Partial and Relational Algebraic Theories
(Extended Abstract)

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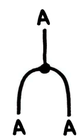
I propose to present the results of my doctoral thesis work [1], which concerns *partial algebraic theories* and *relational algebraic theories*. Partial algebraic theories allow the specification of algebraic structure involving partial functions, and relational algebraic theories allow the specification of algebraic structure involving binary relations. Partial algebraic theories are strictly more expressive than the classical notion of algebraic theory, and relational algebraic theories are strictly more expressive than partial algebraic theories. Both notions of theory admit intuitive presentation via string diagrams, and both admit robust categorical semantics in the form of a syntax-semantics adjunction.

Classical algebraic theories are equivalently cartesian monoidal categories. That is, symmetric monoidal categories in which each object A is equipped with a *copying* morphism $\delta_A : A \to A \otimes A$ and *discarding* morphism $\varepsilon_A : A \to I$. The copying and discarding morphisms must be *natural*, in the sense that for any morphism $f : A \to B$ we have $f\delta_B = \delta_A(f \otimes f)$ and $f\varepsilon_B = \varepsilon_A$. We represent δ_A and ε_A with the following string diagrams:

$\delta_A \rightsquigarrow$ 

$\varepsilon_A \rightsquigarrow$ 

These string diagrams give an alternative method of presenting algebraic theories. Given a signature Σ we represent $f_{/n} \in \Sigma$ as a diagram with n input wires and one output wire. Then string diagrams built from these component diagrams together with the copying and discarding morphisms correspond to tuples of terms over Σ. Each input wire corresponds to a variable x_i, and each output wire corresponds to a term. For example, recall the signature $\Sigma_{\mathsf{Mon}} = \{m_{/2}, e_{/0}\}$ from the classical presentation of the theory of monoids. Let us represent m and e as follows:

$m_{/2} \rightsquigarrow$ 

$e_{/0} \rightsquigarrow$ 

Then for example the following string diagrams correspond to the tuples of terms that label the output wires, in the variables that label the input wires:

Now the equations E_{Mon} of the theory of monoids may be expressed as follows:

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I will call this sort of thing a *Cartesian monoidal presentation*. We construct models of Cartesian monoidal presentations much as we construct models of classical presentations. We must choose a set X to be the carrier, and for each $f_{/n} \in \Sigma$ we must specify a function $X^n \to X$. Then each string diagram over Σ defines a function $X^n \to X^m$, and we have a *model* in case the equations are satisfied. As with classical presentations, models of Cartesian monoidal presentations are equivalently functors into Set that preserve the finite product structure.

The syntax of partial and relational algebraic theories is obtained by modifying the notion of Cartesian monoidal presentation. For partial algebraic theories, we assume an additional string-diagrammatic component $\mu_A : A \otimes A \to A$ which is to be understood as a *partial equality test*. That is, an abstract version of the partial function which, given x and y, returns x if $x = y$ and is otherwise undefined.

$\mu_A \quad \rightsquigarrow$

While for partial algebraic theories we retain the assumption that the copying morphisms are natural in the sense that $f\delta_B = \delta_A(f \otimes f)$, we no longer ask that the discarding morphisms are natural. To obtain the corresponding notion of *model* we interpret our string diagrams in the category of sets and *partial* functions, proceeding much as before. For relational algebraic theories we assume yet another component $\eta_A : I \to A$ which we can think of as *existential quantification*.

$\eta_A \quad \rightsquigarrow$

For relational algebraic theories, the naturality axioms are replaced by the assumption that $\delta_A(f \otimes f)\mu_B = f$ for all $f : A \to B$. To obtain the corresponding notion of *model* we interpret our string diagrams in the category of sets and *relations*.

In the same way that algebraic theories correspond to categories with finite products, partial algebraic theories correspond to *discrete Cartesian restriction (DCR) categories*, and relational algebraic theories correspond to *Carboni-Walters (CW) categories*. I further characterise the *varieties* of partial and relational algebraic theories, being those categories that arise as the category of models of a given theory. For partial algebraic theories, the varieties turn out to be the *locally finitely presentable (LFP)* categories, and for relational algebraic theories one obtains the *definable* categories. This leads to a result concerning *Morita equivalence* of theories, being the situation in which two theories present the same variety. We obtain that for all of classical, partial, and relational algebraic theories, two theories are Morita equivalent if and only if they have equivalent idempotent splitting completions.

From a technical perspective, the primary novelty in all of this consists of two strict 2-equivalences of 2-categories. First, a 2-category of DCR categories and structure-preserving functors is shown to be equivalent to the usual 2-category of categories with finite limits. Second, a 2-category of CW categories and structure-preserving functors is shown to be equivalent to the usual 2-category of regular categories. Crucially, to obtain these equivalences one must take *monoidal lax transformations* as 2-cells both in the 2-category of DCR categories and the 2-category of CW categories. The development also contains significant results concerning DCR and CW categories in and of themselves.

References

[1] Chad Nester (2024): *Partial and Relational Algebraic Theories*. Ph.D. thesis, Tallinn University of Technology.