An operadic view of convexity: entropy and twisted distributions

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The phenomenon of convexity is of key import to a wide variety of fields — optimization, probability theory, and functional analysis. Classically, convexity is a property subsets of $\mathbb{R}$-vector spaces. A subset $C \subset V$ is termed *convex* if it contains every line segment between points in C. However, in practice, the precise nature of the ambient vector space is unimportant, what matters is that, in a convex space, one can take *convex combinations* of elements: sums of the form $\sum_i \alpha_i x_i$ where the α_i are non-negative and sum to 1. Distilling convexity down to this essential concept leads to more abstract approaches to convexity, as in [Neu, Ś74, Fri15, Jac09], in which a convex space is treated as a set with additional algebraic structure, namely the ability to take convex combinations in a consistent way.

This paper ([HOS24]) is concerned with developing operadic methods to analyze and manipulate convex spaces, and, in particular, with using these methods to provide Grothendieck constructions for functors valued in convex sets. The more sophisticated of these takes as input generalizations of lax monoidal functors into the category of convex sets, and thereby generalizes the Set-valued case of the monoidal Grothendieck construction of [MV20]. The Grothendieck construction reformulates the data of such a functor into a kind of monoidal discrete fibration with fibrewise convex structure.

Among the applications of this framework are two which we consider to be particularly potent applications, and which we expose in detail. In the first, we reformulate the categorical characterization of entropy presented in [BFL11] as a structure-preserving functor out of a convex monoidal Grothendieck construction. Both the fibrewise convex structures and the generalized monoidal structures appearing in our Grothendieck construction are essential to this reformulation, suggesting that convex monoidal Grothendieck constructions provide a natural setting to study entropy with categorical methods.

The second of our applications concerns quantum information theory. In quantum theory, *nonlocality* [Bel64] and its cousin, *contextuality* [KS67], are terms for non-classical behaviors of the probability distributions which arise from quantum systems. Contextuality, in particular, provides advantages for quantum computing [HWVE14]. Within the framework of simplicial distributions introduced in [OKI23] to study quantum probabilities, algebraic structures in convex sets called *convex monoids* appear as a way to characterize contextual distributions in terms of invertibility conditions. In [OS], the framework of simplicial distributions is extended to simplicial principal bundles, an extension which, in particular, captures features of projective quantum measurements. In this context, generalizations of convex monoids appear as convex Grothendieck constructions of functors out of the monoidal groupoid of principal bundles over a fixed base.

The convex Grothendieck construction which enables our study of these examples requires two main prerequisites. Firstly, convex sets come equipped with a kind of map, which we term *biconvex*, in analogy with bilinearity for vector spaces. Formally, given convex sets X, Y, and Z, a map

$$f : X \times Y \longrightarrow Z$$

is biconvex when, for any convex combinations $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ in X and Y, respectively, we have

$$f\left(\sum_i \alpha_i x_i, \sum_j \beta_j y_j\right) = \sum_{i,j} \alpha_i \beta_j f(x_i, y_j).$$

In the first section of our paper we develop a *convex tensor product* which represents biconvex maps, and show that it provides a symmetric monoidal structure $\otimes$ on the category CSet of convex sets.

The second component is the theory of $\mathcal{O}$-monoidal structures developed in the companion paper [HS]. Given an operad $\mathcal{O}$ in Set, an $\mathcal{O}$-monoidal category is a category C equipped with operations

$$\mathsf{C}^n \longrightarrow \mathsf{C}$$

for every n-ary operation of $\mathcal{O}$. These must satisfy the composability conditions encoded by $\mathcal{O}$ up to coherent natural isomorphisms. In the special cases of the associative and commutative operads, this notion recovers the (unbiased) notions of monoidal and symmetric monoidal categories, respectively. Using the terminal-ness of the commutative operad, we can consider any symmetric monoidal structure as an $\mathcal{O}$-monoidal structure for any operad $\mathcal{O}$.

After having developed these two pieces of technology, it is easier to state what lives on each side of the Grothendieck construction at the core of this work.

- On one side, we have lax $\mathcal{O}$-monoidal functors from an $\mathcal{O}$-monoidal category I into $(\mathsf{CSet}, \otimes)$.
- On the other side, we have strict $\mathcal{O}$-monoidal discrete fibrations over I with fibrewise convex structure.

We consider the case $\mathcal{O} = \mathsf{QConv}$ for convexity. QConv is the operad with n-ary operations being n-tuples $(\alpha_1, \ldots, \alpha_n)$ such that $\sum_i \alpha_i = 1$. We demonstrate how our convex Grothendieck construction equips the category of finite probability spaces of [BFL11] with the appropriate convex structure. Similarly, we retain relevant convexity information when applying the above tools in the context of twisted distributions.

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