# A Profunctorial Semantics for Quantum Supermaps

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## ABSTRACT

We identify morphisms of strong profunctors as a categorification of quantum supermaps. These black-box generalisations of diagrams-with-holes are hence placed within the broader field of profunctor optics, as morphisms in the category of copresheaves on concrete networks. This enables the first construction of abstract logical connectives such as tensor products and negations for supermaps in a totally theory-independent setting. These logical connectives are found to be all that is needed to abstractly model the key structural features of the quantum theory of supermaps: black-box indefinite causal order, black-box definite causal order, and the factorisation of definitely causally ordered supermaps into concrete circuit diagrams. We demonstrate that at the heart of these factorisation theorems lies the Yoneda lemma and the notion of representability.

## 1 INTRODUCTION

Quantum supermaps have been a major focal point in the field of quantum foundations over the last couple of decades [8]. These supermaps are intended to capture the notion of higher-order quantum processes: a first-order process is a quantum channel evolving quantum states in time, while a second-order process is a map which acts to send first-order processes to first-order processes, that is, quantum channels to quantum channels.

Some simple examples of quantum supermaps include *circuits-with-holes*, also known as *combs* [7, 9]. These are given by incomplete circuits of quantum channels with holes which one may imagine filling with quantum channels to produce a complete circuit. Quantum supermaps also encompass substantially more general notions of higher-order transformation some of which have been demonstrated to exhibit exotic phenomena such as superpositions of causal order, and advantage in computational and information-theoretic tasks [1, 5, 6, 13, 17, 18, 21, 37]. For this reason much of the focus has been on these families of quantum supermaps that go beyond combs. In fact, the most investigated higher-order processes such as the quantum switch [11], the OCB process [25], the Lugano process [2, 3] and the Grenoble process [34] are known to possess no decomposition as a comb and thus are truly beyond the class of maps that could be studied in a framework of combs alone.

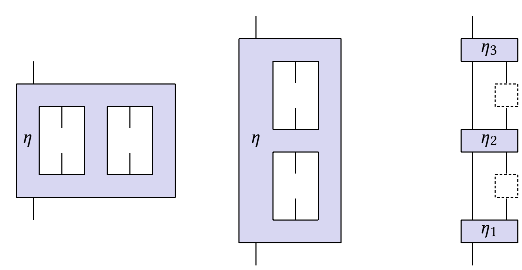
At their heart, supermaps model a simple intuitive idea: a model of first-order processes consists of boxes and wires, while a model of higher-order processes must extend these compositional components to include holes. First-order process theories are understood to have a solid mathematical foundation in terms of monoidal category theory [15], yet there is not a generally accepted and mathematically rigorous foundation that adequately extends these models to include holes.

The lack of such a foundation is a cause of important domain specific problems. Firstly, to combine the study of indefinite causal structure with quantum field theory and ultimately quantum gravity, then the supermap framework will need to be extended to infinite dimensional and even non-separable Hilbert spaces [27]. Current proposals for extension beyond finite dimensions however, are restricted to separable Hilbert spaces, and further to either the 1-input setting [12] or to the Wigner-function representation [20]. Secondly, without a stable framework for supermaps in a general context, it is unclear how to initiate the study of post-quantum causal structures: causal structures compatible with GPTs or OPTs [10], a class of physical theories used to study the special place of quantum physics from information-theoretic principles.

### 1.1 Contributions

In the main article the goal is to produce a categorical model for supermaps which is both structurally well-behaved (with, for instance, suitable tensor products) but still fully general in the categories it can be applied to and the kinds of higher order processes incorporated. The key idea is to make a direct connection between the methods of profunctor optics [14, 16, 22, 26, 28–32] and the diagrammatic approach of [35, 36], by identifying these diagrammatic rules with the notion of morphism in the category $\mathrm{StProf}(C)$ of strong profunctors.

It is known that the category $\mathrm{StProf}(C)$ has two closed tensor products [16, 19, 26] which we use to model the spaces of separable $(\otimes_C)$ and sequenced channels $(\olessthan)$. As a result the morphisms with those domains are shown to faithfully represent the supermaps with indefinite causal order and definite causal order respectively.



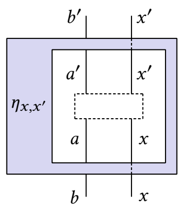

(1)

Indefinite Causal Order $(- \otimes_C -)$ | Abstract Definite Order $(- \olessthan -)$ | Concrete Networks (Representable)

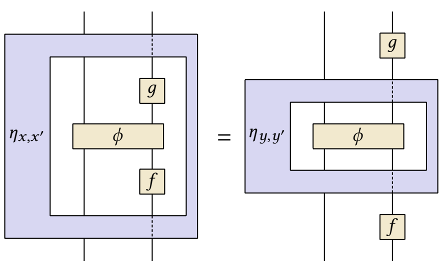
StProf$(C)$ also has a weak dualising functor $(-)^*$, which in general is not involutive but is still strong with respect to the tensor $\otimes_C$. This allows us to define a functorial par operation $\parr$ which has many of the same properties as its analogous one in Caus (for instance it distributes linearly with $\otimes_C$) but it is generally *not* associative or unital and thus fails to be a tensor product. This weakens the $*$-autonomous structure of the Caus-construction [23, 33], meaning we do not have a model of linear logic but of *tensorial logic* [24]. The failure of $(-)^*$ to be involutive has a number of deep connections with the structure of supermaps on the category $C$ and we demonstrate that the involutivity of $(-)^*$ on certain objects of StProf$(C)$ is intimately connected with the fundamental decomposition theorems of quantum supermaps, which demonstrate that abstract definite orders in quantum theory are realisable as concrete networks.

## 2 SUMMARY OF RESULTS

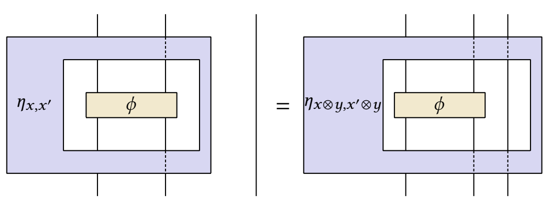
In this work we formalise the notion of a supermap from [36]. Such a supermap consists of a family of functions of the form:




such that the following laws hold.




(2)




(3)

The idea here is to capture the *locality* of the supermap $\eta$: that it should only act locally to the lab $\boldsymbol{a}$ and not on other environment systems $\boldsymbol{x} = (x, x')$. Thus we expect a supermap to commute with the actions of agents on $(x, x')$, in particular with pre- and post-composition by maps on $(x, x')$ (law (2)) and by further tensorial extension of the environment (law (3)). These two laws jointly capture the intuitive idea that supermaps should commute with combs on the environment.

We are able to repackage this definition categorically as follows.

*Definition 2.1.* A single-party locally-applicable transformation is a strong natural transformation of the type

$$\eta : C(a \otimes -, a' \otimes =) \Rightarrow C(b \otimes -, b' \otimes =). \tag{4}$$

This connection with pre-existing categorical notions means we can find a tensor product $\otimes_C$ on StProf$(C)$ on which the multi-partite indefinitely-causally ordered supermaps act.

Theorem 2.2. *On any symmetric monoidal category $C$, the strong natural transformations of type*

$$\eta : \bigotimes_{i=1}^{n}{}_C\, C(a_i \otimes -, a_i' \otimes =) \to C(b \otimes -, b' \otimes =)$$

*are the multi-partite locally-applicable transformations of type $\eta : \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \to \boldsymbol{b}$.*

In the case of quantum channels, this recovers the usual definition of a quantum supermap.

Theorem 2.3. *The quantum supermaps on the non-signalling channels are the morphisms of strong profunctors of type*

$$S : \bigotimes_{\mathsf{CPTP}}^{i} \mathsf{CPTP}(a_i \otimes -, a_i' \otimes =) \to \mathsf{CPTP}(b \otimes -, b' \otimes =).$$

We also find another tensor product $\varovee$ of StProf$(C)$ on which the definitely-causally ordered supermaps act. This captures precisely the supermaps on $n$-combs in the case of quantum theory.

Theorem 2.4. *The quantum supermaps on the $n$-combs are the morphisms of strong profunctors of the following type in* StProf(CPTP).

$$\bigovee_{i=1}^{n} \mathsf{CPTP}(a_i \otimes -, a_i' \otimes =) \to \mathsf{CPTP}(c \otimes -, c' \otimes =) \tag{5}$$

A core theorem on the structure of quantum supermaps is that any supermap with definite causal order can be decomposed into a concrete circuit diagram with holes, such a property can be reframed in StProf$(C)$ in terms of the Yoneda lemma.

*Definition 2.5 (Supermap Decomposition Theorem).* A symmetric monoidal category $C$ has a 1-arity supermap decomposition theorem if

$$\begin{aligned} &\mathrm{StProf}(C)\big(C(a \otimes -, a' \otimes =), C(b \otimes -, b' \otimes =)\big) \\ \cong &\mathrm{StProf}(C)\big(y_{b,b'}, y_{a,a'}\big) \end{aligned}$$

More generally it has an $n$-arity supermap decomposition theorem if

$$\begin{aligned} &\mathrm{StProf}(C)\big(\varovee_i\, C(a_i \otimes -, a_i' \otimes =), C(b \otimes -, b' \otimes =)\big) \\ \cong &\mathrm{StProf}(C)\big(y_{b,b'}, \varovee_i y_{a_i,a_i'}\big) \end{aligned}$$

Here, $y_{a,a'}$ is the Yoneda embedding of coend optics, understood to be the space of combs with $(a, a')$ as input and variable output, and quantum theory indeed has this decomposition property.

Theorem 2.6. *The category* CPTP *has an $n$-arity supermap decomposition theorem for every $n$.*

Finally, we are able to frame this property in terms of involutivity of weak duals in StProf$(C)$.

Proposition 2.7. *A symmetric monoidal category $C$ has a 1-arity decomposition theorem if and only if*

$$C(a \otimes -, a' \otimes =)^* \cong y_{\boldsymbol{a}}, \quad \textit{or equivalently,} \quad y_{\boldsymbol{a}}^{**} \cong y_{\boldsymbol{a}}.$$

*Furthermore, $C$ has an $n$-arity supermap decomposition theorem if and only if*

$$(\varovee_i y_{\boldsymbol{a}_i}^*)^* \cong \varovee_i y_{\boldsymbol{a}_i}.$$

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