Monoidal Optics are Universal

Extended Abstract

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ABSTRACT

Monoidal optics are the free normalization of the cofree produoidal category over an arbitrary monoidal category. This extended abstract is based on "The Produoidal Algebra of Process Decomposition" recently presented at Computer Science Logic 2024 (slides).

1 OPTICS

Definition 1.1 (c.f. [8, 26]). In a monoidal category, an *optic* from (X, Y) to (A, B) is a pair of morphisms $f_1 \colon A \to L \otimes X \otimes R$ and $f_2 \colon L \otimes Y \otimes R \to B$ quotiented by dinaturality on L and R.

Monoidal categories are the algebra of sequential and parallel composition. Optics extend this algebra by permitting holes in the diagrams. However, while optics form a category where composition is defined by nesting, the sequential and parallel operations are not so straightforward to define. Parallel composition of optics is only generally defined if the base category is braided. Even worse, sequential composition of two optics (two morphisms with a single hole) does not yield an optic, but a morphism with two holes.

Let us write the type of an optic with two holes as $(X_1, Y_1) \triangleleft (X_2, Y_2) \to (A, B)$. Our problem is that the operation $(\triangleleft)$, which reads as "and then", is not representable: there is no hole we can put in place of two holes. Optics do not form a monoidal category but a *pro*monoidal category, where the tensor and unit are no longer functors but profunctors. The same problem happens with the parallel compostition in the non-braided case: although we understand what an optic with two parallel holes, $(X_1, Y_1) \otimes (X_2, Y_2) \to (A, B)$, must be, these cannot be represented by a single hole.

1.1 Promonoidal Categories

Definition 1.2 (Day and Street [9, 10]). A *promonoidal category* is a category $\mathbf{V}$, together with two profunctors,

$$\mathbf{V}(\bullet \otimes \bullet; \bullet) \colon \mathbf{V}^{op} \times \mathbf{V}^{op} \times \mathbf{V} \to \mathbf{Set} \text{ and } \mathbf{V}(I; \bullet) \colon \mathbf{V} \to \mathbf{Set},$$

and associator and unitor natural transformations between them satisfying the pentagon and triangle equations.

Intuitively, these profunctors are the non-representable versions of the tensor and the unit of a monoidal category: even when the object $X \otimes Y$ does not exist, the hom-set $\mathbf{V}(X \otimes Y; A)$ does.

Definition 1.3 (Booker and Street [4]). A *produoidal category* is a category $\mathbf{V}$ with two promonoidal structures

$$\mathbf{V}(\bullet \otimes \bullet; \bullet) \colon \mathbf{V}^{op} \times \mathbf{V}^{op} \times \mathbf{V} \to \mathbf{Set} \text{ and } \mathbf{V}(I; \bullet) \colon \mathbf{V} \to \mathbf{Set},$$
$$\mathbf{V}(\bullet \triangleleft \bullet; \bullet) \colon \mathbf{V}^{op} \times \mathbf{V}^{op} \times \mathbf{V} \to \mathbf{Set} \text{ and } \mathbf{V}(N; \bullet) \colon \mathbf{V} \to \mathbf{Set},$$

that distribute laxly over each other with a structure natural transformation $(X \triangleleft Y) \otimes (U \triangleleft V) \to (X \otimes U) \triangleleft (Y \otimes V)$ satisfying the axioms of duoidal categories. A produoidal category is *normal* if the laxator $\phi_0 : \mathbf{V}(I; \bullet) \to \mathbf{V}(N; \bullet)$ is an isomorphism.

Proposition 1.4 (c.f. Pastro and Street [23, 32], c.f. Garner and López Franco [14]). *Monoidal optics, with promonoidal tensors, $(\triangleleft)$ and $(\otimes)$, form a produoidal category.*

Intuitively, we can think of promonoidal categories as *malleable* multicategories, which form an equivalent (but not isomorphic) category. A multicategory is malleable whenever its multicategorical composition is invertible up to dinaturality.

Theorem 1.5 (c.f. Melliès and Zeilberger [21]). *There exists an adjunction between categories and promonoidal categories. The right adjoint takes a category to the promonoidal category whose operations are incomplete arrows of the original category.*

2 UNIVERSALITY

Theorem 2.1. *There exists an adjunction between monoidal categories and produoidal categories: the right adjoint takes a monoidal category to the produoidal category whose operations are incomplete arrows of the original monoidal category (Figure 1).*

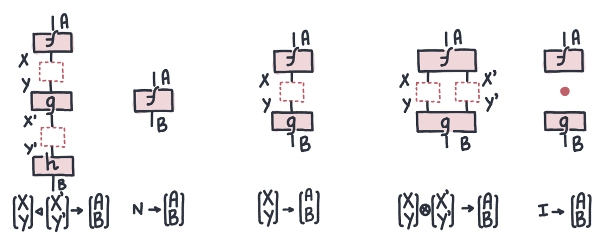

Figure 1: Incomplete arrows of a monoidal category.

Theorem 2.2. *There exists an adjunction between produoidal categories and normal produoidal categories. The free normalization of a produoidal category with hom-sets $\mathbf{C}(A; B)$ is a normal produoidal category with hom-sets $\mathcal{N}\mathbf{C}(A; B) = \mathbf{C}(N \otimes A \otimes N; B)$. Normalization, $\mathcal{N} \colon \mathrm{pDuo} \to \mathrm{pDuo}$, forms an idempotent monad.*

Moreover, there exists a specialized produoidal normalization for the symmetric case, $\mathcal{N}_\sigma \colon \mathrm{spDuo} \to \mathrm{spDuo}$, that again forms an idempotent monad.

Theorem 2.3. *Monoidal optics form the free normalization of the cofree produoidal category over a monoidal category (Figure 2).*

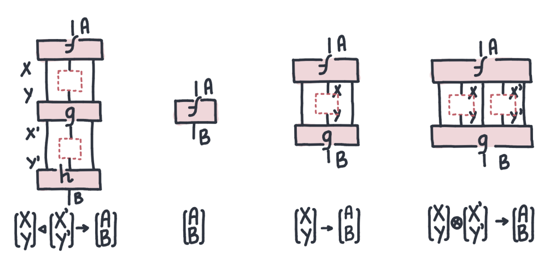

Figure 2: Monoidal optics.

3 CONTRIBUTIONS

The main contribution of this work is to provide a universal characterization of the normal produoidal algebra of optics. This characterization arises from the secondary contribution of two novel adjunctions. The first adjunction is a novel splice-contour adjunction between monoidal and produoidal categories; the second adjunction is the free normalization of produoidal categories. To summarize, the following are novel contributions.

(1) A universal characterization of the normal produoidal category of optics over a monoidal category. Note that, even though the duoidal structure of Tambara modules has been mentioned in the literature, this is its first universal characterization in these terms: Tambara modules are the presheaves of the free normalization of the cofree produoidal category over a monoidal category.
(2) We prove that every produoidal category has a normalization, and we show that normalization of produoidal categories forms an idempotent monad. This induces an adjunction between produoidal categories and normal produoidal categories. In contrast, note that it is known that duoidal categories do not necessarily have a normalization [14], and their normalization – whenever it exists – has not been universally characterized.
(3) Additionally, we show that there exists a specialized free normalization adjunction between symmetric produoidal categories and normal symmetric produoidal categories. Note that symmetric normalization of symmetric duoidals [14] would not induce a monad and it is not defined in general.
(4) We construct a produoidal category of spliced arrows on top of an arbitrary monoidal category; we construct an adjunction between monoidal categories and produoidal categories. Note that the splice-contour adjunction between categories and multicategories was known; we introduce both its coherent counterpart (using promonoidal categories) and its monoidal counterpart (using produoidal categories).

4 RELATED WORK

We are indebted to the previous work of Melliès and Zeilberger [21] on the splice-contour adjunction; but also to the previous work on duoidal categories, both by Aguiar and Mahajan [1] and by Garner and López Franco [14].

Tambara modules. Pastro and Street [23] study the sequential monoidal structure of Tambara modules and its corresponding promonoidal category of optics, under the name of "doubles". The first written record of the correspondence between these Tambara modules and optics in functional programming [20] seems to be due to Milewski [22]: note that the correspondence between strong profunctors for a cartesian monoidal structure and pairs of morphisms $A \to X$ and $A \times Y \to B$ is not trivial; in fact, it took around ten years to be explored in print. Since then, multiple authors have extended this framework [8, 26, 33].

Garner and López Franco [14] describe the duoidal category of bistrong profunctors (or Tambara modules) as the canonical duoidal category of endocells arising from the adjoint pseudomonoid structure of any monoidal category on the monoidal bicategory of profunctors. A generalization of Day's theorem [9] could be used to prove from these results that optics form a produoidal category.

Optics. Functional programmers are to be credited with the development of multiple types of optics and their application: lenses help accessing subfields; prisms recover failing computations; traversals inspect a list uniformly; grates produce an update while taking the view in continuation-passing style; getters, reviews, folds, affine traversals, or kaleidoscopes are all data-accessing patterns [2, 8, 25]. This development was initially independent from the work of Pastro and Street, and it was based on a reinterpretation and a generalization of the previous work on lenses.

Lenses and Wiring Diagrams. In the cartesian case, monoidal optics coincide with lenses, which were developed for database theory, particularly by Johnson and Rosebrugh [13, 18, 19]. Recent work by Clarke, Di Meglio, and others has generalized lenses in this direction [6, 7, 12]. Forgetting about the produoidal structure of lenses, we can recover the operad or multicategory of wiring diagrams over monoidal categories studied by Spivak, Schultz, Vasilakopolou, Vagner, Patterson and others [24, 29, 30].

Applications. The produoidal structure of optics seems to be unknown to most applications of optics, even when sequencing of optics makes an appearance on some of the literature. Among multiple other appearances, optics have been applied to quantum supermaps [16, 17], economic game theory [15], data accessing [25], exact probabilistic conditioning [31], server architecture [34], monoidal process iteration [3, 11], or coend calculus [5, 27].

Our work has been extended with an implementation of the produoidal algebra of monoidal optics in Haskell [28].

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