Effectful Trace Semantics: Extended Abstract

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We introduce effectful streams, a coinductive semantic universe for effectful dataflow programming and traces. In monoidal categories with conditionals and ranges, we show that effectful streams particularize to families of morphisms satisfying a causality condition. Effectful streams allow us to develop notions of trace and bisimulation for effectful Mealy machines; we prove that bisimulation implies effectful trace equivalence. This is an extended abstract of "Effectful Trace Semantics via Effectful Streams", currently submitted for review and available online.

Effectful traces. Traces are sequences that record the outputs of a transition system or a state machine along an execution. They constitute a successful and flexible formalism that can be adapted to the different flavours of transition systems (from *non-deterministic* to *stochastic*). Traces are also a semantic universe of interest for dataflow networks; in fact, they help extending Kahn's original model [Kah74] to a compositional semantics for the non-deterministic case [Jon89].

This work combines Katis, Sabadini and Walters' monoidal semantics of transition systems [KSW97, KSW99, KSW02] with the classical coalgebraic semantics of transition systems [Rut95, Rut00]. Theorem 8 shows that effectful streams give coalgebraic semantics to an effectful analogue of Mealy machines (c.f. [KSW97]). The categorical setting we choose is that of *effectful copy-discard categories*: a triple of categories that allows us to distinguish between values, local computations, and effectful computations as morphisms in a cartesian $\mathbf{V}$, monoidal $\mathbf{P}$, and premonoidal category $\mathbf{C}$, respectively.

Definition 1. An *effectful copy-discard category* is a triple of categories with two identity-on-objects functors, $\mathbf{V} \to \mathbf{P} \to \mathbf{C}$, where *(i)* $\mathbf{V}$ is a cartesian monoidal category; *(ii)* $\mathbf{P}$ is a monoidal category; and *(iii)* $\mathbf{C}$, is a premonoidal category. The identity-on-objects functor $\mathbf{V} \to \mathbf{C}$ must preserve the monoidal structure strictly; the identity-on-objects functor $\mathbf{P} \to \mathbf{C}$ must preserve the premonoidal structure strictly.

Effectful Streams. We introduce a definition of stream in an effectful copy-discard category. A stream, $\boldsymbol{A} \in \mathbf{Stream}(X)$, consists of a first element, $\boldsymbol{A}^\circ \in X$, followed by a stream with the rest of the elements, $\boldsymbol{A}^+ \in \mathbf{Stream}(X)$. Analogously, an effectful stream consists of a process followed by an effectful stream; additionally, a "memory" object communicates these two parts.

Definition 2. An *effectful stream*, $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$, over an effectful category $(\mathbf{P}, \mathbf{C})$ with inputs $\boldsymbol{A} = (A_0, A_1 ...)$ and outputs $\boldsymbol{B} = (B_0, B_1, ...)$ is a triple consisting of *(i)* a *memory*, $M_{\boldsymbol{f}} \in \mathbf{P}_{obj}$; *(ii)* a *head*, or *first action*, $\boldsymbol{f}^\circ \colon \boldsymbol{A}^\circ \rightsquigarrow M_{\boldsymbol{f}} \otimes \boldsymbol{B}^\circ$; and *(iii)* a *tail*, or *rest of the stream*, $\boldsymbol{f}^+ \colon M_{\boldsymbol{f}} \cdot \boldsymbol{A}^+ \rightsquigarrow \boldsymbol{B}^+$.

Effectful streams are quotiented by dinaturality over the memory: the minimal equivalence relation $(\sim)$ such that $(M_{\boldsymbol{f}}, \boldsymbol{f}^\circ, \boldsymbol{f}^+) \sim (M_{\boldsymbol{g}}, \boldsymbol{g}^\circ, \boldsymbol{g}^+)$, for each pure morphism $r \colon M_{\boldsymbol{g}} \to M_{\boldsymbol{f}}$ in $\mathbf{P}$ such that $\boldsymbol{g}^\circ ⨟ r = \boldsymbol{f}^\circ$ and $r \cdot \boldsymbol{f}^+ \sim \boldsymbol{g}^+$. Effectful streams from $\boldsymbol{A}$ to $\boldsymbol{B}$, quotiented by dinaturality form a set, $\mathsf{Stream}(\mathbf{P}, \mathbf{C})(\boldsymbol{A}; \boldsymbol{B})$, and they form the effectful morphisms of the effectful category Stream.

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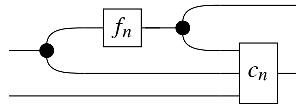
The notion of causal process plays a key role in the theory of streams since the work of Raney [Ran58], who studies them in the cartesian case. The definition of monoidal causal process is more subtle and requires the notion of conditionals [Fri20]. Effectful streams further generalise causal processes to the effectful case; still, when the effects are monoidal, effectful streams coincide with causal processes.

Definition 3. A *causal process*, $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$, in a copy-discard category $(\mathbf{V},\mathbf{P})$, is a family of morphisms $f_n\colon X_0 \otimes \ldots \otimes X_n \to Y_0 \otimes \ldots \otimes Y_n$ that are *causal*: each $f_n \otimes -\!\bullet$ is a marginal of f_{n+1}. Explicitly, there are morphisms $c_n\colon Y_0 \otimes ... \otimes Y_n \otimes X_0 \otimes ... \otimes X_{n+1} \to Y_{n+1}$, the *conditionals*, that satisfy the equation below.

$f_{n+1} =$ 

Causal processes over a copy-discard category $(\mathbf{V},\mathbf{P})$ with quasi-total conditionals [DLR23] form a copy-discard category. When the base category also has ranges [DLdFR22], causal processes are isomorphic to effectful streams.

Theorem 4. *For a copy-discard category $(\mathbf{V},\mathbf{P})$ with quasi-total conditionals and ranges, the category of effectful streams is monoidal and isomorphic to the monoidal category of causal processes,*

$$\mathsf{Stream}(\mathsf{Tot}(\mathbf{P}),\mathbf{P}) \cong \mathsf{Causal}(\mathbf{P}).$$

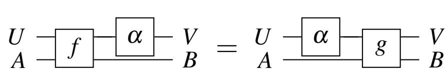
Effectful Mealy Machines. Effectful streams give coalgebraic semantics to effectful state machines: their trace is the effectful stream they generate. The trace defines an effectful functor, which ensures compositionality of the semantics.

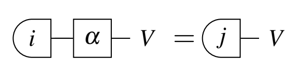
Definition 5. An *effectful Mealy machine* in an effectful copy-discard category, $(\mathbf{V},\mathbf{P},\mathbf{C})$, taking inputs on $A \in \mathsf{Obj}(\mathbf{C})$ and producing outputs in $B \in \mathsf{Obj}(\mathbf{C})$, is a triple (U,i,f) consisting of a *state space* $U \in \mathsf{Obj}(\mathbf{C})$, an initial state $i\colon I \to U$ in $\mathbf{C}$, and a *transition morphism*, $f : U \otimes A \rightsquigarrow U \otimes B$ in $\mathbf{C}$.

Every effectful Mealy machine induces an effectful stream that represents its execution. An object A can be repeated to form a stream $[\![A]\!]$, defined by $[\![A]\!]^{\circ} = A$ and $[\![A]\!]^{+} = [\![A]\!]$. Analogously, a transition morphism, $f\colon U \otimes A \to U \otimes B$ with state space U, can be repeated to form an effectful stream $[\![f]\!]\colon U \cdot [\![A]\!] \to [\![B]\!]$ defined by $[\![f]\!]^{\circ} = f$ and $[\![f]\!]^{+} = [\![f]\!]$. When the transition morphism has an initial state, s_0, it defines a Mealy machine, and we can use it to construct an effectful stream that represents the execution trace of the Mealy machine. The operation $(\cdot)$ attaches the initial state i at the beginning of the execution of the machine f and gives its trace.

Definition 6. The *trace* $\mathsf{tr}(U,i,f)\colon [\![A]\!] \to [\![B]\!]$ of an effectful Mealy machine $(U,i,f)\colon A \to B$ is the effectful stream defined by $\mathsf{tr}(U,i,f) = i \cdot [\![f]\!]$. We say that two Mealy machines are *trace-equivalent* if their traces coincide.

Definition 7. A homomorphism of effectful Mealy machines with the same inputs and outputs, $(U,i,f) \Rightarrow (V,j,g)$, is a value morphism $\alpha\colon U \to V$ such that



and


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Theorem 8. *Mealy machines quotiented by isomorphism of the state space are the morphisms of an effectful category* **Mealy**. *The trace of Mealy machines defines an effectful functor* $\mathsf{tr}\colon \mathbf{Mealy} \to \mathsf{Stream}$.

As in the classical case, bisimulation of effectful Mealy machines implies their trace equivalence.

Theorem 9. *We say that two effectful Mealy machines are* bisimilar *if they are connected by homomorphisms. This coincides with the usual definition in Kleisli categories of monads that preserve weak pullbacks. If two effectful Mealy machines are bisimilar, then they are trace equivalent.*

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