition on the derivative of v. However, if $(\mathbf{s}; \mathbf{t}, \mathbf{x})$ is a singular point of v (e.g. a local extremum), then the Implicit Function Theorem is not applicable (because all derivatives of v are zero). Nevertheless, a locally parameterized fibre could still exist at $(\mathbf{s}; \mathbf{t}, \mathbf{x})$.

In general, we will say that a function v satisfies the *IFT property* if it has a locally parameterized fibre at every point in $\mathcal{T} \times \mathcal{T} \times \mathcal{X}$. As explained in the previous paragraph, the name comes from applying the Implicit Function Theorem at nonsingular points of differentiable functions on differentiable manifolds, but the property is meaningful even for singular points, and indeed, applicable even when $\mathcal{T}$ and $\mathcal{X}$ are arbitrary compact Hausdorff spaces and v is an arbitrary continuous function. Nevertheless, this hypothesis *is* somewhat restrictive. To see this, define $\overline{v} : \mathcal{T} \longrightarrow \mathbb{R}$ by setting $\overline{v}(s) := \max\{v(s; t, x);\ (t, x) \in \mathcal{T} \times \mathcal{X}\}$, for all $s \in \mathcal{T}$. This function is well-defined by the Extreme Value Theorem, and continuous by the Berge Maximal Theorem. If v satisfies the IFT property, then $\overline{v}$ must be constant on each connected component of $\mathcal{T}$, because for all $s \in \mathcal{S}$ there must be a locally parameterized fibre $(\mathcal{Q}, \varphi)$ at s such that $v(q; \varphi(q)) = \overline{v}(s)$ (and hence, $\overline{v}(q) \geqslant \overline{v}(s)$) for all $q \in \mathcal{Q}$. We can now answer the question about Example 1.1.

Proposition B.2 *Let $v : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \longrightarrow \mathbb{R}$ and $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$ be as in Example 1.1. If v satisfies the IFT property, then ϕ is continuous in the co-Vietoris topology.*

Proof of Proposition B.2. Define $\phi^{\complement} : \mathcal{T} \longrightarrow K[(\mathcal{T} \times \mathcal{X}) \times (\mathcal{T} \times \mathcal{X})]$ by setting $\phi^{\complement}(s) :=$

$(\mathcal{T}\times\mathcal{X}\times\mathcal{T}\times\mathcal{X})\backslash\phi(s)$ for all $s\in\mathcal{T}$.[15] Then ϕ is co-Vietoris-continuous if and only if $\phi^{\complement}$ is Vietoris-continuous.

Define the correspondence $\Psi:\mathcal{T}\rightrightarrows(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X})$ by setting $\Psi(s):=\phi(s)^{\complement}$ for all $s\in\mathcal{T}$. The function $\phi^{\complement}:\mathcal{T}\longrightarrow K((\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X}))$ is continuous in the Vietoris topology if and only if the correspondence Ψ is continuous (Aliprantis and Border, 2006, Theorem 17.15, p.563). So we must show that Ψ is both upper and lower hemicontinuous.

Upper hemicontinuous. Define $f:\mathcal{T}\times(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X})\longrightarrow\mathbb{R}$ by $f(s;\,t_1,x_1;\,t_2,x_2):=v(s;\,t_1,x_1)-v(s;\,t_2,x_2)$. This function is clearly continuous. The graph of Ψ is the set $\{(s;\,t_1,x_1;\,t_2,x_2)\in\mathcal{T}\times(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X});\ v(s;\,t_1,x_1)\leqslant v(s;\,t_2,x_2)\}=\{(s;\,t_1,x_1;\,t_2,x_2)\in\mathcal{T}\times(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X});\ \ f(s;\,t_1,x_1;\,t_2,x_2)\leqslant 0\}=f^{-1}(-\infty,0]$. This is a continuous preimage of a closed set, hence closed. Thus, Ψ is upper hemicontinuous, because it has a closed graph and $\mathcal{T}\times\mathcal{X}$ is a compact Hausdorff space (Aliprantis and Border, 2006, Theorem 17.11).

Lower hemicontinuous. For any open subset $\mathcal{V}\subseteq(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X})$, define $\Psi^{\ell}(\mathcal{V}):=\{s\in\mathcal{T};\ \ \Psi(s)\cap\mathcal{V}\neq\emptyset\}$. We will show that $\Psi^{\ell}(\mathcal{V})$ itself is open. Let $s\in\Psi^{\ell}(\mathcal{V})$. Then $\Psi(s)\cap\mathcal{V}\neq\emptyset$, so let $(t_1,x_1;\,t_2,x_2)\in\Psi(s)\cap\mathcal{V}$. Thus, $v(s;\,t_1,x_1)\leqslant v(s;\,t_2,x_2)$. Let $r_1:=v(s;\,t_1,x_1)$ and $r_2=v(s;\,t_2,x_2)$. The IFT yields locally parameterized fibres $(\mathcal{Q}_1,\varphi_1)$ and $(\mathcal{Q}_2,\varphi_2)$ at s such that $v(q;\,\varphi_1(q))=r_1$ for all $q\in\mathcal{Q}_1$ and $v(q;\,\varphi_2(q))=r_2$ for all $q\in\mathcal{Q}_2$. Let $\mathcal{Q}_0:=\mathcal{Q}_1\cap\mathcal{Q}_2$. This is an open neighbourhood of s. Define $\varphi:=(\varphi_1,\varphi_2):\mathcal{Q}\longrightarrow(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X})$. Then for all $q\in\mathcal{Q}_0$, we have $v(q;\,\varphi_1(q))\leqslant v(q;\,\varphi_2(q))$ and hence $\varphi(q)\in\Psi(q)$. The function φ is continuous, so $\mathcal{Q}:=\varphi^{-1}(\mathcal{V})$ is an open subset of $\mathcal{Q}_0$. Furthermore, $s\in\mathcal{Q}$ because $\varphi(s)=(t_1,x_1;\,t_2,x_2)\in\mathcal{V}$. By construction, $\Psi(q)\cap\mathcal{V}\neq\emptyset$ for all $q\in\mathcal{Q}$; thus, $\mathcal{Q}\subseteq\Psi^{\ell}(\mathcal{V})$. Thus $\mathcal{Q}$ is an open neighbourhood around s contained in $\Psi^{\ell}(\mathcal{V})$.

This construction works for any $s\in\Psi^{\ell}(\mathcal{V})$. Thus, $\Psi^{\ell}(\mathcal{V})$ is open. This argument works for any open subset $\mathcal{V}\subseteq(\mathcal{T}\times\mathcal{X})\times(\mathcal{T}\times\mathcal{X})$. Thus, Ψ is lower semicontinuous (Aliprantis and Border, 2006, Lemma 17.5). □

The next result justifies a claim made in Footnote 5 of Example 2.1(c).

Proposition B.3 *Let $\mathcal{T}$ and $\mathcal{X}$ be compact Hausdorff spaces, let $v:\mathcal{T}\times\mathcal{T}\times\mathcal{X}\longrightarrow\mathbb{R}$ and $\psi:\mathcal{T}'\longrightarrow\mathcal{T}$ be continuous, and define $v':=v\circ(\psi\times\psi\times I_{\mathcal{X}}):\mathcal{T}'\times\mathcal{T}'\times\mathcal{X}\longrightarrow\mathbb{R}$. If v satisfies the IFT property, and ψ has local sections, then v' also satisfies the IFT property.*

Proof: Let $(s',t',x)\in\mathcal{T}'\times\mathcal{T}'\times\mathcal{X}$. We must show that v' has a locally parameterized fibre near (s',t',x). Thus, if $r:=v'(s',t',x)$, then we want some open neighbourhood $\mathcal{Q}'\subseteq\mathcal{T}'$ around s' and continuous function $\varphi':\mathcal{Q}'\longrightarrow\mathcal{T}'\times\mathcal{X}$ such that $\varphi'(s')=(t',x)$, while $v'(q';\,\varphi'(q'))=r$ for all $q'\in\mathcal{Q}'$.

Let $s:=\psi(s')$ and $t:=\psi(t')$. Then $v(s,t,x)=v(\psi(s'),\psi(t'),x)=v'(s',t',x)=r$. By hypothesis, v has a locally parameterized fibre near (s,t,x). So there is an open

[15] The set $\phi^{\complement}(s)$ is compact because it is closed, because $\phi(s)$ is open because $\succ_s$ is continuous.

neighbourhood $\mathcal{Q} \subseteq \mathcal{T}$ around s and continuous function $\varphi : \mathcal{Q} \longrightarrow \mathcal{T} \times \mathcal{X}$ such that $\varphi(s) = (t, x)$, while $v(q; \varphi(q)) = r$ for all $q \in \mathcal{Q}$. Meanwhile by hypothesis, ψ has local sections, so there is an open neighbourhood $\mathcal{O} \subseteq \mathcal{T}$ around t and a continuous function $\sigma : \mathcal{O} \longrightarrow \mathcal{T}'$ such that $\sigma(t) = t'$ and $\psi \circ \sigma(o) = o$ for all $o \in \mathcal{O}$. Let $\mathcal{O}^* := \varphi^{-1}(\mathcal{O} \times \mathcal{X})$; this is an open neighbourhood of s in $\mathcal{T}$ (because $\varphi(s) = (t, x)$). Now let $\mathcal{Q}' := \psi^{-1}(\mathcal{Q} \cap \mathcal{O}^*)$; then $\mathcal{Q}'$ is an open neighbourhood around s' in $\mathcal{T}'$ (because $\psi(s') = s$). Define $\varphi' := (\sigma \times I_{\mathcal{X}}) \circ \varphi \circ \psi : \mathcal{Q}' \longrightarrow \mathcal{T}' \times \mathcal{X}$. Then φ' is continuous, and $\varphi'(s') = (\sigma \times I_{\mathcal{X}})(\varphi(\psi(s'))) = (\sigma \times I_{\mathcal{X}})(\varphi(s)) = (\sigma \times I_{\mathcal{X}})(t, x) = (t', x)$.

It remains to show that $v'(q'; \varphi'(q')) = r$ for all $q' \in \mathcal{Q}'$. To see this, let $q' \in \mathcal{Q}'$ and let $q := \psi(q')$. Then $q \in \mathcal{Q}$ (because $q' \in \mathcal{Q}' \subseteq \psi^{-1}(\mathcal{Q})$). Suppose that $\varphi(q) = (t_1, x_1)$ for some $t_1 \in \mathcal{T}$ and $x_1 \in \mathcal{X}$. Then $\varphi'(q') = (t'_1, x_1)$, where $t'_1 = \sigma(t_1)$. Thus, $v'(q'; \varphi'(q')) = v'(q', t'_1, x_1) = v(\psi(q'), \psi(t'_1), x_1) = v(q, t_1, x_1) = v(q; \varphi(q)) = r$, as desired. Thus, $(\mathcal{Q}', \varphi')$ is a locally parameterized fibre for v' at (s', t', x'). □

The proof of Proposition 1.4 requires some background. The Cantor space $\mathbb{K}$ is metrizable. In particular, the topology on $\mathbb{K}$ is generated by the metric d defined as follows: for any $x, y \in \mathbb{K}$, set $d(x, y) := 2^{-N}$ where $N := \min\{n \in \mathbb{N};\ x_n \neq y_n\}$.

Proof of Proposition 1.4. If $\mathcal{X}$ and $\mathcal{T}$ are compact metrizable spaces, then $P(\mathcal{T} \times \mathcal{X})$ is also a compact metrizable space, by Proposition A.1(a,b). There is a continuous surjection from $\mathbb{K}$ onto any compact metrizable space (Willard, 2004, Theorem 30.7, p.217). Thus, in particular, there is a continuous surjection from $\mathbb{K}$ onto $P(\mathbb{K} \times \mathcal{X})$. □

The proofs of Propositions 1.5 and 2.5(c) depend on the next result.

Lemma B.4 *$P(\mathbb{K})$ is homeomorphic to $\mathbb{K}$.*

Proof: See Pivato 2023b, Theorem 4.2. □

Proof of Proposition 1.5. If $\mathcal{T} = \mathcal{X} = \mathbb{K}$, then $\mathcal{T} \times \mathcal{X}$ is also homeomorphic to $\mathbb{K}$. Thus, by Lemma B.4, $P(\mathcal{T} \times \mathcal{X})$ is homeomorphic to $\mathbb{K}$. □

C Proofs from Section 2

Proof of Proposition 2.2. Let $(\mathcal{Y}_1, \succ_1) := \phi_1(t_1^*)$ and $(\mathcal{Y}_2, \succ_2) := \phi_2(t_2^*)$. If (t_1^*, x^*) is recursively optimal for $(\mathcal{T}_1, \phi_1)$, then (t_1^*, x^*) is maximal in $(\mathcal{Y}_1, \succ_1)$. Thus, $(\psi \times I_{\mathcal{X}})(t_1^*, x^*)$ is maximal in $(\psi \times I_{\mathcal{X}})^{\P}(\mathcal{Y}_1, \succ_1)$ (Pivato, 2023b, Theorem 4.5(d)). But $(\psi \times I_{\mathcal{X}})(t_1^*, x^*) = (\psi(t_1^*), x^*) = (t_2^*, x)$, while $(\psi \times I_{\mathcal{X}})^{\P}(\mathcal{Y}_1, \succ_1) = (\mathcal{Y}_2, \succ_2)$, because $t_2^* = \psi(t_1^*)$, and ψ is an RPS morphism. Thus, (t_2^*, x^*) is maximal in $(\mathcal{Y}_2, \succ_2)$; in other words, (t_2^*, x^*) is recursively optimal for $(\mathcal{T}_2, \phi_2)$. □

Proof of Proposition 2.3. This follows immediately from Proposition 3.3. □

The proof of Theorem 2.4 depends on a techical result. Let $\mathcal{X}$ be a compact Hausdorff space. For any space $\mathcal{T} \in \mathsf{CHS}^\circ$, let $R_\mathcal{X}(\mathcal{T}) := P(\mathcal{T} \times \mathcal{X})$. For any continuous function $\phi : \mathcal{S} \longrightarrow \mathcal{T}$, let $R_\mathcal{X}(\phi) = (\phi \times I_\mathcal{X})^\P : R_\mathcal{X}(\mathcal{S}) \longrightarrow R_\mathcal{X}(\mathcal{T})$.

Proposition C.1 *$R_\mathcal{X}$ is an endofunctor on the category* CHS*, and it preserves ω-limits.*

Proof: Observe that $R_\mathcal{X} := P \circ F_\mathcal{X}$, where $F_\mathcal{X}$ is the "product" functor from Example 3.1(a), while P is the functor from Example 3.1(b). Since $R_\mathcal{X}$ is a composition of two functors, it is also a functor. The functor $F_\mathcal{X}$ obviously preserves ω-limits, and Proposition A.3 says that P preserves ω-limits. Thus, $R_\mathcal{X}$ preserves ω-limits. □

Proof of Theorem 2.4. We will use a well-known, general method for constructing terminal coalgebras. Let $\mathcal{C}$ be any category, let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor, and let $\mathcal{T}_0$ be a terminal object of the category $\mathcal{C}$. Let $\mathcal{T}_1 := F(\mathcal{T}_0)$. Let $\phi_0 : \mathcal{T}_1 \longrightarrow \mathcal{T}_0$ be the (unique) terminal morphism. Next, let $\mathcal{T}_2 := F(\mathcal{T}_1) = F^2(\mathcal{T}_0)$, and let $\phi_1 := F(\phi_0) : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$. Let $\mathcal{T}_3 := F(\mathcal{T}_2)$, let $\phi_2 := F(\phi_1) = F^2(\phi_0) : \mathcal{T}_3 \longrightarrow \mathcal{T}_2$, and so forth. For all $n \in \mathbb{N}$, let $\mathcal{T}_n := F^n(\mathcal{T}_0)$ and $\phi_n := F^n(\phi_0) : \mathcal{T}_{n+1} \longrightarrow \mathcal{T}_n$. Finally, let $\breve{\mathcal{T}}$ be the limit of the chain

$$\mathcal{T}_0 \xleftarrow{\phi_0} \mathcal{T}_1 \xleftarrow{\phi_1} \mathcal{T}_2 \xleftarrow{\phi_2} \mathcal{T}_3 \xleftarrow{\phi_3} \cdots \tag{C1}$$

Applying F to (C1) yields a new chain: $F(\mathcal{T}_0) \xleftarrow{F(\phi_0)} F(\mathcal{T}_1) \xleftarrow{F(\phi_1)} F(\mathcal{T}_2) \xleftarrow{F(\phi_2)} \cdots$. But $\mathcal{T}_n := F^n(\mathcal{T}_0)$ and $\phi_n := F^n(\phi_0)$ for all $n \in \mathbb{N}$, so this new chain is simply

$$\mathcal{T}_1 \xleftarrow{\phi_1} \mathcal{T}_2 \xleftarrow{\phi_2} \mathcal{T}_3 \xleftarrow{\phi_3} \mathcal{T}_4 \xleftarrow{\phi_4} \cdots \tag{C2}$$

Clearly, the limit of (C2) is isomorphic to the limit of (C1). Thus if the functor F preserves ω-limits as explained Appendix A.5, then there is an isomorphism from $F(\breve{\mathcal{T}})$ to $\breve{\mathcal{T}}$. In this case, by Theorem 3.21 and Corollary 3.22 of Adámek et al. (2018), the limit object $\breve{\mathcal{T}}$ along with the aforementioned isomorphism $\phi : \breve{\mathcal{T}} \longrightarrow F(\breve{\mathcal{T}})$ is the terminal F-coalgebra (see also Awodey (2010, Proposition 10.12, p.271).

In the case that interests us, $\mathcal{C} = \mathsf{CHS}$, and $\mathcal{T}_0$ is the one-point space $\{*\}$. Having fixed some compact Hausdorff space $\mathcal{X}$, F is the functor $R_\mathcal{X}$ that maps any space $\mathcal{T}$ to the space $P(\mathcal{T} \times \mathcal{X})$. Thus, in the chain (C1) shown above, we have $\mathcal{T}_1 = R_\mathcal{X}(\mathcal{T}_0) = P(\{*\} \times \mathcal{X}) \cong P(\mathcal{X})$, $\mathcal{T}_2 = F(\mathcal{T}_1) = P[P(\mathcal{X}) \times \mathcal{X}]$, $\mathcal{T}_3 = F(\mathcal{T}_2) = P(P[P(\mathcal{X}) \times \mathcal{X}] \times \mathcal{X})$, and so forth. Meanwhile, ϕ_0 is the constant map, and $\phi_{n+1} = R_\mathcal{X}(\phi_n)$ for all $n \in \mathbb{N}$. The functor $R_\mathcal{X}$ preserves ω-limits, by Proposition C.1. Thus, there is a natural homeomorphism $\phi : \breve{\mathcal{T}} \longrightarrow R_\mathcal{X}(\breve{\mathcal{T}})$. (The explicit construction appears above diagram (A7).) As explained in the previous paragraph, this is the terminal $R_\mathcal{X}$-coalgebra.

We now turn to the metrizability statement. If $\mathcal{X}$ and $\mathcal{T}$ are both metrizable, then $R_\mathcal{X}(\mathcal{T})$ is also metrizable, by Proposition A.1(b). The one-point space $\mathcal{T}_0$ is (trivially) metrizable. Thus, if $\mathcal{X}$ is metrizable, then each of the spaces $\mathcal{T}_1, \mathcal{T}_2, \ldots$ is metrizable. Thus, the Tychonoff topology on the countable Cartesian product $\prod_{n=0}^{\infty} \mathcal{T}_n$ is metrizable. Since $\breve{\mathcal{T}} \subseteq \prod_{n=0}^{\infty} \mathcal{T}_n$, the subspace topology on $\breve{\mathcal{T}}$ is also metrizable. □

Lemma C.2 *Let $\mathcal{T}$ and $\mathcal{T}'$ be compact Hausdorff spaces. If $\phi : \mathcal{T} \longrightarrow \mathcal{T}'$ is surjective, then $R_{\mathcal{X}}(\phi) : P(\mathcal{T} \times \mathcal{X}) \longrightarrow P(\mathcal{T}' \times \mathcal{X})$ is also surjective.*

Proof: Observe that $R_{\mathcal{X}} := P \circ F_{\mathcal{X}}$, where $F_{\mathcal{X}}$ is just the "product" functor $F_{\mathcal{X}}(\mathcal{T}) = \mathcal{T} \times \mathcal{X}$ and $F_{\mathcal{X}}(\phi) = \phi \times I_{\mathcal{X}} : \mathcal{T} \times \mathcal{X} \longrightarrow \mathcal{T}' \times \mathcal{X}$. If ϕ is surjective, then $\phi \times I_{\mathcal{X}}$ is obviously surjective. Thus, $F_{\mathcal{X}}(\phi)$ is surjective, by Proposition A.2(b). □

Corollary C.3 *For all $n \in \mathbb{N}$, let $\phi_n : \mathcal{T}_{n+1} \longrightarrow \mathcal{T}_n$ be as in the proof of Theorem 2.4. Then ϕ_n is surjective.*

Proof: The function $\phi_0 : \mathcal{T}_1 \longrightarrow \mathcal{T}_0$ is trivially surjective, because $\mathcal{T}_0$ is a one-point space. Thus, by inductive application of Lemma C.2, ϕ_n is surjective, for all $n \in \mathbb{N}$. □

The proof of Proposition 2.5 depends on a technical result. Recall that a topological space $\mathcal{X}$ is *perfect* if it has no isolated points —in other words, for any $x \in \mathcal{X}$, the singleton $\{x\}$ is not open. Equivalently: every open set containing x contains at least one other point.

Proposition C.4 *Let $\mathcal{X}_\infty$ be the limit of a chain (A2) in which $\mathcal{X}_n$ is a perfect metric space and ϕ_n is a continuous surjection for all $n \in \mathbb{N}$. Then $\mathcal{X}_\infty$ is also perfect.*

Proof: For all $n \in \mathbb{N}$, let d_n be a metric on $\mathcal{X}_n$. The topology on $\mathcal{X}_n$ also generated by the bounded metric $\overline{d}_n(x, y) := \min\{1, d_n(x, y)\}$. So by replacing d_n with $\overline{d}_n$ if necessary, we can assume without loss of generality that $d_n(x, y) \leqslant 1$ for all $x, y \in \mathcal{X}_n$. For any $x \in \mathcal{X}_n$ and $\epsilon > 0$, let $\mathcal{B}_n(x, \epsilon) := \{y \in \mathcal{X}_n;\ d_n(x, y) < \epsilon\}$.

The topology on the product space $\prod_{n=0}^{\infty} \mathcal{X}_n$ is metrizable with the metric $d(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} \frac{1}{2^n} d_n(x_n, y_n)$. Define $\mathcal{X}_\infty \subseteq \prod_{n=0}^{\infty} \mathcal{X}_n$ as in formula (A3), and let $\mathbf{x} = (x_n)_{n=0}^{\infty}$ be a point in $\mathcal{X}$. Let $\epsilon > 0$. We must find some $\mathbf{y} \in \mathcal{X}$ with $d(\mathbf{x}, \mathbf{y}) < \epsilon$, but $\mathbf{y} \neq \mathbf{x}$.

Claim 1: *There is a sequence $(\epsilon_n)_{n=0}^{\infty}$ with $0 < \epsilon_n < \epsilon/2$ for all $n \in \mathbb{N}$, such that $\phi_{n-1}[\mathcal{B}_n(x_n, \epsilon_n)] \subseteq \mathcal{B}_{n-1}(x_{n-1}, \epsilon_{n-1})$ for all $n \in \mathbb{N}$.*

Proof: Let $\epsilon_0 := \epsilon/2$. Since $\phi_0 : \mathcal{X}_1 \longrightarrow \mathcal{X}_0$ is continuous, there is some $\epsilon_1 > 0$ such that $\phi_0[\mathcal{B}_1(x_1, \epsilon_1)] \subseteq \mathcal{B}_0(x_0, \epsilon_0)$. Without loss of generality assume $\epsilon_1 \leqslant \epsilon_0$. Next, since $\phi_1 : \mathcal{X}_2 \longrightarrow \mathcal{X}_1$ is continuous, there is some $\epsilon_2 > 0$ such that $\phi_1[\mathcal{B}_2(x_2, \epsilon_2)] \subseteq \mathcal{B}_1(x_1, \epsilon_1)$. Without loss of generality assume $\epsilon_2 \leqslant \epsilon_0$. Inductively, let $N \in \mathbb{N}$, and suppose we have constructed $\epsilon_1, \ldots, \epsilon_N \leqslant \epsilon_0$ such that $\phi_{n-1}[\mathcal{B}_n(x_n, \epsilon_n)] \subseteq \mathcal{B}_{n-1}(x_{n-1}, \epsilon_{n-1})$ for all $n \in [1 \ldots N]$. Since $\phi_N : \mathcal{X}_{N+1} \longrightarrow \mathcal{X}_N$ is continuous, there is some $\epsilon_N > 0$ such that $\phi_N[\mathcal{B}_{N+1}(x_{N+1}, \epsilon_{N+1})] \subseteq \mathcal{B}_N(x_N, \epsilon_N)$. ◇ Claim 1

Now, find some $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\epsilon}{2}$. Now, $\mathcal{X}_N$ is perfect, so there is some $y_{N+1} \in \mathcal{B}(x_{N+1}, \epsilon_{N+1})$, with $y_{N+1} \neq x_{N+1}$. For all $n \in [0 \ldots N]$, let $y_n := \phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_N(y_{N+1})$. Then $y_n \in \mathcal{B}(x_n, \epsilon_n)$, by the defining property of the sequence $(\epsilon_n)_{n=0}^{\infty}$ in Claim 1.

Since $\phi_{N+1} : \mathcal{X}_{N+2} \longrightarrow \mathcal{X}_{N+1}$ is surjective, there is some $y_{N+2} \in \mathcal{X}_{N+2}$ with $\phi_{N+1}(y_{N+2}) = y_{N+1}$. Inductively, using the surjectivity of the functions $\phi_{N+2}, \phi_{N+3}, \ldots$, we can construct a sequence $y_n \in \mathcal{X}_n$ for all $n \geqslant N+2$ such that $\phi_n(y_{n+1}) = y_n$ for all $n \in \mathbb{N}$.

Now let $\mathbf{y} = (y_n)_{n=0}^{\infty}$. The $\mathbf{y} \in \mathcal{X}_\infty$ by construction, and $\mathbf{y} \neq \mathbf{x}$ (because $y_{N+1} \neq x_{N+1}$). By construction, we have $d_n(x_n, y_n) < \epsilon_n < \epsilon/2$ for all $n \in [0 \ldots N+1]$. Meanwhile, $d_n(x_n, y_n) \leqslant 1$ for all $n \geqslant N+2$, because we have assumed that $(\mathcal{X}_n, d_n)$ has diameter 1. Thus,

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= \sum_{n=0}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) = \sum_{n=0}^{N+1} \frac{1}{2^n} d_n(x_n, y_n) + \sum_{n=N+2}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) \\ &< \sum_{n=0}^{N+1} \frac{1}{2^n} \frac{\epsilon}{2} + \sum_{n=N+2}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2} + \frac{1}{2^{N+1}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

as desired. □

Proof of Proposition 2.5 (a) By Proposition 2.3(a), $\breve{\mathcal{T}}$ is homeomorphic to $P(\breve{\mathcal{T}} \times \mathcal{X})$. By Proposition A.1(d), $P(\breve{\mathcal{T}} \times \mathcal{X})$ contains a subspace homeomorphic to $\breve{\mathcal{T}} \times \mathcal{X}$, and thus, a subspace homeomorphic to $\mathcal{X}$.

This proves (a). To prove (b) and (c), we recall the construction from the proof of Theorem 2.4. Recall that $\breve{\mathcal{T}}$ is the limit of a chain $\mathcal{T}_0 \xleftarrow{\phi_0} \mathcal{T}_1 \xleftarrow{\phi_1} \mathcal{T}_2 \xleftarrow{\phi_2} \cdots$, where $\mathcal{T}_0 = \{*\}$, $\mathcal{T}_1 = P(\mathcal{X})$, and $\mathcal{T}_{n+1} = P(\mathcal{T}_n \times \mathcal{X})$ for all $n \in \mathbb{N}$. Meanwhile, ϕ_0 is the (unique) map into $\mathcal{T}_0$, and for all $n \in \mathbb{N}$, $\phi_n = (\phi_{n-1} \times I_{\mathcal{X}})^{\P} : \mathcal{T}_{n+1} = P(\mathcal{T}_n \times \mathcal{X}) \longrightarrow P(\mathcal{T}_{n-1} \times \mathcal{X}) = \mathcal{T}_n$.

(b) If $\mathcal{X}$ is a finite set with the discrete topology, then $\mathcal{T}_1 = P(\mathcal{X})$ is just the set of all strict partial orders defined on some nonempty subset of $\mathcal{X}$. Thus, $\mathcal{T}_1$ is also finite, and the co-Vietoris topology on $\mathcal{T}_1$ is also discrete. Thus, $\mathcal{X} \times \mathcal{T}_1$ is finite and discrete, and thus, $\mathcal{T}_2 = P(\mathcal{T} \times \mathcal{X})$ (the set of all strict partial orders defined on some nonempty subset of $\mathcal{T}_1 \times \mathcal{X}$) is finite and discrete. Inductively, $\mathcal{T}_n$ is finite and discrete for all $n \in \mathbb{N}$. Thus, the Cartesian product $\prod_{n=1}^{\infty} \mathcal{T}_n$ is a Cantor space, and $\breve{\mathcal{T}}$ is a closed subset of this space. Thus, $\breve{\mathcal{T}}$ is compact, metrizable, and totally disconnected.

Claim 1: *For all $n \in \mathbb{N}$, every element of $\mathcal{T}_n$ has more than one ϕ_n-preimage in $\mathcal{T}_{n+1}$.*

Proof: (by induction on n) To prove the case $n = 0$, recall that $\mathcal{T}_0$ is a one-point set and $\mathcal{T}_1 = P(\mathcal{X})$. Clearly, the (unique) map $\phi : \mathcal{T}_1 \longrightarrow \mathcal{T}_0$ is many-to-one.

Now suppose inductively that every element of $\mathcal{T}_{n-1}$ has more than one ϕ_{n-1}-preimage in $\mathcal{T}_n$, and consider the function $\phi_n : \mathcal{T}_{n+1} \longrightarrow \mathcal{T}_n$. Let $(\mathcal{Y}_n, \succ_n) \in \mathcal{T}_n$, and suppose that $(\mathcal{Y}_{n+1}, \succ_{n+1}) \in \mathcal{T}_{n+1}$ is such that $\phi_n(\mathcal{Y}_{n+1}, \succ_{n+1}) = (\mathcal{Y}_n, \succ_n)$. What sort of structure must $(\mathcal{Y}_{n+1}, \succ_{n+1})$ have? Recall that $\mathcal{T}_n = P(\mathcal{T}_{n-1} \times \mathcal{X})$. Thus, $\mathcal{Y}_n \subseteq \mathcal{T}_{n-1} \times \mathcal{X}$ and $\succ_n$ is a partial order on $\mathcal{Y}_n$. Meanwhile, $\phi_n = (\phi_{n-1} \times I_{\mathcal{X}})^{\P}$. Thus, $\mathcal{Y}_n = (\phi_{n-1} \times I_{\mathcal{X}})(\mathcal{Y}_{n+1})$, and for all $(r, x), (s, y) \in \mathcal{Y}_n$, we have $(r, x) \succ_n (s, y)$ if and only if $(r', x) \succ_{n+1} (s', y)$ for all $r' \in \phi_{n-1}^{-1}\{r\}$ and $s' \in \phi_{n-1}^{-1}\{s\}$ such that $(r', x), (s', y) \in \mathcal{Y}_{n+1}$. (By the induction hypothesis, the preimages $\phi_{n-1}^{-1}\{r\}$ and $\phi_{n-1}^{-1}\{s\}$ are nonempty. So it is always possible to construct an order $\succ_{n+1}$ with this property.) Thus, the order $\succ_n$ totally determines the order $\succ_{n+1}$ on all fibres over pairs of elements in $\mathcal{Y}_n$ that are $\succ_n$-comparable.

However, if $r = s$ and $x = y$, then (r, x) and (s, y) are *not* $\succ_n$-comparable, because $\succ_n$ is strict. Thus, for any $x \in \mathcal{X}$, we can arbitrarily set $(r', x) \succ_{n+1} (s', x)$ for some (distinct) $r', s' \in \phi_{n-1}^{-1}\{s\}$ with $(r', x), (s', x) \in \mathcal{Y}_{n+1}$, while we set $(r', x) \prec_{n+1} (s', x)$ for other $r', s' \in \phi_{n-1}^{-1}\{s\}$ with $(r', x), (s', x) \in \mathcal{Y}_{n+1}$, and make (r', x) $\succ_{n+1}$-incomparable to (s', x) for still other $r', s' \in \phi_{n-1}^{-1}\{s\}$ with $(r', x), (s', x) \in \mathcal{Y}_{n+1}$, *as long as we don't set them all the same way.* (Since the topology of $\mathcal{T}_n \times \mathcal{X}$ is discrete, *any* strict partial order on $\mathcal{Y}_{n+1}$ defines an element of $P(\mathcal{T}_n \times \mathcal{X})$.) By the induction hypothesis, $\phi_{n-1}^{-1}\{s\}$ has more than one element. So if $\mathcal{Y}_n$ is large enough (e.g. if $\mathcal{Y}_n = \mathcal{T} \times \mathcal{X}$), then it contains two or more of these preimages. Thus, there are many partial orders $\succ_{n+1}$ on $\mathcal{Y}_{n+1}$ such that $\phi_n(\mathcal{Y}_{n+1}, \succ_{n+1}) = (\mathcal{Y}_n, \succ_n)$. And of course, there are also generally many subsets $\mathcal{Y}_{n+1} \subseteq \mathcal{T}_n \times \mathcal{X}$ such that $(\phi_{n-1} \times I_{\mathcal{X}})(\mathcal{Y}_{n+1}) = \mathcal{Y}_n$. Thus, $(\mathcal{Y}_n, \succ_n)$ has many ϕ_n-preimages in $\mathcal{T}_{n+1}$. ◇ Claim 1

Claim 2: *$\breve{\mathcal{T}}$ is perfect.*

Proof: Let $t \in \breve{\mathcal{T}}$. Then $t = (t_0, t_1, t_2, \ldots)$ where $t_n \in \mathcal{T}_n$ for all $n \in \mathbb{N}$. To show that t is a cluster point of $\breve{\mathcal{T}} \setminus \{t\}$, it suffices to show that, for all $N \in \mathbb{N}$, there is some $t' \in \breve{\mathcal{T}}$ such that $t'_n = t_n$ for all $n \in [0 \ldots N]$, while $t'_n \neq t_n$ for some $n \geqslant N$. This follows from Claim 1. ◇ Claim 2

Thus, $\breve{\mathcal{T}} \cong \mathbb{K}$, because any compact, perfect, totally disconnected metric space is homeomorphic to $\mathbb{K}$ (Willard, 2004, Corollary 30.4).

(c) From Theorem 2.4, we know that $\breve{\mathcal{T}}$ is a compact metrizable space. Following the steps of the construction, we have $\mathcal{T}_1 = P(\mathbb{K}) \cong \mathbb{K}$ by Lemma B.4, and then, by induction $\mathcal{T}_{n+1} = P(\mathcal{T}_n \times \mathcal{X}) \cong P(\mathbb{K} \times \mathbb{K}) \cong \mathbb{K}$ for all $n \in \mathbb{N}$. Thus, $\breve{\mathcal{T}} \subseteq \prod_{n=0}^{\infty} \mathcal{T}_n \cong \prod_{n=0}^{\infty} \mathbb{K} \cong \mathbb{K}$. So $\breve{\mathcal{T}}$ is a subset of a totally disconnected space, hence totally disconnected.

Furthermore, $\mathbb{K}$ is a perfect metric space, so combining Corollary C.3 and Proposition C.4, we deduce that the limit space $\breve{\mathcal{T}}$ is also perfect. Thus, $\breve{\mathcal{T}} \cong \mathbb{K}$, because any compact, perfect, totally disconnected metric space is homeomorphic to $\mathbb{K}$ (Willard, 2004, Corollary 30.4).

(d) If $\mathcal{X}$ is a continuum, then $\mathcal{T}_1 = P(\mathcal{X})$ is a continuum, by Proposition A.1(c). Thus, $\mathcal{T}_1 \times \mathcal{X}$ is a continuum. Thus, $\mathcal{T}_2 = P(\mathcal{T}_1 \times \mathcal{X})$ is a continuum, by Proposition A.1(c).

Likewise, if $\mathcal{T}_n$ is a continuum, then $\mathcal{T}_{n+1} = P(\mathcal{T}_n \times \mathcal{X})$ is a continuum. Inductively, $\mathcal{T}_n$ is a continuum for all $n \in \mathbb{N}$.

Claim 3: *$\breve{\mathcal{T}}$ is connected.*

Proof: For all $N \in \mathbb{N}$, let $\widehat{\mathcal{T}}_N$ be the projection of $\breve{\mathcal{T}}$ onto the coordinates $[0 \ldots N]$. It is easily verified that this is a continuous image of $\mathcal{T}_N$; thus, it is connected. Thus, the product $\overline{\mathcal{T}}_N := \widehat{\mathcal{T}}_N \times \prod_{n=N+1}^{\infty} \mathcal{T}_n$ is also connected. But $\overline{\mathcal{T}}_1 \supseteq \overline{\mathcal{T}}_2 \supseteq \ldots$ and $\breve{\mathcal{T}} = \bigcap_{n=1}^{\infty} \overline{\mathcal{T}}_n$. Thus, it is the intersection of a nested sequence of connected sets, hence connnected. ◇ `Claim 3`

Finally, $\breve{\mathcal{T}}$ is compact and metrizable by Theorem 2.4, because $\mathcal{X}$ is compact and metrizable. Thus, $\breve{\mathcal{T}}$ is a continuum. □

Proof of Proposition 2.6. These are general facts which are true for terminal coalgebras of *any* functor in *any* category. But for completeness, we will provide a proof.

(a) The terminal morphism ψ satisfies the following commuting diagram.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\phi} & P(\mathcal{T} \times \mathcal{X}) \\ \psi \downarrow & & \downarrow \psi^{\dagger} \\ \breve{\mathcal{T}} & \xrightarrow[\breve{\phi}]{} & P(\breve{\mathcal{T}} \times \mathcal{X}) \end{array} \tag{C3}$$

Now, if $\phi(t_1) = \phi(t_2)$, then $\psi^{\dagger} \circ \phi(t_1) = \psi^{\dagger} \circ \phi(t_2)$. Thus $\breve{\phi} \circ \psi(t_1) = \breve{\phi} \circ \psi(t_2)$, by diagram (C3). But Proposition 2.3 says that $\breve{\phi}$ is bijective. So we conclude that $\psi(t_1) = \psi(t_2)$.

(b) If α is an endomorphism of $(\mathcal{T}, \phi)$, then $\psi \circ \alpha$ is an RPS morphism from $(\mathcal{T}, \phi)$ to $(\breve{\mathcal{T}}, \breve{\phi})$. But ψ is the *unique* morphism from $(\mathcal{T}, \phi)$ to $(\breve{\mathcal{T}}, \breve{\phi})$. So we must have $\psi \circ \alpha = \psi$. Thus, if $\alpha(t_1) = t_2$, then $\psi(t_1) = \psi(t_2)$.

(c) This just extends the argument of (b). Let $\psi' : \mathcal{T} \longrightarrow \breve{\mathcal{T}}$ be the terminal morphism out of $(\mathcal{T}', \phi')$. Then $\psi' \circ \gamma_1$ and $\psi' \circ \gamma_2$ are both morphisms from $(\mathcal{T}, \phi)$ to $(\breve{\mathcal{T}}, \breve{\phi})$. But ψ is the *unique* morphism from $(\mathcal{T}, \phi)$ to $(\breve{\mathcal{T}}, \breve{\phi})$. So we must have $\psi' \circ \gamma_1 = \psi = \psi' \circ \gamma_2$. Thus, if $\gamma_1(t_1) = \gamma_2(t_2)$, and hence $\psi' \circ \gamma_1(t_1) = \psi' \circ \gamma_2(t_2)$, then $\psi(t_1) = \psi(t_2)$. □

Remark. Proposition 2.6(a) is not only true for terminal morphisms. It holds for any RPS $(\breve{\mathcal{T}}, \breve{\phi})$ in which $\breve{\phi}$ is injective (e.g. any perfectly autonomous RPS) and any RPS morphism $\psi : \mathcal{T} \longrightarrow \breve{\mathcal{T}}$.

Proof of Proposition 2.7. Let $\pi_{\mathcal{X}} : \mathcal{T} \times \mathcal{X} \longrightarrow \mathcal{X}$ and $\breve{\pi}_{\mathcal{X}} : \breve{\mathcal{T}} \times \mathcal{X} \longrightarrow \mathcal{X}$ be the projection maps onto the second coordinate. Consider the following commuting diagram:

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{\phi} & P(\mathcal{T}\times\mathcal{X}) & & \\ \psi\big\downarrow & & \psi^{\dagger}\big\downarrow & \searrow^{\pi^{\P}_{\mathcal{X}}} & \\ \breve{\mathcal{T}} & \xrightarrow[\breve{\phi}]{} & P(\breve{\mathcal{T}}\times\mathcal{X}) & \xrightarrow[\breve{\pi}^{\P}_{\mathcal{X}}]{} & P(\mathcal{X}) \end{array} \tag{C4}$$

The rectangle on the left commutes because ψ is an RPS morphism. The triangle on the right commutes because $\psi^{\dagger} = (\psi \times I_{\mathcal{X}})^{\P}$, and thus,

$$\breve{\pi}^{\P}_{\mathcal{X}} \circ \psi^{\dagger} \quad = \quad \breve{\pi}^{\P}_{\mathcal{X}} \circ (\psi \times I_{\mathcal{X}})^{\P} \underset{(F)}{=} [\breve{\pi}_{\mathcal{X}} \circ (\psi \times I_{\mathcal{X}})]^{\P} \quad = \quad \pi^{\P}_{\mathcal{X}},$$

where (F) is by functoriality. Now, for any $t \in \mathcal{T}$, if $\phi(t) = (\mathcal{Y}_t, \succ_t)$, then $\pi^{\P}_{\mathcal{X}} \circ \phi(t) = \pi^{\P}_{\mathcal{X}}(\mathcal{Y}_t, \succ_t) = (\mathcal{X}_t, \gg_t)$, where $\mathcal{X}_t := \pi_{\mathcal{X}}(\mathcal{Y}_t)$. Thus, if $\gg_t$ and $\gg_s$ are distinct, then $\pi^{\P}_{\mathcal{X}} \circ \phi(t) \neq \pi^{\P}_{\mathcal{X}} \circ \phi(s)$. Thus, $\breve{\pi}^{\P}_{\mathcal{X}} \circ \breve{\phi} \circ \psi(t) \neq \breve{\pi}^{\P}_{\mathcal{X}} \circ \breve{\phi} \circ \psi(s)$, because diagram (C4) says that $\breve{\pi}^{\P}_{\mathcal{X}} \circ \breve{\phi} \circ \psi = \pi^{\P}_{\mathcal{X}} \circ \phi$. Thus, we must have $\psi(t) \neq \psi(s)$. □

Proof of Propositions 2.8 and 3.2. The key idea of the proof is that each of the components in the construction of the universal RPS is itself functorial in the argument $\mathcal{X}$. To be precise, let $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{X}''$ be compact Hausdorff spaces. Let $\mathcal{T}_0 = \mathcal{T}'_0 = \mathcal{T}''_0 = \{*\}$ be the one-point space, let $\mathcal{T}_1 := P(\mathcal{X})$, $\mathcal{T}'_1 := P(\mathcal{X}')$, and $\mathcal{T}''_1 := P(\mathcal{X}'')$, and for all $n \geqslant 1$, inductively define $\mathcal{T}_{n+1} := P(\mathcal{T}_n \times \mathcal{X})$, $\mathcal{T}'_{n+1} := P(\mathcal{T}'_n \times \mathcal{X}')$, and $\mathcal{T}''_{n+1} := P(\mathcal{T}''_n \times \mathcal{X}'')$. Let $\phi_0 : \mathcal{T}_1 \longrightarrow \mathcal{T}_0$, $\phi'_0 : \mathcal{T}'_1 \longrightarrow \mathcal{T}'_0$ and $\phi''_0 : \mathcal{T}''_1 \longrightarrow \mathcal{T}''_0$ be the terminal morphisms (i.e. the unique map into the one-point space). Letting $\square$ represent either a blank space, one prime, or two primes, let $\pi_{\mathcal{X}^{\square}} : \mathcal{T}^{\square}_1 \times \mathcal{X}^{\square} \longrightarrow \mathcal{X}^{\square}$ be the projection map onto the second coordinate, and then define

$$\phi^{\square}_1 := \pi^{\P}_{\mathcal{X}^{\square}} : P(\mathcal{T}^{\square}_1 \times \mathcal{X}^{\square}) = \mathcal{T}^{\square}_2 \longrightarrow \mathcal{T}^{\square}_1 = P(\mathcal{X}^{\square}). \tag{C5}$$

Now, for all $n \geqslant 2$, inductively assume that we have already defined $\phi^{\square}_n : \mathcal{T}^{\square}_n \longrightarrow \mathcal{T}^{\square}_{n-1}$, and then define

$$\phi^{\square}_{n+1} := (\phi^{\square}_n \times I_{\mathcal{X}^{\square}})^{\P} : \mathcal{T}^{\square}_{n+1} = P(\mathcal{T}^{\square}_n \times \mathcal{X}^{\square}) \longrightarrow P(\mathcal{T}^{\square}_{n-1} \times \mathcal{X}^{\square}) = \mathcal{T}^{\square}_n. \tag{C6}$$

This yields the chain

$$\mathcal{T}^{\square}_0 \xleftarrow{\phi^{\square}_0} \mathcal{T}^{\square}_1 \xleftarrow{\phi^{\square}_1} \mathcal{T}^{\square}_2 \xleftarrow{\phi^{\square}_2} \mathcal{T}^{\square}_3 \xleftarrow{\phi^{\square}_3} \cdots \tag{C7}$$

Finally, let $\breve{\mathcal{T}}^{\square}$ be the limit of the chain (C7) in the category CHS. Recall from the proof of Theorem 2.4 that this is the type space of the universal RPS over $\mathcal{X}^{\square}$.

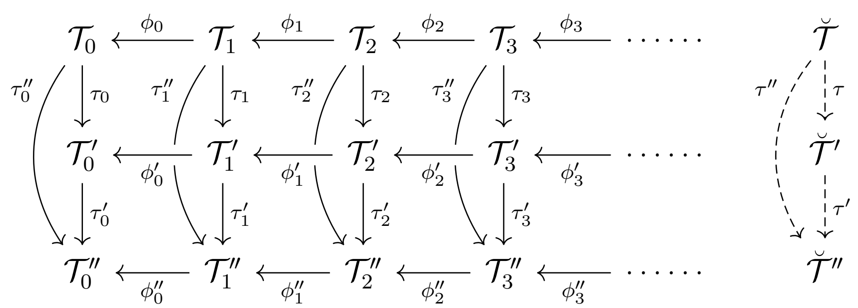
Let $\xi : \mathcal{X} \longrightarrow \mathcal{X}'$ and $\xi' : \mathcal{X}' \longrightarrow \mathcal{X}''$ be continuous, and let $\xi'' := \xi' \circ \xi : \mathcal{X} \longrightarrow \mathcal{X}''$. Define

$$\begin{array}{rrcl} & \tau_1 & := & \xi^{\P} : \mathcal{T}_1 = P(\mathcal{X}) \longrightarrow P(\mathcal{X}') = \mathcal{T}'_1, \\ & \tau'_1 & := & (\xi')^{\P} : \mathcal{T}'_1 = P(\mathcal{X}') \longrightarrow P(\mathcal{X}'') = \mathcal{T}''_1, \\ \text{and} & \tau''_1 & := & (\xi'')^{\P} : \mathcal{T}_1 = P(\mathcal{X}) \longrightarrow P(\mathcal{X}'') = \mathcal{T}''_1. \end{array} \tag{C8}$$

Then $\tau_1'' = \tau_1' \circ \tau_1$, because P is a functor (Example 3.1(b)) and $\xi'' = \xi' \circ \xi$. Inductively, for all $n \geqslant 2$, suppose we have defined $\tau_n : \mathcal{T}_n \longrightarrow \mathcal{T}_n'$, $\tau_n' : \mathcal{T}_n' \longrightarrow \mathcal{T}_n''$ and $\tau_n'' : \mathcal{T}_n \longrightarrow \mathcal{T}_n''$ such that $\tau_n'' = \tau_n' \circ \tau_n$. Now define

$$\begin{array}{rrcl} & \tau_{n+1} & := & (\tau_n \times \xi)^{¶} : \mathcal{T}_{n+1} = P(\mathcal{T}_n \times \mathcal{X}) \longrightarrow P(\mathcal{T}_n' \times \mathcal{X}') = \mathcal{T}_{n+1}', \\ & \tau_{n+1}' & := & (\tau_n' \times \xi')^{¶} : \mathcal{T}_{n+1}' = P(\mathcal{T}_n' \times \mathcal{X}') \longrightarrow P(\mathcal{T}_n'' \times \mathcal{X}'') = \mathcal{T}_{n+1}'', \\ \text{and} & \tau_{n+1}'' & := & (\tau_n'' \times \xi'')^{¶} : \mathcal{T}_{n+1} = P(\mathcal{T}_n \times \mathcal{X}) \longrightarrow P(\mathcal{T}_n'' \times \mathcal{X}'') = \mathcal{T}_{n+1}''. \end{array} \tag{C9}$$

Then $\tau_{n+1}'' = \tau_{n+1}' \circ \tau_{n+1}$ because P is a functor, $\tau_n'' = \tau_n' \circ \tau_n$, and $\xi'' = \xi' \circ \xi$. At this point, we have the diagram shown in the left part of (C10) below:



(C10)

Claim 1: *The diagram in the left part of (C10) commutes.*

Proof: We have already shown that $\tau_n'' = \tau_n' \circ \tau_n$ for all $n \in \mathbb{N}$. Now consider the squares in diagram (C10). First note that ϕ_0, ϕ_0', ϕ_0'', τ_0, τ_0', and τ_0'' are all instances of the (unique) terminal morphism from their respective domains. So the squares in the farthest left column commute, trivially. Next,

$$\tau_1 \circ \phi_1 \underset{\text{(C5,C8)}}{=\!=\!=} \xi^{¶} \circ \pi_{\mathcal{X}}^{¶} \underset{(F)}{=\!=} (\xi \circ \pi_{\mathcal{X}})^{¶} \underset{(*)}{=\!=} \Big(\pi_{\mathcal{X}'} \circ (\tau_1 \times \xi)\Big)^{¶} \underset{(F)}{=\!=} \pi_{\mathcal{X}'}^{¶} \circ (\tau_1 \times \xi)^{¶} \underset{\text{(C5,C9)}}{=\!=\!=} \phi_1' \circ \tau_2,$$

so the top square in the second column of (C10) commutes. Here, the equalities marked (F) are because P is a functor, while $(*)$ is because $\xi \circ \pi_{\mathcal{X}}(t,x) = \xi(x) = \pi_{\mathcal{X}'} \circ (\tau_1 \times \xi)(t,x)$ for all $(t,x) \in \mathcal{T}_1 \times \mathcal{X}$, and thus, $\xi \circ \pi_{\mathcal{X}} = \pi_{\mathcal{X}'} \circ (\tau_1 \times \xi)$. The other equalities come from substituting the definitions from equations (C5), (C8), and (C9). By a similar argument, the bottom square in the second column commutes.

By induction, suppose that we have shown that the squares of the nth column of (C10) commute —i.e. that

$$\tau_{n-1} \circ \phi_{n-1} = \phi_{n-1}' \circ \tau_n \quad \text{and} \quad \tau_{n-1}' \circ \phi_{n-1}' = \phi_{n-1}'' \circ \tau_n'. \tag{C11}$$

Then

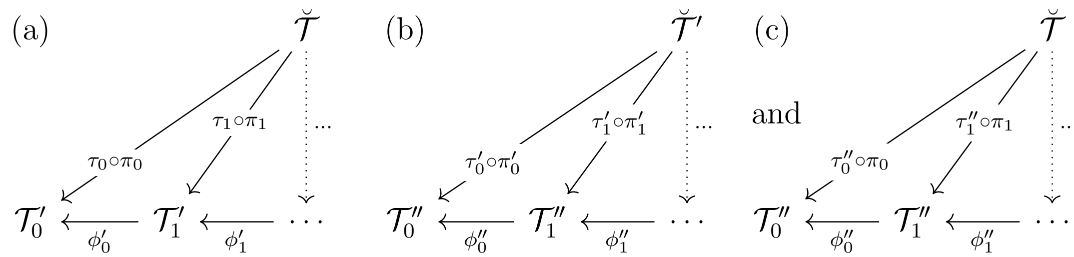
$$\begin{aligned} \tau_n \circ \phi_n &\underset{(*)}{=\!=} (\tau_{n-1} \times \xi)^{¶} \circ (\phi_{n-1} \times I_{\mathcal{X}})^{¶} \underset{(F)}{=\!=} \Big((\tau_{n-1} \times \xi) \circ (\phi_{n-1} \times I_{\mathcal{X}})\Big)^{¶} \\ &= \Big((\tau_{n-1} \circ \phi_{n-1}) \times (\xi \circ I_{\mathcal{X}})\Big)^{¶} \underset{(\dagger)}{=\!=} \Big((\phi_{n-1}' \circ \tau_n) \times \xi\Big)^{¶} \\ &= \Big((\phi_{n-1}' \times I_{\mathcal{X}'}) \circ (\tau_n \times \xi)\Big)^{¶} \underset{(F)}{=\!=} (\phi_{n-1}' \times I_{\mathcal{X}'})^{¶} \circ (\tau_n \times \xi)^{¶} \underset{(*)}{=\!=} \phi_n' \circ \tau_{n+1}, \end{aligned}$$

as desired. Here, both $(*)$ are by defining formulae (C6) and (C9), both (F) are because P is a functor, and $(\dagger)$ is by the induction hypothesis (C11).

By a similar argument, we obtain $\tau'_n \circ \phi'_n = \phi''_n \circ \tau'_{n+1}$. By induction, the whole diagram commutes. ◇ Claim 1

For all $n \in \mathbb{N}$, let $\pi_n : \breve{\mathcal{T}} \longrightarrow \mathcal{T}_n$, $\pi'_n : \breve{\mathcal{T}}' \longrightarrow \mathcal{T}'_n$, and $\pi''_n : \breve{\mathcal{T}}'' \longrightarrow \mathcal{T}''_n$ be the projection maps associated with the limit of the chains (C7). Composing these with τ_n, τ'_n and τ''_n yields the maps $\tau_n \circ \pi_n : \breve{\mathcal{T}} \longrightarrow \mathcal{T}'_n$, $\tau'_n \circ \pi'_n : \breve{\mathcal{T}}' \longrightarrow \mathcal{T}''_n$, and $\tau''_n \circ \pi_n : \breve{\mathcal{T}} \longrightarrow \mathcal{T}''_n$, for all $n \in \mathbb{N}$.

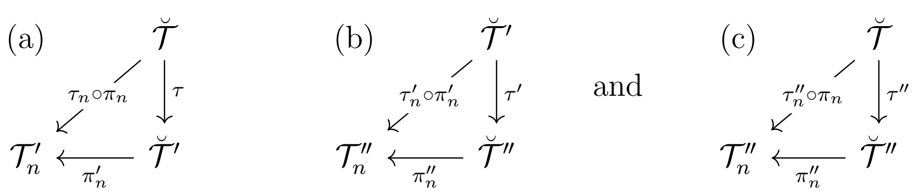
Claim 2: *The following three diagrams commute:*



(C12)

Proof: We will prove the claim for diagram (C12)(a). (The proofs for the other diagrams follow the same pattern). We have $\phi'_n \circ (\tau_{n+1} \circ \pi_{n+1}) = (\phi'_n \circ \tau_{n+1}) \circ \pi_{n+1} \underset{(*)}{=} (\tau_n \circ \phi_n) \circ \pi_{n+1} = \tau_n \circ (\phi_n \circ \pi_{n+1}) \underset{(\dagger)}{=} \tau_n \circ \pi_n$, where $(*)$ is by Claim 1 and $(\dagger)$ is by the property of the limit cone defining $\breve{\mathcal{T}}$. ◇ Claim 2

In light of Claim 2, the universal property of limits yields unique continuous functions $\tau : \breve{\mathcal{T}} \longrightarrow \breve{\mathcal{T}}'$, $\tau' : \breve{\mathcal{T}}' \longrightarrow \breve{\mathcal{T}}''$, and $\tau'' : \breve{\mathcal{T}} \longrightarrow \breve{\mathcal{T}}''$ such that, for all $n \in \mathbb{N}$, the following diagrams commute:



(C13)

Using these diagrams, we can finally prove that the "limit" diagram at the right end of (C10) also commutes.

Claim 3: $\tau'' = \tau' \circ \tau$.

Proof: For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \pi''_n \circ (\tau' \circ \tau) &= (\pi''_n \circ \tau') \circ \tau \underset{(*)}{=} (\tau'_n \circ \pi'_n) \circ \tau = \tau'_n \circ (\pi'_n \circ \tau) \\ &\underset{(\dagger)}{=} \tau'_n \circ (\tau_n \circ \pi_n) = (\tau'_n \circ \tau_n) \circ \pi_n \underset{(\diamond)}{=} \tau''_n \circ \pi_n, \end{aligned}$$

where $(*)$ is by diagram (C13)(b), $(\dagger)$ is by diagram (C13)(a), and $(\diamond)$ is by Claim 1.

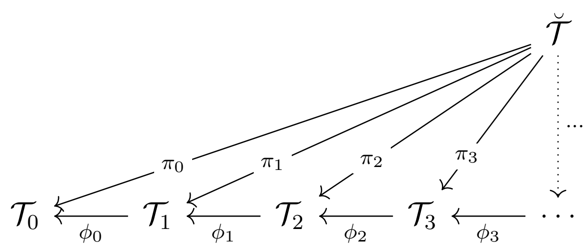
Thus, for all $n \in \mathbb{N}$, we have $\pi''_n \circ (\tau' \circ \tau) = \tau''_n \circ \pi_n$. In other words, $\tau' \circ \tau$ satisfies the defining property of τ'', as expressed in diagram (C13)(c). Thus, we must have $\tau'' = \tau' \circ \tau$. ◇ `Claim 3`

Now, let $\breve{\phi} : \breve{\mathcal{T}} \xrightarrow{\sim} P(\breve{\mathcal{T}} \times \mathcal{X})$, $\breve{\phi}' : \breve{\mathcal{T}}' \xrightarrow{\sim} P(\breve{\mathcal{T}}' \times \mathcal{X}')$, and $\breve{\phi}'' : \breve{\mathcal{T}}'' \xrightarrow{\sim} P(\breve{\mathcal{T}}'' \times \mathcal{X}'')$ be the homeomorphisms constructed in the proof of Theorem 2.4. The next claim establishes the first statement in Proposition 2.8.

Claim 4: *The following diagrams commute:*

$$\text{(a)}\quad \begin{array}{ccc} \breve{\mathcal{T}} & \overset{\breve{\phi}}{\rightsquigarrow} & P(\breve{\mathcal{T}} \times \mathcal{X}) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle (\tau\times\xi)^{¶}} \\ \breve{\mathcal{T}}' & \underset{\breve{\phi}'}{\rightsquigarrow} & P(\breve{\mathcal{T}}' \times \mathcal{X}') \end{array} \qquad \text{(b)}\quad \begin{array}{ccc} \breve{\mathcal{T}}' & \overset{\breve{\phi}'}{\rightsquigarrow} & P(\breve{\mathcal{T}}' \times \mathcal{X}') \\ {\scriptstyle \tau'}\downarrow & & \downarrow{\scriptstyle (\tau'\times\xi')^{¶}} \\ \breve{\mathcal{T}}'' & \underset{\breve{\phi}''}{\rightsquigarrow} & P(\breve{\mathcal{T}}'' \times \mathcal{X}'') \end{array} \quad \text{and} \quad \text{(c)}\quad \begin{array}{ccc} \breve{\mathcal{T}} & \overset{\breve{\phi}}{\rightsquigarrow} & P(\breve{\mathcal{T}} \times \mathcal{X}) \\ {\scriptstyle \tau''}\downarrow & & \downarrow{\scriptstyle (\tau''\times\xi'')^{¶}} \\ \breve{\mathcal{T}}'' & \underset{\breve{\phi}''}{\rightsquigarrow} & P(\breve{\mathcal{T}}'' \times \mathcal{X}'') \end{array}$$

Proof: We will prove that diagram (a) commutes. (The proofs for diagrams (b) and (c) follow the same pattern.) For all $n \in \mathbb{N}$, let $\pi_n : \breve{\mathcal{T}} \longrightarrow \mathcal{T}_n$ be the projection maps associated with the limit of chain (C7). We thus have a limit cone

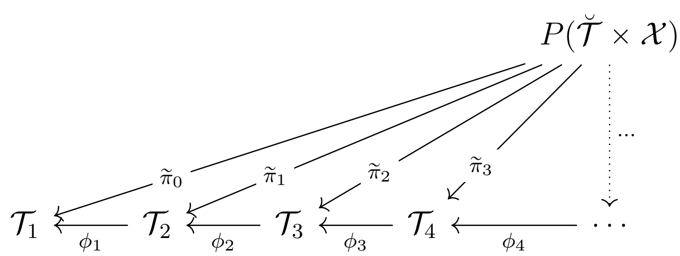


(C14)

Then let

$$\begin{array}{rrcl} & \widetilde{\pi}_n & := & (\pi_n \times I_{\mathcal{X}})^{¶} : P(\breve{\mathcal{T}} \times \mathcal{X}) \longrightarrow P(\mathcal{T}_n \times \mathcal{X}) \;=\; \mathcal{T}_{n+1} \quad \text{(C15)} \\ \text{and} & \widetilde{\pi}'_n & := & (\pi'_n \times I_{\mathcal{X}'})^{¶} : P(\breve{\mathcal{T}}' \times \mathcal{X}') \longrightarrow P(\mathcal{T}'_n \times \mathcal{X}') \;=\; \mathcal{T}'_{n+1}. \quad \text{(C16)} \end{array}$$

We thus get another cone:



(C17)

and a corresponding cone involving the functions $\{\widetilde{\pi}'_n\}_{n=0}^{\infty}$. (To see that (C17) commutes, note that $\phi_{n+1} = R_{\mathcal{X}}(\phi_n)$ and $\widetilde{\pi}_n = R_{\mathcal{X}}(\pi_n)$ for all $n \in \mathbb{N}$, and recall $R_{\mathcal{X}}$

is an endofunctor by Proposition C.1. Thus, the diagram (C17) commutes because it is the result of applying the endofunctor $R_{\mathcal{X}}$ to the commuting diagram (C14).) Thus, the universal property of the limits $\breve{\mathcal{T}}$ and $\breve{\mathcal{T}}'$ yields unique continuous functions $\gamma : P(\breve{\mathcal{T}} \times \mathcal{X}) \longrightarrow \breve{\mathcal{T}}$ and $\gamma' : P(\breve{\mathcal{T}}' \times \mathcal{X}') \longrightarrow \breve{\mathcal{T}}'$ such that, for all $n \in \mathbb{N}$, the following diagrams commute:

$$\text{(i)}\quad \begin{array}{ccc} & & P(\breve{\mathcal{T}} \times \mathcal{X}) \\ & \swarrow^{\widetilde{\pi}_n} & \downarrow{\scriptstyle \gamma} \\ \mathcal{T}_{n+1} & \xleftarrow[\pi_{n+1}]{} & \breve{\mathcal{T}} \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} & & P(\breve{\mathcal{T}}' \times \mathcal{X}') \\ & \swarrow^{\widetilde{\pi}'_n} & \downarrow{\scriptstyle \gamma'} \\ \mathcal{T}'_{n+1} & \xleftarrow[\pi'_{n+1}]{} & \breve{\mathcal{T}}' \end{array} \tag{C18}$$

Now $R_{\mathcal{X}}$ preserves ω-limits (by Proposition C.1), so γ and γ' are actually homeomorphisms. In the proof of Theorem 2.4, we then defined $\breve{\phi} := \gamma^{-1}$ and $\breve{\phi}' := (\gamma')^{-1}$. Thus, $\breve{\phi}$ and $\breve{\phi}'$ are the unique functions such that the following diagrams commute, for all $n \in \mathbb{N}$:

$$\text{(i)}\quad \begin{array}{ccc} & & P(\breve{\mathcal{T}} \times \mathcal{X}) \\ & \swarrow^{\widetilde{\pi}_n} & \Big\uparrow{\scriptstyle \breve{\phi}} \\ \mathcal{T}_{n+1} & \xleftarrow[\pi_{n+1}]{} & \breve{\mathcal{T}} \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} & & P(\breve{\mathcal{T}}' \times \mathcal{X}') \\ & \swarrow^{\widetilde{\pi}'_n} & \Big\uparrow{\scriptstyle \breve{\phi}'} \\ \mathcal{T}'_{n+1} & \xleftarrow[\pi'_{n+1}]{} & \breve{\mathcal{T}}' \end{array} \tag{C19}$$

To prove that diagram (a) commutes, it is equivalent to prove that $\gamma' \circ (\tau \times \xi)^{\P} \circ \breve{\phi} = \tau$, (because $\gamma' = (\breve{\phi}')^{-1}$). To do this, it is sufficient to show that

$$\pi'_m \circ \gamma' \circ (\tau \times \xi)^{\P} \circ \breve{\phi} \quad = \quad \tau_m \circ \pi_m \tag{C20}$$

for all $m \in \mathbb{N}$, because this shows that $\gamma' \circ (\tau \times \xi)^{\P} \circ \breve{\phi}$ satisfies the defining property of τ, as expressed in diagram (C13)(a). The case $m = 0$ of equation (C20) is trivially true, because π'_0 and τ_0 are both the (constant) function into the one-point space $\mathcal{T}'_0$. It remains to prove (C20) for $m \geqslant 1$. To achieve this, first note that

$$\begin{aligned} \pi'_{n+1} \circ \gamma' \circ (\tau \times \xi)^{\P} \underset{(*)}{=\!=} \widetilde{\pi}'_n \circ (\tau \times \xi)^{\P} \underset{(\ddagger)}{=\!=} (\pi'_n \times I_{\mathcal{X}'})^{\P} \circ (\tau \times \xi)^{\P} \\ \underset{(F)}{=\!=} \Big((\pi'_n \times I_{\mathcal{X}'}) \circ (\tau \times \xi)\Big)^{\P} = \Big((\pi'_n \circ \tau) \times (I_{\mathcal{X}'} \circ \xi)\Big)^{\P} \\ \underset{(\dagger)}{=\!=} \Big((\tau_n \circ \pi_n) \times (\xi \circ I_{\mathcal{X}}\Big)^{\P} = \Big((\tau_n \times \xi) \circ (\pi_n \times I_{\mathcal{X}})\Big)^{\P} \\ \underset{(F)}{=\!=} (\tau_n \times \xi)^{\P} \circ (\pi_n \times I_{\mathcal{X}})^{\P} \underset{(\diamond)}{=\!=} \tau_{n+1} \circ \widetilde{\pi}_n, \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{C21}$$

Here, $(*)$ is by diagram (C18)(ii), $(\ddagger)$ is by formula (C16), $(\dagger)$ is by diagram (C13)(a), $(\diamond)$ is by formulae (C9) and (C15), and both (F) are because P is a functor. Thus,

$$\pi'_{n+1} \circ \gamma' \circ (\tau \times \xi)^{\P} \circ \breve{\phi} \underset{(*)}{=\!=} \tau_{n+1} \circ \widetilde{\pi}_n \circ \breve{\phi} \underset{(\dagger)}{=\!=} \tau_{n+1} \circ \pi_{n+1}, \quad \text{for all } n \in \mathbb{N}. \tag{C22}$$

Here $(*)$ is by equation (C21) and $(\dagger)$ is by diagram (C19)(i). Setting $n := m - 1$ in (C22) yields equation (C20) for all $m \geqslant 1$. We conclude that $\gamma' \circ (\tau \times \xi)^{\P} \circ \breve{\phi} = \tau$, as claimed. Similar arguments show that diagrams (b) and (c) commute. ◇ `Claim 4`

Claim 4 shows that τ, τ', and τ'' all satisfy the relevant version of diagram (2). Thus, τ, τ', and τ'' are the functions promised in Proposition 2.8.

Functoriality. Define $\mathrm{URPS}(\xi) := \tau$, $\mathrm{URPS}(\xi') := \tau'$, and $\mathrm{URPS}(\xi) := \tau''$. Then Claim 3 yields $\mathrm{URPS}(\tau' \circ \tau) = \mathrm{URPS}(\tau') \circ \mathrm{URPS}(\tau)$; hence URPS is a functor, as claimed by Proposition 3.2.

Surjectivity. Suppose ξ is surjective. Let $\breve{\mathbf{t}}' \in \breve{\mathcal{T}}'$. We want $\breve{\mathbf{t}} \in \breve{\mathcal{T}}$ such that $\tau(\breve{\mathbf{t}}) = \breve{\mathbf{t}}'$.

Claim 5: $\tau_n : \mathcal{T}_n \longrightarrow \mathcal{T}'_n$ *is surjective for all* $n \in \mathbb{N}$.

Proof: If ξ is surjective then τ_1 is surjective, by defining formula (C8) and Proposition A.2(b). Now let $n \in \mathbb{N}$, and suppose that τ_n is surjective. Then $\tau_n \times \xi$ is surjective. Thus, τ_{n+1} is surjective, by defining formula (C9) and Proposition A.2(b). Inductively, $\tau_n : \mathcal{T}_n \longrightarrow \mathcal{T}'_n$ is surjective for all $n \in \mathbb{N}$. ◇ Claim 5

Suppose that $\breve{\mathbf{t}}' = (t'_n)_{n=0}^{\infty}$.

Claim 6: *For all* $n \in \mathbb{N}$, *there exists* $\breve{\mathbf{t}}^{(n)} \in \breve{\mathcal{T}}$ *such that, if we write if* $\breve{\mathbf{t}}^{(n)} = (t_m^{(n)})_{m=0}^{\infty}$, *then* $\tau_m(t_m^{(n)}) = t'_m$ *for all* $m \in [1 \ldots n]$.

Proof: Claim 5 yields some $t_n \in \mathcal{T}_n$ such that $\tau_n(t_n) = t'_n$. For all $m \in [0 \ldots n-1]$, define $t_m := \phi_m \circ \phi_{m+1} \circ \cdots \circ \phi_{n-1}(t_n)$. Then $\tau_m(t_m) = t'_m$ via the commuting diagram (C10) (established by Claim 1). Meanwhile, Corollary C.3 says there is some $t_{n+1} \in \mathcal{T}_{n+1}$ such that $\phi_n(t_{n+1}) = t_n$, Inductively, for all $m \geqslant n+1$ we obtain $t_{m+1} \in \mathcal{T}_{m+1}$ such that $\phi_m(t_{m+1}) = t_m$. Now let $\breve{\mathbf{t}}^{(n)} := (t_m)_{m=0}^{\infty}$. Then by construction, $\breve{\mathbf{t}}^{(n)} \in \breve{\mathcal{T}}$, and $\tau_m(t_m) = t'_m$ for all $m \in [0 \ldots n]$. ◇ Claim 6

Let $\breve{\mathbf{t}}$ be a cluster point of the sequence $\{\breve{\mathbf{t}}^{(n)}\}_{n=1}^{\infty}$; this exists because $\breve{\mathcal{T}}$ is compact. Suppose that $\breve{\mathbf{t}} = (t_m)_{m=0}^{\infty}$. For all $m \in \mathbb{N}$, we have $\tau_m(t_m) = t'_m$, because τ_m is continuous and $\tau_m(t_m^{(n)}) = t'_m$ for all $n \geqslant m$. Thus, $\tau(\breve{\mathbf{t}}) = \breve{\mathbf{t}}'$, as desired.

Homeomorphism. Functors preserve isomorphisms. So if $\xi : \mathcal{X} \longrightarrow \mathcal{X}'$ is a homeomorphism (i.e. an isomorphism in the category CHS), then $\tau : \breve{\mathcal{T}} \longrightarrow \breve{\mathcal{T}}'$ is also a homeomorphism. □

D The dual model: recursive quasipreferences

This appendix briefly presents another model of recursive preferences that is "dual" to the one presented in the rest of the paper. It has a somewhat different economic interpretation. But all of the earlier results are easily translated into this new model, by means of duality.

Let $\mathcal{X}$ be a set and let $\trianglerighteq$ be a binary relation on $\mathcal{X}$. Recall that $\trianglerighteq$ is *complete* if for all $x, y \in \mathcal{X}$, either $x \trianglerighteq y$ or $x \trianglelefteq y$. Let $\triangleright$ be the asymmetric part of $\trianglerighteq$ (i.e. $x \triangleright y$ if $x \trianglerighteq y$ and $y \not\trianglerighteq x$). We say that that $\trianglerighteq$ is *quasitransitive* if $\triangleright$ is transitive. In other words, for all $x, y, z \in \mathcal{X}$, if $x \triangleright y$ and $y \triangleright z$, then $x \triangleright z$. However, if $x \trianglerighteq y$ and $y \trianglerighteq z$, it is not necessarily the case that $x \trianglerighteq z$. In particular, if $\equiv$ is the symmetric part of $\trianglerighteq$, then we may have $x \equiv y$ and $y \equiv z$, but $x \triangleright z$. But we cannot have $x \triangleright y$ and $y \equiv z$ while $x \triangleleft z$ —this would violate quasitransitivity. Thus, the most we can say is that if $x \triangleright y$ and $y \trianglerighteq z$, then $x \trianglerighteq z$. Quasiorders describe preferences that exhibit "sorites paradoxes", in which a sequence of tiny, unnoticeable changes (e.g. individual grains of sugar in a cup of coffee) can add up to a noticeable change.

Now suppose that $\mathcal{X}$ is a topological space. A quasiorder $\trianglerighteq$ is *continuous* if the set $\{(x,y) \in \mathcal{X} \times \mathcal{X};\ \ x \trianglerighteq y\}$ is closed in $\mathcal{X} \times \mathcal{X}$. A *local continuous quasiorder* is an ordered pair $(\mathcal{Y}, \trianglerighteq)$, where $\mathcal{Y}$ is a closed subset of $\mathcal{X}$, and $\trianglerighteq$ is a continuous quasiorder on $\mathcal{Y}$. Let $[\![\mathcal{Y}, \trianglerighteq]\!] := \{(x,y) \in \mathcal{X} \times \mathcal{X};\ \ x \trianglerighteq y\}$; this is a closed subset of $\mathcal{X} \times \mathcal{X}$, because $\mathcal{Y}$ is closed and $\trianglerighteq$ is continuous. Let $Q(\mathcal{X})$ be the set of all continuous quasiorders on $\mathcal{X}$. Then the injective function $Q(\mathcal{X}) \ni (\mathcal{Y}, \trianglerighteq) \mapsto [\![\mathcal{Y}, \trianglerighteq]\!] \in K(\mathcal{X})$ identifies $Q(\mathcal{X})$ with a subset of $K(\mathcal{X})$, and the Vietoris topology on $K(\mathcal{X})$ pulls back to define a topology on $Q(\mathcal{X})$, which we will also call the *Vietoris topology*.

Duality. Let $\mathcal{X}$ be a set, and let $\succ$ be a binary relation on $\mathcal{X}$. The *dual* of $\succ$ is the binary relation $\trianglerighteq$ defined as follows:

$$\text{For all } x, y \in \mathcal{X}, \qquad \Big(x \trianglerighteq y\Big) \iff \Big(x \not\prec y\Big). \tag{D1}$$

Clearly, if $\trianglerighteq$ is dual to $\succ$, then $\succ$ is dual to $\trianglerighteq$. So we will say they are dual to each other.

Lemma D.1 *Let $\mathcal{X}$ be a topological space, let $\mathcal{Y} \subseteq \mathcal{X}$ be a closed subset, and let $\trianglerighteq$ and $\succ$ be two binary relations on $\mathcal{Y}$ that are dual to each other. Then $(\mathcal{Y}, \trianglerighteq)$ is a local continuous quasiorder if and only if $(\mathcal{Y}, \succ)$ is a local continuous strict partial order. Thus, the duality relation defines a bijection between $P(\mathcal{X})$ and $Q(\mathcal{X})$. This bijection is a homeomorphism between the co-Vietoris topology on $P(\mathcal{X})$ and the Vietoris topology on $Q(\mathcal{X})$.*

Proof: See Pivato 2023b, Lemma B. □

In light of Lemma D.1, Proposition A.1 can immediately be translated into an equivalent statements for $Q(\mathcal{X})$. Thus, if $\mathcal{X}$ is a compact Hausdorff space, then so is $Q(\mathcal{X})$. If $\mathcal{X}$ is metrizable, then so is $Q(\mathcal{X})$. If $\mathcal{X}$ is a continuum, then so is $Q(\mathcal{X})$.

Forward images of quasiorders. Let $\mathcal{X}$ and $\mathcal{X}'$ be compact Hausdorff spaces, and let $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ be continuous. For any local continuous quasiorder $(\mathcal{Y}, \trianglerighteq)$ on $\mathcal{X}$, let $\phi^{@}(\mathcal{Y}, \trianglerighteq) := (\mathcal{Y}', \trianglerighteq')$, where $\mathcal{Y} = \phi(\mathcal{Y}) \subseteq \mathcal{X}$, and where the binary relation $\trianglerighteq'$ is defined on $\mathcal{Y}'$ by stipulating, for all $x', y' \in \mathcal{X}'$, that $x' \trianglerighteq' y'$ if $x \trianglerighteq y$ for some $x \in \phi^{-1}\{x'\}$ and $y \in \phi^{-1}\{y'\}$. It can be shown that $(\mathcal{Y}', \trianglerighteq')$ is itself a continuous quasiorder on $\mathcal{X}'$; thus, we get a function $\phi^{@} : Q(\mathcal{X}) \longrightarrow Q(\mathcal{X}')$.

Now let $(\mathcal{Y}, \succ)$ be a local continuous strict partial order on $\mathcal{X}$, and let $(\mathcal{Y}, \trianglerighteq)$ be a local continuous quasiorder. Let $\mathcal{X}'$ be another topological space, and let $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ be continuous. If $(\mathcal{Y}, \succ)$ is dual to $(\mathcal{Y}, \trianglerighteq)$, then it is easily verified that $\phi^{\P}(\mathcal{Y}, \succ)$ is dual to $\phi^{@}(\mathcal{Y}, \trianglerighteq)$. Thus, Proposition A.2 implies that the function $\phi^{@} : Q(\mathcal{X}) \longrightarrow Q(\mathcal{X}')$ is continuous in the Vietoris topology. If ϕ is surjective, then so is $\phi^{@}$.

Recursive quasipreference structures. In light of this duality between quasiorders and strict partial orders, we can introduce a "dual" version of recursive preference structures. Let $\mathcal{X}$ be a compact Hausdorff space. A *recursive quasipreference structure* (RQS) over $\mathcal{X}$ is an ordered pair $(\mathcal{T}, \phi)$, where $\mathcal{T}$ is a compact Hausdorff space, and $\phi : \mathcal{T} \longrightarrow Q(\mathcal{T} \times \mathcal{X})$ is a continuous function. Thus, for any type $t \in \mathcal{T}$, $\phi(t)$ is a continuous quasiorder on $\mathcal{T} \times \mathcal{X}$. By Lemma D.1 and the homeomorphism $P(\mathcal{T} \times \mathcal{X}) \cong Q(\mathcal{T} \times \mathcal{X})$, we see that any recursive quasipreference structure $\phi : \mathcal{T} \longrightarrow Q(\mathcal{T} \times \mathcal{X})$ can be transformed into a dual recursive preference structure $\phi : \mathcal{T} \longrightarrow P(\mathcal{T} \times \mathcal{X})$, and vice versa. Thus, the two models are equivalent, and can be used interchangeably.

Every concept and theorem for recursive preference structures has an equivalent version for recursive quasipreference structures. For example, a type-outcome pair (t^*, x^*) in $\mathcal{T} \times \mathcal{X}$ is *recursively optimal* for an RQS $(\mathcal{T}, \phi)$ if (t^*, x^*) is maximal in the local continuous quasiorder $\phi(t^*)$. To be precise: if $\phi(t^*) = (\mathcal{Y}, \trianglerighteq)$, then (t^*, x^*) is *recursively optimal* if $(t^*, x^*) \trianglerighteq (t, x)$ for all $(t, x) \in \mathcal{Y}$. If an RQS and an RPS are dual to each other, then a type-outcome pair is recursively optimal for one if and only if it is recursively optimal for the other one. (This follows from the last statement in Lemma B of Pivato 2023b.)

Let $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$ be recursive quasipreference structures over $\mathcal{X}$. A *RQS-morphism* from $(\mathcal{T}_1, \phi_1)$ to $(\mathcal{T}_2, \phi_2)$ is a continuous function $\psi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_1 & \xrightarrow{\phi_1} & Q(\mathcal{T}_1 \times \mathcal{X}) \\ \psi \downarrow & & \downarrow (\psi \times I_{\mathcal{X}})^{@} \\ \mathcal{T}_2 & \xrightarrow[\phi_2]{} & Q(\mathcal{T}_2 \times \mathcal{X}) \end{array}$$

If $(\mathcal{T}_1, \phi_1')$ and $(\mathcal{T}_2, \phi_2')$ are the RPS's that are dual to $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$, then ψ is a RQS-morphism from $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$ if and only if it is an RPS-morphism from $(\mathcal{T}_1, \phi_1')$ and $(\mathcal{T}_2, \phi_2')$. Thus, there is an RQS version of Proposition 2.2.

An RQS $(\breve{\mathcal{T}}, \breve{\phi})$ over $\mathcal{X}$ is *universal* if, for any other RQS $(\mathcal{T}, \phi)$ over $\mathcal{X}$, there is a *unique* RQS-morphism $\psi : \mathcal{T} \longrightarrow \breve{\mathcal{T}}$. Lambek's Theorem (Proposition 3.3) yields an RQS version of Proposition 2.3. Finally, Theorem 2.4 and Proposition 2.5 yield the following result:

Theorem D.2 *For any compact Hausdorff space $\mathcal{X}$, there is a universal RQS over $\mathcal{X}$. The type space $\breve{\mathcal{T}}$ of this universal RQS is a compact Hausdorff space, which contains a subspace homeomorphic to $\mathcal{X}$. If $\mathcal{X}$ is metrizable, then so is $\breve{\mathcal{T}}$. If $\mathcal{X}$ is a continuum, then so is $\breve{\mathcal{T}}$. If $\mathcal{X}$ is homeomorphic to Cantor space, then so is $\breve{\mathcal{T}}$.*

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