Extended abstract: Graphical Symplectic Algebra

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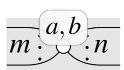
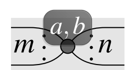
In this article we give a presentation for the prop of affine Lagrangian relations over an arbitrary field. This gives semantics for the phase-space picture of both quantum and classical mechanics, depending on the choice of field. Therefore, the presentations we give here yield a unifying treatment for restrictions of both classical mechanical and quantum circuits.

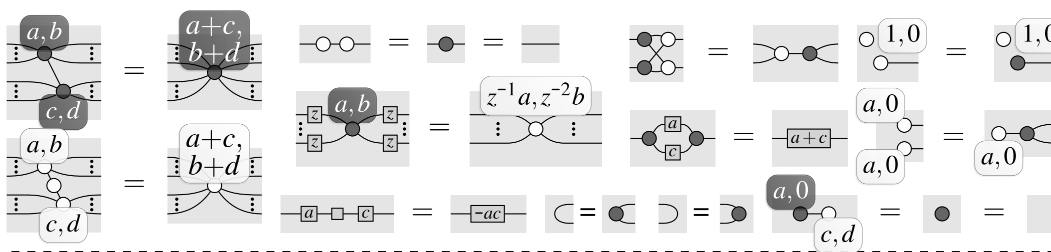
Affine Lagrangian relations over a field $\mathbb{K}$ is generated by white and grey undirected graphs, with m inputs and n outputs, for all $m, n \in \mathbb{N}$ (ie. flexsymmetric spiders) labeled by elements of $\mathbb{K}^2$:



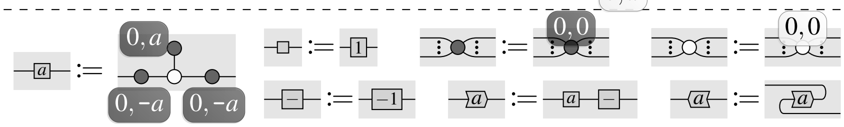
and



modulo the equations,



With derived generators:



Semantically, a wire represents the identity on the affine plane $\mathbb{K}^2$ with a distinguished position and momentum basis. The first component of the white/grey spiders shifts the origin of the momentum/position. The second component of the white/grey spiders shears in the momentum/position direction. The a-labelled arrow shaped generator squeezes the affine plane: scaling the position by a and momentum by $1/a$. The 1 and -1-labelled boxes rotate the plane counterclockwise and clockwise, representing the symplectic Fourier transform and its inverse. The spiders with trivial phase acts as generalized Dirac deltas in the position and momentum bases.

Taking $\mathbb{K} := \mathbb{F}_p$, for p an odd prime, this is restricts to the presentation of the odd prime dimensional stabiliser ZX-calculus of Poór et al. [1]. The position corresponds to the Pauli Z observable, and the momentum corresponds to the Pauli X observable.

By working in the strictification of our presentation, where multiple wires can be bundled together and unbundled, we are able to construct gates which correspond to squeezing by matrices, rather than merely scalars; similarly for higher dimensional Fourier transforms. We draw these higher dimensional wires and operations with thickened borders. We also can define higher-dimensional spiders, which have different labels than in the single-arity setting. We define $\mathbb{K}^k \times \mathrm{Sym}_k(\mathbb{K})$-labelled k-coloured spiders inductively; for the inductive step, take $n, m \in \mathbb{N}$, $a, b \in \mathbb{K}$, $\vec{v}, \vec{w} \in \mathbb{K}^k$ and $A \in \mathrm{Sym}_k(\mathbb{K})$, we define

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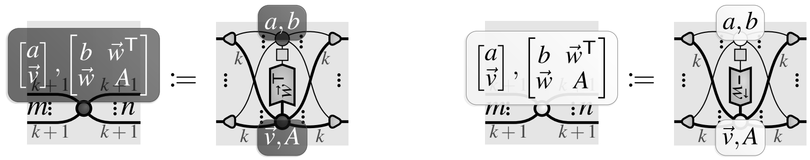
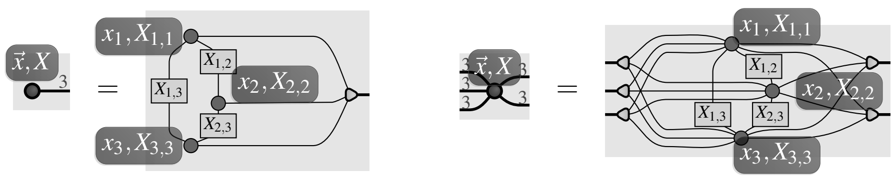
$k+1$-coloured spiders as:



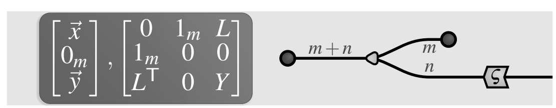
(1)

A k-coloured $n \to m$ spider parametrizes an undirected open graph with edges labelled by $\mathbb{K}$, vertices labelled by $\mathbb{K}^2$, with n / m distinguished inputs / outputs. For example, on the left hand side of equation (2) we have a "graph state" with $n=0$, $m=1$ and $k=3$. On the right hand side, we have a spider with $m=3$, $n=2$ and $k=3$:



(2)

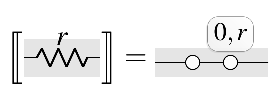
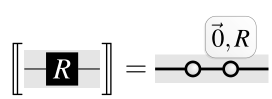
These higher dimensional spiders allow us to succinctly state our normal form, in terms of specific kinds "partially open graphs," where all vertices of the graph need not be connected to inputs/outputs. Specifically, every nonempty state $0 \to n$ is reducible to a unique reduced-AP form, represented by a 7-tuple $(L, \Sigma, \vec{x}, \vec{y}, Y, \varsigma)$, where $m \leqslant n \in \mathbb{N}$, $\vec{x} \in \mathbb{K}^m$, $\vec{y} \in \mathbb{K}^{n-m}$, $L \in \mathrm{Mat}_{\mathbb{K}}(m, n-m)$, $Y \in \mathrm{Sym}_{n-m}(\mathbb{K})$, with permutation $\varsigma \in \mathrm{Mat}_{\mathbb{K}}(n,n)$:



(3)

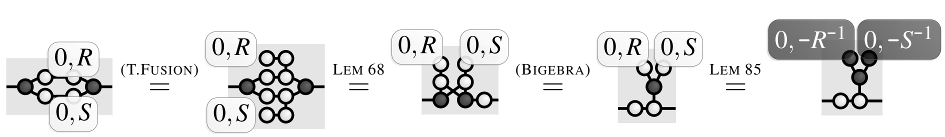
By working over $\mathbb{K} := \mathbb{F}_p$ for p an odd-prime, this thickened notation allows one to mix the stabiliser tableau notation with quantum tensor networks: taking advantage of matrix algebra as well as the topologically intuitive properties of string diagrams. For example, the matrix on the LHS of the state in equation (2) is the stabilizer tableau for the quantum graph-state represented on the RHS.

This elucidates connections between stabiliser quantum mechanics and classical mechanics. For example, working over $\mathbb{K} := \mathbb{R}$ the analogue of the phase-shift gate in electrical circuits is the resistor, represented by a spider labeled by some $r \in \mathbb{R}$. However, when we thicken things, as we did for quantum graph states, we can represent a reciprocal network of passive electrical circuits whose black-boxed behaviour is determined by its impedence matrix $R \in \mathrm{Sym}_k(\mathbb{R})$ (see Pozar [2, section 4.2], for reference):

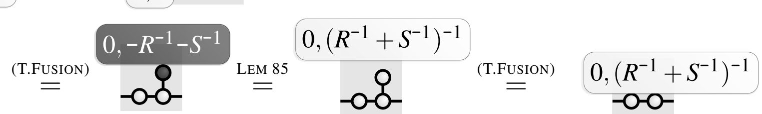
 and 

(4)

This thickened notation is very powerful, because it allows us to rewrite large-scale networks. For example given two classical mechanical systems on the same space with invertible impdence matrices $R, S \in \mathrm{Sym}_k(\mathbb{R})$, we can compose them in parallel and simplify them, just as for resistors:



(5)



(6)

References

[1] Boldizsár Poór, Robert I. Booth, Titouan Carette, John Van De Wetering, and Lia Yeh. "The Qupit Stabiliser ZX-travaganza: Simplified Axioms, Normal Forms and Graph-Theoretic Simplification". In: *Electronic Proceedings in Theoretical Computer Science*. Twentieth International Conference on Quantum Physics and Logic. Vol. 384. Paris, France, Aug. 29, 2023, pp. 220–264. DOI: 10.4204/EPTCS.384.13.

[2] David M Pozar. *Microwave Engineering*. en. 4th ed. Chichester, England: John Wiley & Sons, Nov. 2011.