Convergence of martingales via enriched dagger categories

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Summary. Martingales are a cornerstone concept in probability theory, formalizing the idea of a process where, as time progresses, we learn more and more about the system, while 'averaging' over what we do not know yet. One of the most important result of probability theory, the martingale convergence theorem, roughly says that as we learn more about a system, this 'averaging' process becomes a better and better approximation, tending to a state of perfect knowledge.

Here we give a categorical proof of martingale convergence in mean, using enriched category theory. The enrichment we use is in topological spaces, with their canonical closed monoidal (not cartesian) structure. We work in a topologically enriched dagger category of probability spaces and Markov kernels up to almost sure equality. In this category we can describe conditional expectations exactly as dagger-split idempotent morphisms, and filtrations can be encoded as directed nets of split idempotents, with their canonical partial order structure. As we show, every increasing (or decreasing) net of idempotents tends topologically to its supremum (or infimum).

Random variables on a probability space form contravariant functors into categories of Hilbert and Banach spaces, which we can enrich topologically in a canonical way. Martingales, in this formalism, live in the image of these functors, indexed by a filtration. Since enriched functors preserve convergence, we obtain, almost for free, that martingales and backward martingales converge in the L^p norms. This abstract method does not rely specifically on real numbers, and so it gives convergence for very general notions of martingales: they can take values in arbitrary Banach spaces, and they can be indexed by arbitrary nets.

This work seems to be the first application ever of topologically enriched categories to analysis and probability. We hope that this enrichment, so often overlooked in the past, will become a standard way to obtain convergence results categorically in the future.

Probability spaces, kernels, and random variables. Among the most important structures of probability theory are probability spaces and random variables. In this work we give a categorical account of both: probability spaces as a category, and random variables as a functor.

In probability theory it is customary to fix a single probability space, the outcome space, and to work only on that space. However, one often considers several related sigma-algebras on that space at once, for forming a filtration. The categorical description that we use formalizes this intuition: we want probability spaces, with fixed sigma-algebras, to be the objects of a category, and refinements and coarse-grainings of these spaces to be the morphisms. Concretely we work with a dagger category EKrn whose objects are (well-behaved) probability spaces, and whose morphisms are measure-preserving Markov kernels quotiented under a version of almost sure equality. This category, in the standard Borel case, was first defined in [6].

Given a probability space $(X, \mathcal{A}, p)$, one can consider the random variables on it which are measurable for $\mathcal{A}$ and integrable for p. Choosing a different probability space (for example, a different sigma-algebra), one obtains a different set of random variables. Moreover, refinements and coarse-grainings of the probability spaces have an effect on random variables, for example via conditional expectations. This is encoded, category-theoretically, by modelling random variables as a *functor* on the category of probability spaces. This idea was first developed in [6] and extended in [3], and here we give a further generalization of it. This approach also lets us consider more general functors of random variables, for example Banach-space-valued ones.

Sub-sigma-algebras, filtrations, and martingales. Our first step is to express the coarse-grainings to sub-sigma-algebras as (dagger) idempotent morphisms. This is analogous to how a closed subspace of a Hilbert space is both a subspace and a quotient in a compatible way:

Theorem 1. *Let $(X, \mathcal{A}, p)$ be a standard Borel probability space. There is a bijection between idempotent morphisms $e : (X, \mathcal{A}, p) \to (X, \mathcal{A}, p)$ and sub-sigma-algebras $\mathcal{B} \subseteq \mathcal{A}$, up to null sets.*

Moreover, this bijection is order-preserving, in the sense that $\mathcal{B} \subseteq \mathcal{C}$ almost surely if and only if $e_{\mathcal{B}} \circ e_{\mathcal{C}} = e_{\mathcal{C}} \circ e_{\mathcal{C}} = e_{\mathcal{B}}$.

A *filtration* on a probability space $(X, \mathcal{A}, p)$ is now a sequence, or a net, of sub-sigma-algebras $\mathcal{B}_i \subseteq \mathcal{A}$ which are increasingly finer, that is, $\mathcal{B}_i \subseteq \mathcal{B}_j$ whenever $i \leq j$. Usually the filtration is indexed by a total order, which we can see as time: as time progresses, we are able to make finer and finer distinctions. The *join* sigma algebra $\mathcal{B}_\infty = \bigvee_i \mathcal{B}_i$, which is the limit of the subobjects, can be interpreted as encoding all the knowledge that one can possibly learn from the process.

A *martingale* is a collection of random variables f_i which, intuitively, follow the refinement of a filtration $(\mathcal{B}_i)$. The *martingale convergence theorem* says that, under some conditions,

- The f_i admits a common universal refinement f, measurable for $\mathcal{B}_\infty$, and moreover,
- The f_i tend to f topologically.

In other words, as our knowledge increases, our refinements become better and better "approximations". This "approximation" part is achieved, categorically, by means of a topological enrichment.

Topological enrichment. Ordinary category theory allows to form what intuitively are "approximations of spaces", via particular limits and colimits (hence the name *limit*). For example, the diagram in the category of sets formed by finite sets and inclusion maps

$$\{0\} \hookrightarrow \{0,1\} \hookrightarrow \{0,1,2\} \hookrightarrow \{0,1,2,3\} \hookrightarrow \dots$$

has as colimit the set of natural numbers $\mathbb{N}$. Similarly, in the case of martingales one has a filtration of sigma-algebras, and its limit gives the universal common refinement of the filtration.

If we enrich our categories in topological spaces, we moreover incorporate a notion of "approximating morphisms" into an otherwise purely algebraic, equational environment.

The martingale convergence theorem, categorically, can be expressed as the fact that *if we express an object as a categorical limit, we also have a topological convergence associated with it.*

This correspondence between categorical limits and topological ones is a phenomenon that, in different ways, has been noticed before, for example, in the category of Hilbert spaces [4, 5]. In this work we make this correspondence mathematically precise as follows:

Theorem 2. *In the categories* Hilb *and* EKrn*, monotone nets of dagger idempotents tend topologically to their supremum (or infimum, if directed downwards).*

As a consequence, purely from the fact that the L^1 and L^2 functors are enriched, we get a version of the convergence theorem for martingales and backward martingales:

Theorem 3 (Martingale convergence theorem in mean)**.** *Let $(X, \mathcal{A}, p)$ be a standard Borel probability space, let $(B_\lambda)_{\lambda\in\Lambda}$ be a filtration, and let $(f_\lambda)_{\lambda\in\Lambda}$ be a martingale with either uniform L^2 bound, or a common $\mathcal{A}$-measurable refinement. Then there exists $f_\infty \in L^1(X, \mathcal{B}_\infty, p)$ refining all the f_i (i.e. such that $f_\lambda = \mathbb{E}[f_\infty|\mathcal{B}_\lambda]$ for all λ), and the f_λ tend to f_∞ both in L^1 and in L^2.*

The dual statement can be made for decreasing filtrations and backward martingales.

The result also holds for martingales with values in an arbitrary Banach space. As far as we know, the convergence in mean of backward martingales indexed by arbitrary nets is a new result.

References

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