Extended abstract: Complete equational theories for classical and quantum Gaussian relations

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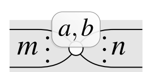
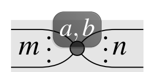
In this paper, we give generators and equations for Gaussian quantum processes with infinite squeezing. We use complex affine Lagrangian relations as our semantics, rather than infinite dimensional Hilbert spaces, which allows us to account for infinitely squeezed operations such as dirac deltas and Frobenius algebras in a compositional matter. More specifically, we pick out a sub-category of *positive* complex affine Lagrangian relations in order to establish a bijection with the syntax which we provide.

We represent Gaussian processes by tensor networks of white and grey undirected graphs, with m inputs and n outputs, whose nodes are labeled by pairs $a \in \mathbb{R}, b \in \mathbb{C}$ with $\mathrm{Im}(b) \geqslant 0$:



and


, (1)

The equations for the language, as well as some derived generators (or syntactic sugar), are given in the following figure :

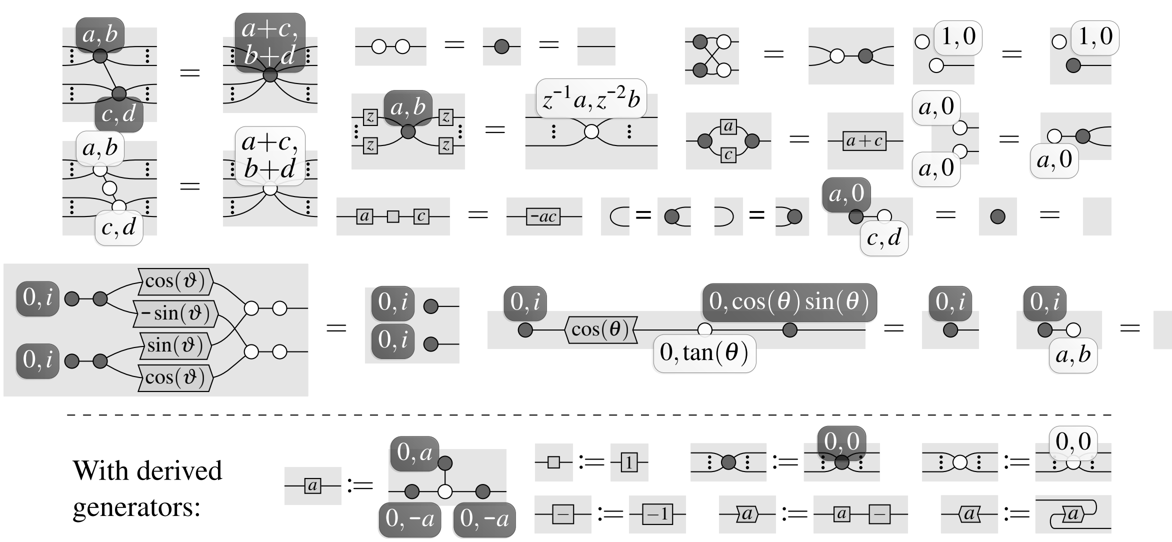


Figure 1: Equations for our graphical language. Here, $a,c,z \in \mathbb{R}$, $b,d \in \mathbb{C}$, $\theta, \vartheta \in [0,2\pi)$ with $\mathrm{Im}(b) \geqslant 0$, $\mathrm{Im}(d) \geqslant 0$, $z \neq 0$, $\theta \notin \{\pi/2, 3\pi/2\}$.

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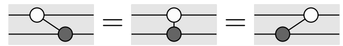
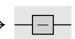
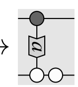
We emphasize that the fact that our generators are undirected graphs means that only the connectivity of diagrams matters; for example, all of the three following circuits are equivalent:



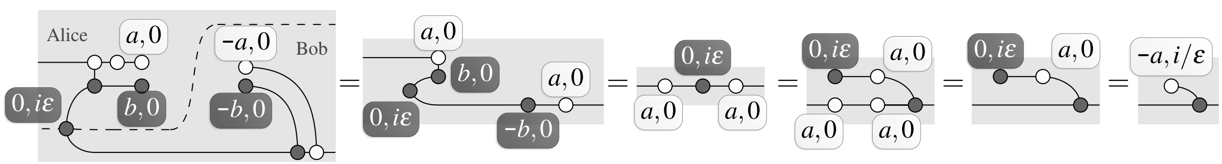
(2)

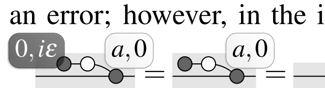
In other words, we take the presentation for real affine Lagrangian relations given by Booth et al. [1] and add a single generator $0,i$: $0 \to 1$ which freely codiscards symplectic rotations as well as non-zero effects. From a physical point of view, this new state corresponds to the quantum vacuum state, representing the standard bivariate Gaussian distribution of positions and momentums in the phase space. On the other hand, infinitely-squeezed eigenstates of the position and momentum displacement operators $\hat{p}$ and $\hat{q}$ are represented by $|p : \hat{p}\rangle \mapsto$ $p,0$ and $|q : \hat{q}\rangle \mapsto$ $q,0$. We can also translate a universal gate set for Gaussian unitaries into our language; consisting of the **displaced shear** of position, the **Fourier transform** and **weighted CNOT** [2] with $a,b \in \mathbb{R}$:

$\exp\big(i(a\hat{q}+b\hat{q}^2)\big) \mapsto$ 

$\exp\Big(i\frac{\pi}{2}(\hat{q}^2+\hat{p}^2)\Big) \mapsto$ 
$\exp(ia(\hat{q}\otimes\hat{p})) \mapsto$  (3)

We can now give a straightforward graphical proof of the continuous-variable quantum teleportation protocol [3] (to be compared with the graphical proof in finite dimensions eg. [4, §5.4]), where Alice and Bob share a Gaussian Bell state with covariance of position $0 < \varepsilon \in \mathbb{R}$. Alice records the homodyne measurement outcome $(a,b) \in \mathbb{R}^2$ in the Bell basis, and sends it to Bob, who performs the phase correction $\hat{p}^{-b}\hat{q}^{-a}$. This is stated graphically and simplified as follows:



The result is a quantum channel with an error; however, in the infinitely-squeezed limit of $\varepsilon = 0$ Alice teleports a perfect channel to Bob: 

. In the literature, the infinitely-squeezed Bell state is often haphazardly represented by the non-convergent integral $\int_{\mathbb{R}} |p : \hat{p}\rangle \otimes |p : \hat{p}\rangle\, dp$. However, we reiterate that this expression *can* be represented in our calculus by an unlabeled 2-legged grey node; and it is interpreted *soundly* in our semantics by replacing the integral over position eigenstates $|p : \hat{p}\rangle$ with the *relational composition* in the phase-space.

In the full paper, we discuss the relation to different works existing in the literature. For example we show how our work extends Menicucci et al's representation of Gaussian states of [5] in a compositional matter. We also discuss the relationship to quantum optics, and in particular, the graphical language LOv for passive quantum optics [6]. We also show how Stein and Samuelson's prop of Gaussian relations [7] can be presented by taking real affine relations and adding a generator which freely codiscards rotations and nonzero effects; a result which was also independently discovered [8]!

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