No-go theorems: Polynomial comonads that do not distribute over distribution monads

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Abstract

Monads and comonads are important constructions from category theory which find widespread application in computer science and other related disciplines. Distributive laws allow these constructions to interact compositionally. Such laws are not guaranteed to exist, and even when they do, finding them can be a difficult task. Inspired by recent results which establish conditions under which no distributive laws can exist between pairs of monads, we present a family of no-go theorems for the existence of distributive laws of a comonad over a monad. We begin by showing that, in the category of sets, every polynomial functor has a unique Kleisli law over the non-empty powerset monad. We then show that this Kleisli law only extends to a comonad-monad distributive law if the comonad is a linear polynomial comonad, i.e. coreader comonad. Consequently, every other polynomial comonad does not distribute over the powerset monad. Next, we generalise our results to a large class of monads, which we call uniform choice monads. Examples of monads in this class include any multiset or distribution monad parameterised by a suitable semiring and the filter monad. Finally, we give string-diagrammatic proofs of 'transfer theorems', that allow us show when a distributive law of (co)monads over categories which contain the category of sets as a (co)reflective subcategory restrict to related (co)monads over the category of sets. Using these transfer theorems, we show that several game comonads, recently introduced in the context of finite model theory, fail to distribute over variants of the powerset and distribution monads, which are used to capture relaxations of the constraint satisfaction problem.

Possibilistic monads and polynomial comonads Amongst the many applications of monads is their use in giving semantics for effectful computation [Mog89]. An effectful computation from inputs of type A to outputs of type B is modeled as a morphism of type $A \to MB$. For instance, if M is the powerset monad, then a morphism $A \to MB$ is non-deterministic computation from inputs A to values in B. Similarly, if M is the discrete probability distribution monad, a morphism $A \to MB$ computes from inputs in A probability distributions over random variables in B. The Kleisli category of the monad M ensures that the non-deterministic/probabilistic computations can be composed. On the other hand, comonads (W, ε, δ) whose underlying functor is a polynomial endofunctor capture data structures which have 'directed' or contextual-folding structure, i.e. they are equipped with coherent notions of 'subshape' and 'root position' [ACU14]. For polynomial comonads, a morphism of type $WA \to B$ represents a contextual fold of a data structure filled with elements from A to compute a value in B. The coKleisli category of the comonad W ensures that these contextual computations can be composed.

No-go theorems for mixed distributive laws Given a monad M and a comonad W, when can two contextual nondeterministic computations $WA \to MB$ and $WB \to MC$ be composed? Power and Watanabe in [PW02] demonstrated that the existence of a mixed distributive law $\kappa\colon WM \to MW$ gives a sufficient (but not necessary) condition for obtaining a biKleisli category where morphisms of type $WA \to MB$ are composed. A mixed distributive law $\kappa\colon WM \to MW$ of a comonad (W, ε, δ) over a monad (M, η, μ)

is a natural transformation satisfying some coherence axioms with the M's unit η, W's counit ε, M's multiplication μ, and W's comultiplication δ. Understanding when such mixed distributive laws exist is important for understanding the limitations of compositionality in computation. Similar questions have been asked for distributive laws of monads over monads. General-purpose techniques were developed to come up with such laws (e.g. [BHKR13, MM07, MM08, Par20, DPS18]). However, recent research has also focused on the non-existence of monad-monad laws [KS18, ZM22]. Inspired by these no-go theorems, the paper we will present (**attached below**) contains a no-go theorem demonstrating that there is no distributive law of type $\kappa\colon WM \to MW$ for all non-linear polynomial comonads W and all monads M which have a meaningful notion of 'uniform sampling distribution'. The class of monads M includes a wide-class of distribution and multiset monads parameterised by a semiring, and the filter monad. The proof of the no-go theorem exhibits a mixture of various techniques including Plotkin-style naturality diagram chases, supported endofunctors, and the algebraic notion of an n-ary open term of a monad.

Transfer theorems To widen the scope of applicability of our no-go theorem, the paper we will present also includes a two part *transfer theorem*. This transfer theorem dictates conditions under which the existence of a mixed distributive law in a category C implies the existence of a mixed distributive law in **Set**. The first part of this theorem is a generalised comonad-monad variant of [MM07, Theorem 3.1.3], which considers transfer theorems for monad-monad distributive laws defined on the same category. The additional ingredient for this generalisation requires a Yang-Baxter equation to be satisfied. The second part of this theorem demonstrates that given a $\iota \dashv U$ coreflection including **Set** into C we can produce a mixed distributive law of (co)monads over **Set** from a distributive law of (co)monads over C. We then apply this transfer theorem to cases where $C = \mathcal{R}(\sigma)$ is the category of relational structures. In particular, we demonstrate there cannot exist a distributive law of the Ehrenfeucht-Fraïssé or pebbling comonads [ADW17, AS20], which capture equivalence in fragments of infinitary first-order logic, over the tree-duality and fractional isomorphism monads [NDM12, Con22], which capture relaxations of constraint satisfaction problems.

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No Go Theorems: Directed Containers That Do Not Distribute Over Distribution Monads

Abstract—**Monads and comonads are important constructions from category theory which find widespread application in computer science and other related disciplines. Distributive laws allow these constructions to interact compositionally. Such laws are not guaranteed to exist, and even when they do, finding them can be a difficult task.**

Inspired by recent results which establish conditions under which no distributive laws can exist between pairs of monads, we present a family of no-go theorems for the existence of distributive laws of a comonad over a monad.

We begin by showing that in the category of sets every container has a unique Kleisli law over the non-empty powerset monad. We then show that this Kleisli law only extends to a comonad-monad distributive law if the comonad is a coreader comonad. Consequently, every other directed container does not distribute over the powerset monad. Next, we generalise our results to a large class of monads, which we call uniform choice monads. Examples of monads in this class include any multiset or distribution monad parameterised by a suitable semiring. Finally, we extend our results to the category of relational structures where we show that several game comonads, recently introduced in the context of finite model theory, fail to distribute over variants of the powerset and distribution monads, which are used to capture relaxations of the constraint satisfaction problem. Overall, our no-go results cover a diverse range of (co)monads that are of interest in many areas of mathematics and computer science, such as probability theory, programming languages, and finite model theory.

I. Introduction

Monads and comonads are ubiquitous throughout mathematics and theoretical computer science. Amongst the many applications of monads is their use in giving semantics for effectful computation [35]. An effectful computation from inputs of type A to outputs of type B is modeled as a morphism of type $A \to MB$. In this paper we will be concerned with monads M that model a form of nondeterminism, e.g. possiblistic, probabilistic, or quantum. Nondeterministic computations $A \to MB$ are composed in the Kleisli category associated to the monad M. Similarly, comonads are used to give semantics for contextual computations where inputs are paired with a context in a larger data structure [37], [42]. Contextual computations from inputs of type A to outputs of type B are modeled as morphisms of type $WA \to B$ where W is a comonad that models contexts in a data structure, e.g. prefixes of a list, indices of a list, nodes in a tree. Dual to the notion of a Kleisli category over a monad M, contextual computations $WA \to B$ are composed in the coKleisli category associated to the comonad W. Given a monad M and a comonad W, when can two contextual nondeterministic computations $WA \to MB$ and $WB \to MC$ be composed? Power and Watanabe in [40] demonstrated that the existence of a mixed distributive law gives a sufficient (but not necessary) condition for obtaining a biKleisli category where morphisms of type $WA \to MB$ are composed.

A mixed distributive law from a comonad W to a monad M is a natural transformation $\kappa\colon WM \to MW$ satisfying the four axioms in Definition II.15. An example for such a law is to take W to be the prefix list comonad L^+ and M to be the partiality monad, i.e. the Haskell 'Maybe' monad [37]. Intuitively, this law takes a non-empty list of potentially undefined values and returns either (1) undefined if the last value of the list is undefined, or (2) the list of those values that are defined. In a sense, the partiality monad is a rudimentary form of nondeterminism–either a value is determined or undetermined. As we will show, for certain other nondeterministic monads M no such distributive law can exist. The motivation that led us to discover this family of no-go theorems came from two different avenues of research.

The first avenue comes from recent no-go theorems on a different notion of distributive laws. Mixed distributive laws are one among many different notions of distributive laws where the structure on W and M are varied, e.g. monads over comonads, pointed endofunctors over endofunctors, and many others detailed in [31]. Chief among these notions is the monad-monad distributive law. A monad to monad distributive law of type $TS \to ST$ provides a sufficient condition for when two monads on functors S and T determine a monad structure on the composed functor $S \circ T$. Many general-purpose techniques were developed to come up with such laws (e.g. [10], [33], [34], [38], [14]). However, recent research has also shown the non-existence of such laws [29], [45]. In particular, a result in [43] attributed to Plotkin showed that the powerset monad does not distribute over the distribution monad. Zwart and Marsden [44], [45] vastly generalise this result to obtain sufficient conditions for the non existence of a distributive law between pairs of monads. Unfortunately, the conditions that appear in [45] rely on algebraic presentations of monads. Since the analogous notion of coalgebraic presentations of comonads is an active area of research [15], these conditions cannot be readily adapted to studying mixed distributive laws. This paper is a first step towards a general theory for demonstrating the non-existence of mixed distributive laws.

The second avenue comes from applications of (co)monads to finite model theory. In work by Abramsky et. al [4], [5], Spoiler-Duplicator games, a key tool in finite model theory, are internalized as families of indexed comonads over the category of relational structures $\mathcal{R}(\sigma)$ for some relational signature σ. In model theory, a Spoiler-Duplicator game is used to show when two relational structures satisfy the same sentences in a logic graded by some resource associated to the game. Consequently, by internalizing a game as a comonad, Abramsky et. al provide categorical semantics to the corresponding logic. For example, the k-pebble comonad $\mathbb{P}_k$ is obtained by putting a relational structure on the non-empty list of previous Spoiler moves in a one-sided variant of the k-pebble game. Parallel to the development of Spoiler-Duplicator comonads, a paper by Abramsky et. al [3] demonstrates that the notion of quantum homomorphism, phrased originally in terms of non-local games from quantum information theory, is a morphism in the Kleisli category of a graded quantum monad $\mathbb{Q}_d$ over relational structures. An open question that was first alluded to in [3], and later explicated in [12] asks if a mixed distributive law exists between $\mathbb{P}_k$ and $\mathbb{Q}_d$. A positive answer to this question would give rise to a biKleisli category which could be used to define compositional quantum winning strategies for Duplicator in the k-pebble game. This open question will be addressed in a related paper that is currently in preparation. The present paper is instead dedicated to detailing the no-go theorems we discovered in the process of answering this question.

Because $\mathbb{P}_k$ and $\mathbb{Q}_d$ are complicated constructions, we decided to first explore the question of whether mixed distributive laws exists between simpler (co)monads which share some similarities with $\mathbb{P}_k$ and $\mathbb{Q}_d$. To this end, we first considered the non-empty list comonad L^+ and the powerset monad $\mathcal{P}$. In this case, L^+ can be seen as a version of $\mathbb{P}_k$ defined on $\mathbf{Set}$ rather than $\mathcal{R}(\sigma)$ and without the complication of pebbles, while $\mathcal{P}$ can be seen as a possibilistic version of $\mathbb{Q}_d$ which does not assign probabilities or measurement outcomes to these possibilities. We discovered that it is impossible to construct a distributive law $\kappa : L^+\mathcal{P} \to \mathcal{P}L^+$ between this pair of (co)monads. This result will be the cornerstone upon which the other no-go theorems in this paper are built.

The remainder of the paper is organised as follows: We introduce the preliminary mathematical background in section II. Section III contains our first contribution, a "Plotkin-style" argument which shows that any container (equivalently, polynomial endofunctor) has a unique Kleisli law over the non-empty powerset monad. We then show in section IV that for a large class of comonads this law does not extend to a comonad-monad distributive law over the (non-empty) powerset monad. In particular, there is no distributive law of the non-empty prefix list comonad over the (non-empty) powerset monad. This is in contrast with a well-known distributive law of the list monad over the (non-empty) powerset monad [33]. In section V we extend our no-go result to monads which admit a sensible notion of 'uniform distribution'. We then determine conditions under which the distribution and multiset monads $\mathcal{D}_{\mathscr{S}}$ and $\mathcal{M}_{\mathscr{S}}$ belong to this class of monads. Finally, in section VI we prove a transfer theorem which allows us to extend our results to the category of relational structures. Using this theorem we prove that many of the game comonads introduced to study finite model theory do not distribute over variants of the powerset and distribution monads used to capture relaxations of the constraint satisfaction problem.

II. Preliminaries

In this section, we establish some notational preliminaries and provide a short introduction to the relevant concepts in category theory that we use.

A. Category theory

We assume familiarity with the standard category-theoretic notions of category, functor, natural transformation, and adjunction (see e.g. [39] for definitions). Given a category $\mathfrak{C}$, we will denote its class of objects $\mathfrak{C}_0$ and the class of morphisms $\mathfrak{C}_1$. Given two objects $X, Y \in \mathfrak{C}_0$, the class of morphisms of type $X \to Y$, is denoted $\mathfrak{C}(X, Y)$.

The category of endofunctors with morphisms as natural transformations will be denoted $[\mathfrak{C}, \mathfrak{C}]$. This category is strict monoidal with the monoidal product given by functor composition and the monoidal unit given by the identity functor. In particular, this means that given natural transformations $\nu \colon F \to G$ and $\nu' \colon F' \to G'$, we can 'horizontally compose' them to obtain a natural transformation $\nu' \star \nu \colon F' \circ F \to G' \circ G$ defined as $\nu' \star \nu = \nu' G \circ F'\nu = G'\nu \circ \nu' F$. Note these two definitions are indeed equivalent by naturality of ν' applied to ν's components. Since we will not use any other aspect of the monoidal structure on the category $[\mathfrak{C}, \mathfrak{C}]$, we do not define monoidal categories and refer the reader to chapter 1 in [23].

We write $\mathbf{Set}$ for the category of sets and functions. For a set X, $|X|$ denotes the (potentially infinite) cardinality of X. For $n \in \mathbb{N}$, $[\mathsf{n}] = \{1, \ldots, n\}$.

B. Monads and comonads

We recall the definition of a *monad on* $\mathfrak{C}$ is a triple (M, η, μ) where $M \colon \mathfrak{C} \to \mathfrak{C}$ is an endofunctor and $\eta^M : \mathbf{Id}_{\mathfrak{C}} \to M$, $\mu^M : MM \to M$ are natural transformations. such that the following equations hold: $(1) : \mu \circ M\mu = \mu \circ \mu$ and $(2) : \mu \circ M\eta = \mu \circ \eta = \mathbf{id}_M$. We may omit the superscripts and write η and μ whenever the functor M is clear.

A *monad map from* (M, η^M, μ^M) *to* $(M', \eta^{M'}, \mu^{M'})$ is a natural transformation $\rho \colon M \to M'$ such that $\rho \circ \mu^M = \mu^{M'} \circ (\rho \star \rho)$; $\rho \circ \eta^M = \eta^{M'}$. There is a category $\mathbf{Mon}(\mathfrak{C})$ of monads and monads maps on $\mathfrak{C}$. The category $\mathbf{Mon}(\mathfrak{C})$ has a notion of subobject. A monad M is a *submonad of* M' if there exists a monic monad map $\iota \colon M \to M'$. In many cases, we will not need to consider the multiplication μ^M of a monad, and instead consider pointed endofunctors (M, η^M) where M is an endofunctor and $\eta \colon \mathbf{id}_{\mathfrak{C}} \to M$ is a unit natural transformation. A map between two pointed endofunctors $\rho \colon M \to M'$ is a natural transformation satisfying $\rho \circ \eta^M = \eta^{M'}$.

Dually, a *comonad on* $\mathfrak{C}$ is a triple $(W, \varepsilon^W, \delta^W)$ where $W \colon \mathfrak{C} \to \mathfrak{C}$ is an endofunctor and $\varepsilon^W : W \to \mathbf{Id}_{\mathfrak{C}}$, $\delta^W : W \to WW$ are natural transformations such that the following equations hold: $(1) : W\delta^W \circ \delta^W = \delta^W W \circ \delta^W$ and $(2) : W\varepsilon^W \circ \delta^W = \varepsilon^W W \circ \delta^W = \mathbf{id}_W$. We may omit the superscripts and write ε and δ whenever the functor W is clear.

C. Multiset and Distribution Monads

A semiring is given by the data $\mathscr{S} = (S, 0_{\mathscr{S}}, 1_{\mathscr{S}}, +, \cdot)$ where $(S, 0_{\mathscr{S}}, +)$ is an 'additive' commutative monoid and $(S, 1_{\mathscr{S}}, \cdot)$ is a 'multiplicative' monoid where multiplication $\cdot$ distributes over addition $+$, i.e. for all x, y, z $\;x \cdot (y + z) = x \cdot y + x \cdot z$, $(y + z) \cdot x = y \cdot x + z \cdot x$, and $0_{\mathscr{S}} \cdot x = 0_{\mathscr{S}} = x \cdot 0_{\mathscr{S}}$. A semiring morphism is a set function preserving the addition, multiplication, additive unit, and multiplicative unit. The *multiset monad over* $\mathscr{S}$ is $(\mathcal{M}_{\mathscr{S}}, \eta, \mu)$ where $\mathcal{M}_{\mathscr{S}} \colon \mathbf{Set} \to \mathbf{Set}$ is the endofunctor such that:

- for a set X, $\mathcal{M}_{\mathscr{S}}(X)$ is the set of all functions of type $\varphi \colon X \to S$ where for all but finitely many $x \in X$, $\varphi(x) = 0_{\mathscr{S}}$ and and
- for a function, $f \colon X \to Y$, $\mathcal{M}_{\mathscr{S}}(f) \colon \mathcal{M}_{\mathscr{S}}(X) \to \mathcal{M}_{\mathscr{S}}(Y)$ maps φ to $\lambda y.\Sigma_{x \in f^{-1}(y)} \varphi(x)$.
- The unit has components defined as $\eta_X(x) = \Delta_x$ where $\Delta_x(x) = 1_{\mathscr{S}}$ and $\Delta_x(y) = 0_{\mathscr{S}}$ for $y \neq x$.
- The multiplication has components defined as $\mu_X(\varphi) = \lambda x.\Sigma_{\psi \in \mathcal{M}_{\mathscr{S}}(X)} \varphi(\psi) \cdot \psi(x)$.

We define the support of a multiset $\varphi \in \mathcal{M}_{\mathscr{S}}(X)$ as the set $\mathbf{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0_{\mathscr{S}}\}$. Elements $\varphi \in \mathcal{M}_{\mathscr{S}}(X)$ can be written as finite formal sums $\varphi = \Sigma s_i x_i$ with $x_i \in \mathbf{supp}(\varphi)$ and $\varphi(x_i) = s_i$.

The distribution monad $(\mathcal{D}_{\mathscr{S}}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ over semiring $\mathscr{S}$ underlying functor $\mathcal{D}_{\mathscr{S}} \colon \mathbf{Set} \to \mathbf{Set}$ where $\mathcal{D}_{\mathscr{S}}(X)$ is the subset of $\mathcal{M}_{\mathscr{S}}(X)$ such that elements $\varphi \in \mathcal{M}_{\mathscr{S}}(X)$ satisfy the normalisation condition $\sum_{x \in X} s_i = 1_{\mathscr{S}}$. The unit and multiplication of $(\mathcal{D}_{\mathscr{S}}, \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$ are obtained by restricting the corresponding maps of the mulitset monad to $\mathcal{D}_{\mathscr{S}}(X)$.

Example II.1. *The ordinary multiset or 'bag' monad $\mathcal{M}$ is recovered as $\mathcal{M}_{\mathscr{S}} = \mathcal{M}_{\mathbb{N}}$ for the semiring of natural numbers $(\mathbb{N}, 0, 1, +, *)$.*

Example II.2. *The ordinary probability distribution monad $\mathcal{D}$ is recovered as $\mathcal{D}_{\mathscr{S}} = \mathcal{D}_{\mathbb{R}_{\geq 0}}$ for the semiring of non-negative real numbers $(\mathbb{R}_{\geq 0}, 0, 1, +, *)$.*

Example II.3. *Given a ring R, the free R-module and R-convex space monad are recovered as $\mathcal{M}_{\mathscr{S}}$ and $\mathcal{D}_{\mathscr{S}}$ for ring $\mathscr{S} = R$ considered as a semiring.*

Example II.4. *The multiset $\mathcal{M}_{\mathscr{S}}$ and distribution $\mathcal{D}_{\mathscr{S}}$ monads for the min-plus semiring $(\mathbb{R} \cup \{\infty\}, \infty, 0, \min, +)$ are useful for modeling spaces in tropical geometry and logical queries with confidence scores [21].*

Example II.5. *If $\mathscr{S} = (\mathbb{B}, \bot, \top, \vee, \wedge)$ is the Boolean semiring, then $\mathcal{M}_{\mathscr{S}}$ and $\mathcal{D}_{\mathscr{S}}$ are isomorphic to the finite powerset monad $\mathcal{P}_f$ and finite non-empty powerset monad $\mathcal{P}_f^+$.*

In the case where the addition operation $+$ of $\mathscr{S}$ can be extended to an arbitrary sum operation which also distributes over multiplication, $\mathscr{S}$ is a complete semiring. As this sum operation is well-defined for infinite subsets, we can remove the finite support restriction on the functions in $\mathcal{M}_{\mathscr{S}}(X)$ and $\mathcal{D}_{\mathscr{S}}(X)$ to obtain analogous monads $\mathcal{M}_{\mathscr{S}}^{\infty}$ and $\mathcal{D}_{\mathscr{S}}^{\infty}$.

Example II.6. *If $\mathscr{S}$ is the Boolean semiring $\mathbb{B}$, then $\mathcal{M}_{\mathscr{S}}^{\infty}$ and $\mathcal{D}_{\mathscr{S}}^{\infty}$ are isomorphic to the full powerset monad $\mathcal{P}$ and the full non-empty powerset monad $\mathcal{P}^+$.*

D. Directed Containers

All of the $\mathbf{Set}$ comonads we consider in this work have the property that their underlying functor is a *container* [1]. Intuitively, a container captures data structures which have a set of shapes and addressed positions for the data stored within those shapes.

Formally, a *container* $S \triangleleft P$, is a set S and a functor $P \colon S \to \mathbf{Set}$ where we consider S as a discrete category, i.e. P defines an S-indexed family of sets. We consider S to be a set of shapes, and for each $s \in S$, $P(s)$ is the set of positions associated with a shape $s \in S$. The *induced endofunctor on* $S \triangleleft P$ is $[S \triangleleft P] \colon \mathbf{Set} \to \mathbf{Set}$ where

- for a set X, $[S \triangleleft P]X = \{(s, l) \mid s \in S, l \colon P(s) \to X\}$; and
- for a function $g \colon X \to Y$, $[S \triangleleft P]g \colon [S \triangleleft P]X \to [S \triangleleft P]Y$ is defined as $(s, l) \mapsto (s, g \circ l)$.

We abuse terminology and say an endofunctor $F \colon \mathbf{Set} \to \mathbf{Set}$ is a container if $F \cong [S \triangleleft P]$ for some set S and functor $P \colon S \to \mathbf{Set}$.

Example II.7. *For a fixed set S, the product by S endofunctor $S \times (\cdot) \colon \mathbf{Set} \to \mathbf{Set}$ is a container such that for all $s \in S$, $P(s)$ is a singleton $\{\top\}$. Dually, for a set T, the exponentiation by T endofunctor $(\cdot)^T$ is a container such that $S = \{\bot\}$ is a singleton and $P(\bot) = T$.*

In fact, containers are equivalent to polynomial functors [30], i.e. $[S \triangleleft P] = \sum_{s \in S} (\cdot)^{P(s)}$.

Example II.8. *The list endofunctor L is a container where $S = \mathbb{N}$, $P(0) = \varnothing$, and for every positive $n \in S$, $P(n) = \{1, \dots, n\}$. The non-empty list endofunctor L^+ is defined similarly where $S = \mathbb{N}^+$ is the set of positive integers.*

Example II.9. *The rooted binary tree endofunctor $B \colon \mathbf{Set} \to \mathbf{Set}$ is a container where S is the set of full unlabelled binary trees and $P(s)$ is the set of internal nodes in tree $s \in S$. The rooted non-empty binary tree endofunctor $B^+ \colon \mathbf{Set} \to \mathbf{Set}$ is obtained removing the empty tree from S.*

Ahman, Chapman, and Uustalu, showed that comonads whose underlying endofunctor is a container are equivalent to containers which are equipped with additional 'directed' structure [6]. Intuitively, directed containers are containers where each position has an associated 'sub container' and every such 'sub container' has a root. Formally, a *directed container* consists of

- a container $F = [S \triangleleft P]$;
- for each shape $s \in S$, a root position $\mathsf{o}_s \in P(s)$;
- for each position $p \in P(s)$, a subshape $s \downarrow p \in S$;
- for each position $p' \in P(s \downarrow p)$ in the subshape $s \downarrow p$, a translation into a position $p \oplus_s p' \in P(s)$ in the global shape $s \in S$;

satisfying the equations

$$s \downarrow \mathsf{o}_s = s \quad s \downarrow (p \oplus_s p') = (s \downarrow p) \downarrow p' \tag{1}$$

$$p \oplus_s \mathsf{o}_{s \downarrow p} = p = \mathsf{o}_s \oplus_s p \tag{2}$$

$$(p \oplus_s p') \oplus_s p'' = p \oplus_s (p' \oplus_{s \downarrow p} p''). \tag{3}$$

A directed container $D = ([S \triangleleft P], \mathsf{o}, \downarrow, \oplus)$, has an *induced comonad* $([S \triangleleft P], \varepsilon, \delta)$ *of* D where:

- The counit has components defined as $\varepsilon_X(s, l \colon P(s) \to X) = l(\mathsf{o}_s)$
- The comultiplication has components defined as $\delta_X(s, l \colon P(s) \to X) = (s, \lambda p.(s \downarrow p, \lambda q.l(p \oplus_s q)))$

We will abuse terminology and say a comonad (W, ε, δ) is a directed container if it is isomorphic to the induced comonad of a directed container.

Example II.10. *Extending example II.7, the coreader comonad $(S \times (\cdot), \varepsilon, \delta)$ on a fixed set S has counit with components defined as $\varepsilon(s, x) = x$ and comultiplication with components defined as $\delta(s, x) = (s, (s, x))$. The identity comonad is recovered as the case where S is a singleton. The coreader comonad on set S is isomorphic to the induced comonad of a directed container on $[S \triangleleft P]$ where for every $s \in S$, $P(s)$ is the singleton set $\{\top\}$. As $P(s)$ is trivial for every $s \in S$, the directed container structure is such that $\mathsf{o}_s = \top$, $s \downarrow \top = s$, and $\top \oplus_s \top = \top$.*

Example II.11. *The cowriter comonad $((\cdot)^T, \varepsilon, \delta)$ on a fixed monoid $(T, e, *)$ has counit defined as $\varepsilon(f) = f(e)$ and comultiplication defined as $\delta(f) = \lambda m \lambda n.f(m * n)$. For a monoid $(T, e, *)$, the cowriter comonad is isomorphic to the induced comonad on the directed container $([\{\bot\} \triangleleft P], \mathsf{o}, \downarrow, \oplus)$ where $P(\bot) = T$, $\mathsf{o}_\bot = e$, for every $m \in M$, $\bot \downarrow m = \bot$, and $\oplus_\bot = *$.*

Example II.12. *The list container L from Example II.8 cannot be extended to a directed container as $P(0) = \varnothing$ and so there is no possible root $\mathsf{o}_0 \in P(0)$. However, the non-empty list container L^+ is commonly extended to a directed container in two non-isomorphic ways. The first way induces the prefix list comonad with counit defined as $\varepsilon([x_1, \dots, x_n]) = x_n$ and comultiplication defined as $\delta([x_1, \dots, x_n]) = [[x_1], [x_1, x_2], \dots, [x_1, \dots, x_n]]$.*

In this case, for every shape $n \in \mathbb{N}^+$, $\mathsf{o}_n = n$, for every position $i \in P(n) = \{1, \dots, n\}$, $n \downarrow i = i$, and for every $i \in P(n)$, $j \in P(n \downarrow i)$, $i \oplus_n j = j \in P(n)$. The suffix list comonad is isomorphic to the prefix list comonad via the 'reverse' natural transformation. The second non-isomorphic way induces the cyclic list comonad, see e.g. [37, Example E.24] for details.

Example II.13. *The non-empty binary tree container B^+ can be extended to a directed container [7]. This induces the comonad where the counit sends an X-labelled tree t to the label at t's root. The comultiplication sends an X-labelled tree t to the $B^+(X)$-labelled tree t' where node v of t' is replaced with the X-labelled subtree t_v of t rooted at node v.*

Example II.14. *For every comonad (W, ε, δ), the composed functor $W \circ S \times (\cdot)$ has a comonad structure. In the case of a set $S = [\mathsf{k}]$ for some $k \in \mathbb{N}$ and $W = L^+$ is the prefix list comonad, we obtain a pebble list comonad $(L^+[k], \varepsilon, \delta)$. This comonad has counit with components defined as $\varepsilon([(p_1, a_1), \ldots, (p_n, a_n)]) = a_n$ and comultiplication $\delta([(p_1, a_1), \ldots, (p_n, a_n)] = [(p_1, s_1), \ldots, (p_n, s_n)]$ where for all $i \in [\mathsf{n}]$, $s_i = [(p_1, a_1), \ldots, (p_i, a_i)]$. This comonad is the pebbling comonad of [4] over* $\mathbf{Set}$.

E. Distributive Laws

Definition II.15. *A mixed distributive law of comonad (W, ε, δ) over monad (M, η, μ) is a natural transformation $\kappa \colon WM \to MW$ satisfying the following diagrams:*

$$\begin{tikzcd} & W \arrow[dl, "W\eta"'] \arrow[dr, "\eta W"] & \\ WM \arrow[rr, "\kappa"'] & & MW \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} & WM \arrow[dl, "\kappa"'] \arrow[dr, "\varepsilon M"] & \\ MW \arrow[rr, "M\varepsilon"'] & & M \end{tikzcd} \tag{5}$$

$$\begin{tikzcd} WMM \arrow[r, "\kappa M"] \arrow[d, "W\mu"'] & MWM \arrow[r, "M\kappa"] & MMW \arrow[d, "\mu W"] \\ WM \arrow[rr, "\kappa"'] & & MW \end{tikzcd} \tag{6}$$

$$\begin{tikzcd} WM \arrow[rr, "\kappa"] \arrow[d, "\delta M"] & & MW \arrow[d, "M\delta"'] \\ WWM \arrow[r, "W\kappa"] & WMW \arrow[r, "\kappa W"] & MWW \end{tikzcd} \tag{7}$$

We call (4), (5), (6), (7) the unit, counit, multiplication, and comultiplication axioms respectively. Note that a given natural transformation can satisfy each axiom independently of the others. Thus, one can consider relaxations of the above definition where some of the diagrams are excluded. Many of these relaxations have been studied in existing literature, e.g. see [27]. One relaxation that will be important in our work is the case where κ is only required to satisfy the unit axiom. We refer to this case as a *pointed law*.

Definition II.16. *A pointed law of a functor W over the pointed endofunctor (M, η) is a natural transformation $\kappa \colon WM \to MW$ satisfying* (4)*.*

Note that there is no requirement in this definition for W to have a comonad structure, we only require that it is an endofunctor and M does not need to have an associated multiplication operation. If we do require compatibility with the multiplication in (M, η, μ), we obtain the definition for a Kleisli law [27]. A *Kleisli law of an endofunctor W over the monad (M, η, μ)* is a natural transformation $\kappa \colon WM \to MW$ satisfying (4) and (6).

In fact, Kleisli laws can be defined more generally when one has two monads defined on different base categories together with a functor between these categories. We can think of such laws as a generalised notion of morphism between monads on different categories.

Definition II.17. *Let (M, η^M, μ^M), $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$ be monads defined on categories $\mathfrak{C}, \mathfrak{D}$, and let $U : \mathfrak{D} \to C$ be a functor. A Kleisli law is a natural transformation $\lambda : U\mathbb{M} \to MU$ satisfying the following axioms:*

$$\lambda \circ U\eta^{\mathbb{M}} = \eta^M U$$
$$\lambda \circ U\mu^{\mathbb{M}} = \mu^M U \circ M\lambda \circ \lambda\mathbb{M}$$

We also require the dual concept of a coKleisli law.

Definition II.18. *Let $(W, \varepsilon^W, \delta^W)$, $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$ be comonads defined on categories $\mathfrak{C}, \mathfrak{D}$, and let $U : \mathfrak{D} \to C$ be a functor. A coKleisli law is a natural transformation $\lambda : WU \to U\mathbb{W}$ satisfying the following axioms:*

$$U\varepsilon^{\mathbb{W}} \circ \lambda = \varepsilon^W U$$
$$U\delta^{\mathbb{W}} \circ \lambda = \lambda\mathbb{W} \circ W\lambda \circ \delta^W U$$

III. Containers over Powerset

In this section, we show that if a container $F \colon \mathbf{Set} \to \mathbf{Set}$ has a pointed law κ over $(\mathcal{P}, \eta)$, then κ sends elements in a container with non-empty sets at every position to the set of all possible ways to sample at each position. As a corollary, we prove that there exists a unique pointed law of a container F over $(\mathcal{P}^+, \eta^+)$. As every distributive law is also a pointed law, this theorem prunes the space of possible distributive laws, and is therefore an important building block for our no-go theorems.

Theorem III.1. *If $F = [S \triangleleft P]$ is a container and there exists a pointed law $\kappa \colon F\mathcal{P} \to \mathcal{P}F$, then for all sets X and elements $(s, l \colon P(s) \to \mathcal{P}^+(X)) \in F(\mathcal{P}^+(X)) \subseteq F(\mathcal{P}(X))$,*

$$\kappa_X(s, l) = \{(s, j \colon P(s) \to X) \mid \forall p \in P(s), j(p) \in l(p)\}. \tag{8}$$

The uniqueness theorem is proved in three stages using a 'Plotkin-style' argument. This style of argument involves, at each stage, chasing specific elements along naturality squares for cleverly chosen functions. We then draw conclusions either from the direct image or pre-image of the element under a component of the pointed law. The first two stages involve demonstrating that equation (8) holds for all elements (s, l) which satisfy the following pairwise disjoint condition:

(PD) For all $p, q \in P(s)$, if $p \neq q$, then $l(p) \cap l(q) = \varnothing$.

The first stage uses the unit axiom and involves chasing the κ-naturality square for a 'collapse' function c. In order to convey the intuition of this first stage, we first sketch the case where $F = L^+$. Consider $\mathbf{L} = [X_1, \ldots, X_n]$ in $L^+\mathcal{P}^+(X)$ For $X_i = \{x_i\}$ singletons, $\kappa_X([\{x_1\}, \ldots, \{x_n\}) = \{[x_1, \ldots, x_n]\}$ follows directly from the unit axiom. More generally, as each X_i is in $\mathcal{P}^+X$, we can choose some $y_i \in X_i$. We consider a "collapse the X_i" function $c \colon X \to X$ which maps every $x_i \in X$ to y_i and is the identity otherwise. c is indeed a total function, i.e. single-valued, by the (PD) condition. Chasing the κ-naturality square of c and utilizing the unit axiom allows us to conclude that $\kappa_X(\mathbf{L}) \subseteq \{[x_1, \ldots, x_n] \mid \forall i \in [\mathsf{n}], x_i \in c^{-1}(y_i) = X_i\}$. Intuitively, this argument generalises to containers as the way defined the collapse only depended on the set at each position.

Lemma III.2 (Collapse)**.** *Suppose F and κ satisfy the hypotheses of Theorem III.1, then for all sets X and $(s, l) \in F(\mathcal{P}^+(X))$ satisfying (PD), $\varnothing \neq \kappa_X(s, l) \subseteq \{(s, j) \mid \forall p \in P(s), j(p) \in l(p)\}$.*

Proof. The argument proceeds in two steps.

1) Unpacking the unit axiom, we show that equation (8) holds if (s, l) is such that for all $p \in P(s)$, there exists a $x_p \in X$, $l(p) =$

$\{x_p\}$. There is function $j = \lambda p.x_p \colon P(s) \to X$. Consider the singleton $\{(s, \lambda p.x_p)\} \in \mathcal{P}^+(F(X))$

$$\begin{aligned}
\{(s, \lambda p.x_p)\} &= \eta_{F(X)}(s, \lambda p.x_p) && \text{definition of } \eta \\
&= \kappa_X \circ F(\eta_X)(s, \lambda p.x_p) && \text{unit axiom} \\
&= \kappa_X(s, \eta_X \circ \lambda p.x_p) && F \text{ on morphisms} \\
&= \kappa_X(s, \lambda p.\eta_X(x_p)) && \text{composition} \\
&= \kappa_X(s, \lambda p.\{x_p\}) && \text{definition of } \eta \\
&= \kappa_X(s, l) && \text{definition of } l
\end{aligned}$$

2) For all $(s, l) \in F(\mathcal{P}^+(X))$ and $p \in P(s)$, since $l(p) \neq \varnothing$, there exists a $y_p \in l(p)$. We construct a 'collapse the $l(p)$' function $c \colon X \to X$. The definition of $c \colon X \to X$ is

$$c(x) = \begin{cases} y_p & \text{if } x \in l(p) \\ x & \text{otherwise} \end{cases}.$$

By the (PD) assumption, the function c is well-defined. Observe that $\mathcal{P}^+(c) \circ l = \lambda p.\{y_p\}$. Chasing (s, l) along the κ-naturality square of c, we obtain

$$\begin{tikzcd}
(s,l) \arrow[r, mapsto, "\kappa_X"] \arrow[d, mapsto, "F(\mathcal{P}^+(c))"'] & \kappa_X(s,l) \arrow[d, mapsto, "\mathcal{P}^+(F(c))"] \\
(s, \lambda p.\{y_p\}) \arrow[r, mapsto, "\kappa_X"] & \{(s, \lambda p.y_p)\}
\end{tikzcd}$$

where the bottom arrow follows from the first step. Since $\mathcal{P}^+(F(c))$ maps the set $\kappa_X(s, l)$ to the singleton $\{(s, \lambda p.y_p)\}$, we can make two observations. The first observation is that since $\mathcal{P}^+(F(c))(\kappa_X(s, l)) \neq \varnothing$, it must be the case that $\varnothing \neq \kappa_X(s, l)$. The second observation is that for every $(t, j) \in \kappa_X(s, l)$, $F(c)(t, j) = (s, \lambda p.y_p)$. Since the function $F(c)$ preserves shape, for every $(t, j) \in \kappa_X(s, l)$, $t = s$. Moreover, from the definition of $\mathcal{P}(F(c))$, we conclude that

$$\kappa_X(s, l) \subseteq \{(s, j) \mid \forall p \in P(s), j(p) \in c^{-1}(y_p) = l(p)\}.$$

□

This second stage involves chasing the κ-naturality square of a 'swap' bijection b to obtain the opposite inclusion. In the case where $F = L^+$ and $\mathbf{L} = [X_1, \ldots, X_n] \in F(\mathcal{P}^+(X))$ satisfying (PD), by Lemma III.2, there must exist $[z_1, \ldots, z_n] \in \kappa_X(\mathbf{L})$ with $z_i \in X_i$. For every $[x_1, \ldots, x_n]$ with $x_i \in X_i$, consider the bijection $b \colon X \to X$ which for each i swaps z_i and x_i. Chasing the κ-naturality square of b and utilizing (PD) allows us to conclude that $\kappa_X(\mathbf{L}) \supseteq \{[x_1, \ldots, x_n] \mid \forall i \in [\mathsf{n}], x_i \in X_i\}$.

Lemma III.3 (Swap)**.** *Suppose F and κ satisfy the hypotheses of Theorem III.1, then for all sets X and $(s, l) \in F(\mathcal{P}^+(X))$ satisfying (PD), $\kappa_X(s, l) \supseteq \{(s, j) \mid \forall p \in P(s), j(p) \in l(p)\}$.*

Proof. For all $(s, l) \in F(\mathcal{P}^+(X))$ satisfying condition (PD), we construct a 'swap' bijection $b \colon X \to X$. Suppose $(s, j) \in F(X)$ such that $j(p) \in l(p)$. We need to show that $(s, j) \in \kappa_X(s, l)$. By the previous case and the fact that $\kappa_X(s, l) \in \mathcal{P}^+(F(X))$ is a non-empty subset of $F(X)$, we know that there exists at least one $(s, j') \in \kappa_X(s, l)$ such that $j'(p) \in l(p)$. For every $p \in P(s)$, let $s_p \colon X \to X$ be the permutation which swaps $j'(p) \in l(p)$ and $j(p) \in l(p)$ while fixing every other element of X. As s_p fixes the set $l(p)$ and, by the (PD) condition, leaves $l(q)$ unchanged for all $q \neq p \in P(s)$, we can conclude that $\mathcal{P}^+(s_p) \circ l = l$ and $(s, l) = F(\mathcal{P}^+(s_p))(s, l)$. Hence, by the κ-naturality square of s_p, we have that $\mathcal{P}^+(F(s_p))(\kappa_X(s, l)) = \kappa_X(s, l)$. Therefore, we can compose all these swapping bijections (in any order) $\{s_p\}_{p \in P(s)}$ to obtain a bijection $b \colon X \to X$ such that $\mathcal{P}^+(F(b))(\kappa_X(s, l)) = \kappa_X(s, l)$ and for all $p \in P(s)$, $b(j'(p)) = j(p)$. This means that:

$$\begin{aligned}
(s, j') \in \kappa_X(s, l) &\Rightarrow F(b)(s, j') \in \kappa_X(s, l) \\
&\Rightarrow (s, b \circ j') \in \kappa_X(s, l) \\
&\Rightarrow (s, j) \in \kappa_X(s, l)
\end{aligned}$$

Since $(s, j') \in \kappa_X(s, l)$, we can conclude that $(s, j) \in \kappa_X(s, l)$ as desired. □

The final stage of the proof of Theorem III.1 is a 'relabel' argument which demonstrates that the condition (PD) does not constitute a loss of generality. In the case where $F = L^+$ and $\mathbf{L} = [X_1, \ldots, X_n] \in F(\mathcal{P}^+(X))$ (the X_i are not necessarily pairwise-disjoint), we first consider the set $Y = [\mathsf{n}] \times X$. There is a list $\mathbf{L}' = [X'_1, \ldots, X'_n] \in F(\mathcal{P}(Y))$ where $X'_i = \{(i, x_i) \mid x_i \in X_i\} \in \mathcal{P}(Y)$ satisfying (PD). By construction, $F(\mathcal{P}(\pi_2))(\mathbf{L}') = \mathbf{L}$ where $\pi_2 \colon Y \to X$ is the projection onto the second component. Since $\mathbf{L}'$ satisfies (PD), we can use Lemma III.2, Lemma III.3 and the naturality square of π_2 to compute that

$$\begin{aligned}
\kappa_X(\mathbf{L}) &= \kappa_X(F\mathcal{P}\pi_2(\mathbf{L}')) \\
&= \mathcal{P}F\pi_2(\kappa_Y(\mathbf{L}')) \\
&= \mathcal{P}F\pi_2(\{[(1, x_1), \ldots, (n, x_n)] \mid \forall i \in [\mathsf{n}], x_i \in X_i\}) \\
&= \{[x_1, \ldots, x_n] \mid \forall i \in [\mathsf{n}], x_i \in X_i\}.
\end{aligned}$$

Generalising this argument to arbitrary container completes the proof of Theorem III.1.

Proof. Suppose $(s, l \colon P(s) \to \mathcal{P}^+(X)) \in F(\mathcal{P}^+(X))$ and let $Y = P(s) \times X$. We factor l as $l = \mathcal{P}^+(t) \circ z$ where $z \colon P(s) \to \mathcal{P}^+(Y)$ is defined as $z(p) = \{(p, x) \mid x \in l(p)\}$ and $t \colon Y \to X$ is the projection onto the second component. By construction (s, z) satisfies the (PD) condition, so applying Lemma III.2 and Lemma III.3

$$\begin{aligned}
\kappa_Y(s, z) &= \{(s, m) \mid \forall p \in P(s), m(p) \in z(p)\} \\
&= \{(s, m) \mid \forall p \in P(s) \exists x \in l(p), m(p) = (p, x)\}
\end{aligned}$$

Since every $(s, j \colon P(s) \to X)$ such that $j(p) \in l(p)$ can be factored as $j = t \circ m$ for $(s, m) \in \kappa_Y(s, z)$ with $m(p) = (p, j(p))$, we obtain

$$\kappa_X(s, l) = \{(s, j) \mid \forall p \in P(s), j(p) \in l(p)\}$$

as desired. □

It is easy to check that $\kappa^+ \colon F\mathcal{P}^+ \to \mathcal{P}^+F$ with components having the same elementwise definition as κ in Equation (8) satisfies diagram (4) yielding the following consequence from the proof of Theorem III.1.

Corollary III.4. *For every container $F = [S \triangleleft P]$, there exists a unique pointed law $\kappa^+ \colon F\mathcal{P}^+ \to \mathcal{P}^+F$ of F over $(\mathcal{P}^+, \eta^+)$, where $\kappa^+(s, l) = \kappa(s, l)$ defined in* (8)*.*

Remark III.5. *Equation* (8) *is sometimes referred to as the Jacobs law [9], though its definition appears in Barr [8]. For every weak-pullback preserving functor $T \colon \mathbf{Set} \to \mathbf{Set}$, the Jacobs law determines the unique 'monotone' Kleisli law $\kappa \colon T\mathcal{P}^+ \to \mathcal{P}^+T$ [26], [41], [11]. As containers are weak-pullback preserving, it might ostensibly appear that Corollary III.4 is a consequence of this fact. However by restricting to containers, Corollary III.4 strengthens this consequence by showing that κ^+ is the unique law of F over $(\mathcal{P}^+, \eta^+, \mu^+)$* sine conditione *rather than merely the unique* monotone *law. There are weak-pullback preserving functors T which are not containers, such* *as the powerset endofunctor, for which the Jacobs law is not the only Kleisli law of T over monad $(\mathcal{P}^+, \eta^+, \mu^+)$, e.g. see [18, Example 2.14].*

Since it may be difficult to parse the equation (8) for an arbitrary container, the following examples provide additional intuition for how this transformation works.

Example III.6. *Given a set S, for the product endofunctor $F = S \times (\cdot)$ of example II.7, the pointed laws $\kappa\colon F\mathcal{P} \to \mathcal{P}F$ and $\kappa^+\colon F\mathcal{P}^+ \to \mathcal{P}^+F$ have components satisfying $\kappa_X(s, Y) = \kappa^+_X(s, Y) = \{(s, y) \mid y \in Y \subseteq X\}$ for all $Y \neq \varnothing$.*

Example III.7. *For the infinite streams endofunctor $L^\infty = (-)^{\mathbb{N}}$, a special case of example II.7, the pointed law $\kappa\colon L^\infty\mathcal{P} \to \mathcal{P}L^\infty$ has components satisfying $\kappa_X((X_1, X_2, \dots)) = \{(x_1, x_2, \dots) \mid x_i \in X_i\}$ for all streams $(X_1, X_2, \dots)$ such that every $X_i \neq \varnothing$.*

Example III.8. *For the non-empty list container of example II.8, the pointed law $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$ has components satisfying $\kappa_X([X_1, \dots, X_n]) = \{[x_1, \dots, x_n] \mid x_i \in X_i\}$ for all lists $[X_1, \dots, X_n]$ such that every $X_i \neq \varnothing$.*

As each of these example illustrates, the action of κ on a container with non-empty sets is to output the set of all containers which sample an element from each position. This allows us to easily compute the (possibly infinite) cardinality of the subset $\kappa(s, l) \in \mathcal{P}(F(X))$.

Lemma III.9. *If $F = [S \triangleleft P]$ is a container and $\kappa\colon F\mathcal{P} \to \mathcal{P}F$ satisfies equation (8) for $(s, l) \in F\mathcal{P}^+(X) \subseteq F\mathcal{P}(X)$, then $|\kappa_X(s, l)| = \prod_{p \in P(s)} |l(p)|$.*

Proof. Follows from a simple counting argument. For each position $p \in P(s)$, sample an element from $l(p)$. Each sampling is independent. $\square$

Remark III.10. *The proof of Theorem III.1 also applies to pointed laws $\kappa\colon F\mathcal{P}_f \to \mathcal{P}_fF$ and $\kappa\colon F\mathcal{P}^+_f \to \mathcal{P}^+_fF$ of containers F over finite powerset $(\mathcal{P}_f, \eta)$ and finite non-empty powerset $(\mathcal{P}^+_f, \eta)$, respectively. However, unlike with full non-empty powerset, the analogue of Corollary III.4 for $(\mathcal{P}^+_f, \eta)$ does not always hold. For containers $[S \triangleleft P]$ where there exists an $s \in S$ such that $P(s)$ is infinite, a pointed law $\kappa\colon F\mathcal{P}^+_f \to \mathcal{P}^+_fF$ satisfying equation (8) does not exist as the set $\kappa_X(s, l)$ would be necessarily infinite.*

IV. Directed Containers over Powerset

In this section, we investigate, given a directed container (W, ε, δ), when does the pointed law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ of W over $(\mathcal{P}, \eta)$ extend to a distributive law of (W, ε, δ) over $(\mathcal{P}, \eta, \mu)$.

For instance, the pointed law $\kappa^+\colon S\times\mathcal{P}^+(\cdot) \to \mathcal{P}^+(S \times (\cdot))$ described in example III.6 does extend to a comonad-monad distributive law of the coreader comonad of example II.10 over the non-empty powerset monad. Moreover, it follows from the counit axiom (5) of Definition II.15 that any element $(s, \varnothing) \in S\times\mathcal{P}(X)$ must be mapped by a distributive law $\kappa\colon S\times\mathcal{P}(\cdot) \to \mathcal{P}(S \times (\cdot))$ to $\varnothing \in \mathcal{P}(S\times X)$. By Theorem III.1, it must be the case that κ is equal to κ^+ for elements (s, Y) with $Y \neq \varnothing$, obtaining a uniqueness result.

Proposition IV.1. *For a fixed set S, the coreader comonad $W = (S \times (\cdot), \varepsilon, \delta)$ has a unique distributive law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ over the powerset monad $(\mathcal{P}, \eta, \mu)$ and a unique distributive law $\kappa^+\colon W\mathcal{P}^+ \to \mathcal{P}^+W$ over the non-empty powerset monad $(\mathcal{P}^+, \eta^+, \mu^+)$.*

This distributive law appears as [40, Example 7.6]. However, as we will see at the end of this section, the coreader comonad is the only directed container where a distributive law over the powerset monad exists. To illustrate the issue, we first show that the pointed law $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$ in Theorem III.1 does not extend to a comonad-monad distributive law of the the prefix list comonad $(L^+, \varepsilon, \delta)$ over the powerset monad $(\mathcal{P}, \eta, \mu)$.

Theorem IV.2. *There is no distributive law $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$ of $(L^+, \varepsilon, \delta)$ over $(\mathcal{P}, \eta, \mu)$.*

Proof. Suppose for contradiction there exists a distributive law $\kappa\colon L^+\mathcal{P} \to \mathcal{P}L^+$. As κ must satisfy the unit axiom (4), κ is a pointed law of L^+ over $(\mathcal{P}, \eta)$. Theorem III.1 implies that for lists which contain only non-empty sets, the components of κ satisfy equation (8).

Considering the list $\mathbf{L} = [\{a, b\}, \{b\}] \in L^+(\mathcal{P}(X))$ for $X = \{a, b\}$, we obtain the following inequality contradicting the comultiplication axiom:

$$\begin{aligned}\mathcal{P}\delta_X \circ \kappa_X(\mathbf{L}) &= \{[[a], [a, b]], [[b], [b, b]]\} \\ &\neq \{[[a], [a, b]], [[b], [a, b]], [[a], [b, b]], [[b], [b, b]]\} \\ &= \kappa_{L^+X} \circ L^+\kappa_X \circ \delta_{\mathcal{P}X}(\mathbf{L})\end{aligned}$$

$\square$

Interestingly, chasing the list $\mathbf{L}' = [\{b\}, \{a, b\}]$ rather than $\mathbf{L}$ would have shown that $\mathcal{P}\delta_X \circ \kappa_X(\mathbf{L}') = \kappa_{L^+X} \circ L^+\kappa_X \circ \delta_{\mathcal{P}X}(\mathbf{L}')$. The contrast between these two cases is because the set $\{a, b\}$ is placed in a root position in $\mathbf{L}'$ whereas $\{a, b\}$ is in a non-root position in $\mathbf{L}$. The existence of a non-root position in a shape $s \in S$ of a directed container is the key property that forbids the pointed law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ satisfying equation (8) from extending to a distributive law of (W, ε, δ) over $(\mathcal{P}, \eta, \mu)$.

We use $\mathcal{C}_W$ to denote the class of directed containers (W, ε, δ) with $W = [S \triangleleft P]$ such that there exists an $s \in S$ with $\lvert P(s)\rvert \geq 2$.

Since every distributive law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$ must satisfy the unit axiom, then by Theorem III.1, for elements $(s, l) \in W\mathcal{P}^+(X) \subseteq W\mathcal{P}(X)$, κ satisfies equation (8). However, a simple diagram chase of κ applied to a specific $(s, l) \in W\mathcal{P}^+(X)$ for $W \in \mathcal{C}_W$ demonstrates that κ cannot satisfy diagram (7). Let $(\wp, \eta, \mu)$ be either either the powerset monad $(\mathcal{P}, \eta, \mu)$, non-empty powerset monad $(\mathcal{P}^+, \eta^+, \mu^+)$, finite powerset monad $(\mathcal{P}_f, \eta, \mu)$, or finite non-empty powerset monad $(\mathcal{P}^+_f, \eta^+, \mu^+)$.

Theorem IV.3. *If $(W, \varepsilon, \delta) \in \mathcal{C}_W$, then it has no distributive law $\kappa\colon W\wp \to \wp W$ over $(\wp, \eta, \mu)$.*

Proof. Suppose for contradiction, there exists a distributive law $\kappa\colon W\mathcal{P} \to \mathcal{P}W$. By Theorem III.1, κ must satisfy (8). We show that κ satisfying equation (8) does not satisfy the comultiplication axiom (7).

If $|P(s)| \geq 2$, then there exists a position $v \in P(s)$ such that $v \neq \mathsf{o}_s$. Consider a set X with cardinality of 2 and $(s, l) \in W(\mathcal{P}(X))$ where $l(v) = X \in \mathcal{P}(X)$ and for all other $q \neq v \in P(s)$, $l(q)$ is a singleton. We have that

$$|\mathcal{P}(\delta_X) \circ \kappa_X(s, l)| = |\kappa_X(s, l)| = \prod_{p \in P(s)} |l(p)| = |l(v)| = 2$$

where the first equality follows from δ being injective and second equality from Lemma III.9. We know that δ is injective since the axioms of a comonad imply that δ is a split monomorphism and monomorphisms in **Set** are injective functions.

By contrast, consider $(s, m) = W(\kappa_X) \circ \delta_{\mathcal{P}(X)}(s, l)$. From the definition of δ and Lemma III.9, we have that $|m(p)| = \prod_{q \in P(s \downarrow p)} |l(p \oplus_s q)|$. Applying Lemma III.9 again, we obtain that

$$\begin{aligned}|\kappa_{W(X)} \circ W(\kappa_X) \circ \delta_{\mathcal{P}(X)}(s, l)| &= \prod_{p \in P(s)} |m(p)| \\ &= \prod_{p \in P(s)} \prod_{q \in P(s \downarrow p)} |l(p \oplus_s q)| \\ &\geq |l(v \oplus_s \mathsf{o}_{s\downarrow v})||l(\mathsf{o}_s \oplus_s v)| \\ &= |l(v)||l(v)| = 4\end{aligned}$$

where the substitutions $v = v \oplus_s \mathsf{o}_{s\downarrow v} = \mathsf{o}_s \oplus_s v$ are equation (2). By our supposition, κ satisfies the comultiplication axiom, so

$$2 = |\mathcal{P}(\delta_X) \circ \kappa_X(s, l)| = |\kappa_{W(X)} \circ W(\kappa_X) \circ \delta_{\mathcal{P}(X)}(s, l)| \geq 4.$$

Contradiction, $2 \not\geq 4$. The proofs for the non-empty powerset monad, finite powerset monad, and finite non-empty powerset monad are similar. □

It is worth noting that the above proof only involves the unit and comultiplication axioms of a distributive law. Thus, we have actually proven as stronger statement.

Theorem IV.4. *If $(W, \varepsilon, \delta) \in \mathcal{C}_W$, then there is no natural transformation $\kappa \colon W\wp \to \wp W$ which simultaneously satisfies the unit and comultiplication axioms.*

If a directed container W is not in $\mathcal{C}_W$, then by the existence of a root $\mathsf{o}_s \in P(s)$ for every $s \in S$, we have that $|P(s)|$ is non-empty and $|P(s)| = 1$. However, directed containers which satisfy this condition are isomorphic to the coreader comonads described in Example II.10. This allows us to phrase Theorem IV.3 positively and characterise coreader comonads in terms of distributive laws.

Theorem IV.5. *Let (W, ε, δ) be a directed container with $W = [S \triangleleft P]$.*

$W = S \times (\cdot)$ is the coreader comonad on S if and only if there exists a distributive law $\kappa \colon W\mathcal{P}^+ \to \mathcal{P}^+W$.

Proof. $\Rightarrow$ Suppose $W = S \times (\cdot)$ is the coreader comonad. Let $\kappa \colon W\mathcal{P}^+ \to \mathcal{P}^+W$ be the natural transformation with components defined as

$$\kappa_X(s, Y) := \{(s, t) \mid t \in Y\}$$

for every $Y \in \mathcal{P}^+(X)$ and $s \in S$. It is easy to check that κ satisfies the diagrams (4),(5),(6),(7).

$\Leftarrow$ Conversely, suppose that there exists a distributive law $\kappa \colon W\mathcal{P}^+ \to \mathcal{P}^+W$. By Theorem IV.3, $W \notin \mathcal{C}_W$. Negating the definition of the class $\mathcal{C}_W$, for all $s \in S$, $|P(s)| < 2$. By W being a directed container, for every $s \in S$, there is root position $\mathsf{o}_s \in P(s)$, so $P(s)$ is non-empty. Therefore, $|P(s)| = 1$, so we can take $P(s) = \{\top\}$. This is precisely the definition of the coreader comonad given in Example II.10. □

V. Directed Containers over Uniform Choice Monads

In this section, we widen the scope of Theorem IV.3 by showing no distributive law of the form $\rho \colon WM \to MW$ exists for any comonad $W \in \mathcal{C}_W$ and any monad, in fact pointed endofunctor, $M \colon \mathbf{Set} \to \mathbf{Set}$ which has meaningful notion of 'uniform distribution of size ≥ 2'. In order to formally define this class of monads M, we take inspiration from the equational presentations of monads which arise from universal algebra. From this perspective, we define, given an endofunctor $M \colon \mathbf{Set} \to \mathbf{Set}$, a *$n$-ary term for M* as a natural transformation $\beta \colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to M$ where the domain of β is the endofunctor on $\mathbf{Set}$ mapping a set X to its n-th power $X^n = X \times \cdots \times X$. Beyond this algebraic portion of the definition, we also need to restrict to monads M which have a meaningful notion of support, i.e. there exists a natural transformation $\mathbf{supp} \colon M \to \mathcal{P}$. With these notions in place, we can now define what it means for any pointed endofunctor (M, η) to have a 'uniform distribution'.

Definition V.1. *Given a pointed endofunctor (M, η) with a natural transformation $\mathbf{supp} \colon M \to \mathcal{P}$, a n-ary term $\beta \colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to M$ for M is a n-*uniform choice term *if*

1) β is idempotent: *For all $X \in \mathbf{Set}$ and $x \in X$,*
$$\beta(x, \ldots, x) = \eta(x);$$
2) β is commutative: *For all $X \in \mathbf{Set}$, $x_1, \ldots, x_n \in X$, and permutations $\pi \colon [\mathsf{n}] \to [\mathsf{n}]$,*
$$\beta(x_1, \ldots, x_n) = \beta(x_{\pi(1)}, \ldots, x_{\pi(n)});$$
3) $\mathbf{supp}$ preserves β: *For all $X \in \mathbf{Set}$ and $x_1, \ldots, x_n \in X$,*
$$\mathbf{supp}(\beta(x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}.$$

We will say (M, η) is a *n-uniform choice pointed endofunctor* if there exists a natural transformstion $\mathbf{supp} \colon M \to \mathcal{P}$ and a n-uniform choice term β for (M, η). We will say a monad (M, η, μ) is a *n-uniform choice monad* if (M, η) is a n-uniform choice pointed endofunctor. Since the the powerset monad $(\mathcal{P}, \eta, \mu)$ has support $\mathbf{id}_{\mathcal{P}} \colon \mathcal{P} \to \mathcal{P}$ and a n-uniform choice term $\beta^{\mathcal{P}}(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ for every $n > 0$, the terminology for condition 3 in Definition V.1 is justified. In fact, for every n-uniform choice monad M, it follows that $\mathbf{supp} \colon M \to \mathcal{P}$ is a pointed endofunctor morphism:

$$\mathbf{supp}(\eta(x)) = \mathbf{supp}(\beta(x, \ldots, x)) = \{x\} = \eta^{\mathcal{P}}(x). \tag{9}$$

Moreover, every pointed endofunctor (M, η) such that the natural transformation $\mathbf{supp} \colon M \to \mathcal{P}$ is a pointed endofunctor morphism is a 1-uniform choice endofunctor where $\beta = \eta$.

For a n-uniform choice monad (M, η, μ) with n-uniform choice term β and $X \in \mathbf{Set}_0$, we define the set of uniform terms as

$$U_\beta(X) := \{\beta(x_1, \ldots, x_n) \mid \forall x_1, \ldots, x_n \in X\} \subseteq M(X).$$

The set $U_\beta(X)$ generalises the set of uniform distributions on n elements. To illustrate, take M to be the discrete probability distribution monad and define β as

$$\beta(x_1, \ldots, x_n) = \sum_{i \in [\mathsf{n}]} \frac{1}{n} x_i.$$

In this case, the set $U_\beta(X)$ is precisely the uniform distributions on subsets of X with cardinality n.

We proceed by first proving a generalisation of Theorem III.1, demonstrating that every pointed law $\rho \colon FM \to MF$ of a satisfies an analogue of equation (8) on supports when restricted to uniform distributions.

Theorem V.2. *If $F = [S \triangleleft P]$ is a container, (M, η, μ) is a n-uniform choice monad, and there exists a pointed law $\rho \colon FM \to MF$, then for all sets X and elements $(s, l \colon P(s) \to U_\beta(X)) \in F(U_\beta(X)) \subseteq F(M(X))$,*

$$\begin{aligned}&\mathbf{supp}(\rho_X(s, l)) \\ &= \{(s, j \colon P(s) \to X) \mid \forall p \in P(s), j(p) \in \mathbf{supp}(l(p))\}.\end{aligned} \tag{10}$$

From Theorem V.2, we can conclude an analogue of Lemma III.9.

Lemma V.3. *If $F = [S \lhd P]$ is a container and $\rho\colon FM \to MF$ satisfies equation (10) for $(s,l) \in F(U_\beta(X)) \subseteq F(M(X))$, then* $|\mathbf{supp}(\rho(s,l))| = \prod_{p\in P(s)} |\mathbf{supp}(l(p))|$.

Proof. Follows from a simple counting argument. For each position $p \in P(s)$, sample an element from $\mathbf{supp}(l(p))$. Each sampling is independent. $\square$

The proof of Theorem IV.3 derived the contradiction $2 \geq 4$ by chasing element $(s,l) \in W(\mathcal{P}^+(X))$ where $|l(v)| = 2$ for some $v \neq \mathsf{o}_s \in P(s)$ around diagram (7) using κ from equation (8). Inspired by this argument, we chase an element $(s,l) \in W(U_{\mathscr{S}}(X)) \subseteq W(\mathcal{D}_{\mathscr{S}}(X))$ where $l(v)$ is a uniform distribution for a $v \neq \mathsf{o}_s \in P(s)$ such that $|\mathbf{supp}(l(v))| \geq 2$. We define our class $\mathcal{C}_M$ to be those distribution monads that allow us to build this counterexample.

We use $\mathcal{C}_M$ to denote the class of n-uniform choice monads (M, η^M, μ^M) where $n \geq 2$.

Theorem V.4. *If $(W,\varepsilon,\delta) \in \mathcal{C}_W$ and $(M,\eta,\mu) \in \mathcal{C}_M$, then there is no distributive law $\rho\colon WM \to MW$ of (W,ε,δ) over (M,η,μ).*

Proof. Suppose for contradiction, there exists a distributive law $\rho\colon WM \to MW$. By $M \in \mathcal{C}_M$, M has a n-uniform choice term β for $n \geq 2$. Consider the set $X = \{x_1,\ldots,x_n,x\} \in \mathbf{Set}_0$. By Theorem V.2, ρ_X must satisfy (10) for every $(s,l) \in W(U_\beta(X))$. We show that ρ satisfying equation (10) does not satisfy the comultiplication axiom (7).

As $W \in \mathcal{C}_W$, we have that $|P(s)| \geq 2$. Hence, there exists a position $v \in P(s)$ such that $v \neq \mathsf{o}_s$. Let $l(v) = \beta(x_1,\ldots,x_n)$ and for all $q \neq v \in P(s)$, $l(q) = \eta(x)$. By construction, $|\mathbf{supp}(l(v))| = n$ and for all $q \neq v \in P(s)$, $|\mathbf{supp}(l(q))| = 1$.

$$
\begin{aligned}
&|\mathbf{supp} \circ M(\delta) \circ \rho(s,l)| \\
&= |\mathcal{P}(\delta) \circ \mathbf{supp} \circ \rho(s,l)| && \mathbf{supp}\text{-naturality of } \delta_X \\
&= |\mathbf{supp}(\rho(s,l))| && \delta_X \text{ injective} \\
&= \prod_{p\in P(s)} |\mathbf{supp}(l(p))| && \text{Lemma V.3} \\
&= |\mathbf{supp}(l(v))| = n && |\mathbf{supp}(l(q)| = 1 \text{ for } q \neq v
\end{aligned}
$$

By contrast, consider $(s,m) = W(\rho_X) \circ \delta_{M(X)}(s,l)$ and note by the functorial action of W,

$$(s, \mathbf{supp} \circ m) = W(\mathbf{supp}_X \circ \rho_X) \circ \delta_{M(X)}(s,l).$$

From the definition of δ and Lemma V.3, we have that $|\mathbf{supp}(m(p))| = \prod_{q\in P(s\downarrow p)} |\mathbf{supp}(l(p \oplus_s q))|$. Applying Lemma V.3 again, we obtain that

$$
\begin{aligned}
&|\mathbf{supp} \circ \rho_{W(X)} \circ W(\rho) \circ \delta_{M(X)}(s,l)| \\
&= \prod_{p\in P(s)} |\mathbf{supp}(m(p))| = \prod_{p\in P(s)} \prod_{q\in P(s\downarrow p)} |\mathbf{supp}(l(p \oplus_s q))| \\
&\geq |\mathbf{supp}(l(v \oplus_s \mathsf{o}_{s\downarrow v}))||\mathbf{supp}(l(\mathsf{o}_s \oplus_s v))| \\
&= |\mathbf{supp}(l(v))||\mathbf{supp}(l(v))| = n^2
\end{aligned}
$$

where the substitutions $v = v \oplus_s \mathsf{o}_{s\downarrow v} = \mathsf{o}_s \oplus_s v$ are equation (2). By our supposition, ρ satisfies the comultiplication axiom, so

$$M(\delta) \circ \rho(s,l) = \rho \circ W(\rho) \circ \delta_{M(X)}(s,l)$$

Composing with the support map on both sides of this equation, we obtain that the cardinality of $\mathbf{supp} \circ M(\delta_X) \circ \rho_X(s,l)$ is equal to the cardinality of $\mathbf{supp} \circ \rho_{W(X)} \circ W(\rho_X) \circ \delta_{M(X)}(s,l)$. However,

$$|\mathbf{supp} \circ M(\delta_X) \circ \rho_X(s,l)| = n; \text{ and}$$
$$|\mathbf{supp} \circ \rho_{W(X)} \circ W(\rho_X) \circ \delta_{M(X)}(s,l)| \geq n^2.$$

Contradiction, $n \geq 2$ implies that $n \ngeq n^2$. $\square$

Once again, we have actually proven a stronger result.

Theorem V.5. *If $(W,\varepsilon,\delta) \in \mathcal{C}_W$ and $(M,\eta,\mu) \in \mathcal{C}_M$, then there is no natural transformation $\rho\colon WM \to MW$ which simultaneously satisfies the unit and comultiplication axioms.*

To illustrate the generality and limitations of Theorem V.4, we recall the following examples from the discussion after Definition V.1.

Example V.6. *The ordinary discrete probability distribution monad $\mathcal{D}$ of Example II.2 is in $\mathcal{C}_M$. This monad has a natural transformation $\mathbf{supp}\colon \mathcal{D} \to \mathcal{P}$ which maps a probability distribution to its underlying support, i.e. $\mathbf{supp}(\varphi) = \{x \mid \varphi(x) \neq 0\}$, and a 2-uniform choice term $\beta\colon \mathbf{Id_{Set}} \times \mathbf{Id_{Set}} \to \mathcal{D}$ defined as*

$$\beta(x_1,x_2) = \frac{1}{2}x_1 + \frac{1}{2}x_2.$$

Example V.7. *Every variation of the powerset $\wp$ monad, e.g. full, finite, non-empty, has a transformation $\mathbf{supp}\colon \wp \to \mathcal{P}$ given by the inclusion into the full powerset functor. Moreover, for every $n > 0$, $\wp$ has a n-uniform choice term $\beta\colon \mathbf{Id_{Set}} \times \cdots \times \mathbf{Id_{Set}} \to \wp$ defined as*

$$\beta(x_1,\ldots,x_n) = \{x_1,\ldots,x_n\}.$$

Thus, we recover Theorem IV.3 as an application of Theorem V.4.

Both these examples are part of wider class of distribution and multiset monads over semirings $\mathscr{S}$ which fall under the scope of Theorem V.4. In particular, Example V.6 is the case where $\mathscr{S}$ is the semiring of non-negative reals $(\mathbb{R}_{\geq 0}, 0, 1, +, *)$ and Example V.7 is the case where $\mathscr{S}$ is the Boolean semiring $(\{0,1\}, 0, 1, \vee, \wedge)$.

We enumerate sufficient conditions for when a distribution $\mathcal{D}_{\mathscr{S}}$ and multiset monad $\mathcal{M}_{\mathscr{S}}$ over a semiring $\mathscr{S}$ is in the class of monads $\mathcal{C}_M$ To state these conditions, recall that the initial object in the category of semirings is the semiring of natural numbers. Therefore, for every semiring $\mathscr{S}$, there is a unique semiring morphism $\top^{\mathscr{S}}\colon \mathbb{N} \to \mathscr{S}$ where $\top^{\mathscr{S}}(n)$ is mapped to the n-fold sum of $1_{\mathscr{S}}$.

(S1) *$\mathscr{S}$ is zero-sumfree*: If $a + b = 0_{\mathscr{S}}$, then $a = 0_{\mathscr{S}}$ and $b = 0_{\mathscr{S}}$.

(S2) *$\mathscr{S}$ has a natural non-trivial unit $n_{\mathscr{S}}$:* There exists some $n \geq 2$ such that $n_{\mathscr{S}} = \top^{\mathscr{S}}(n)$ is a unit. i.e. there exists $t \in \mathscr{S}$ such that $n_{\mathscr{S}} t = t n_{\mathscr{S}} = 1_{\mathscr{S}}$. If such a $t \in \mathscr{S}$ exists, then t is unique and so we can denote t as $\frac{1}{n_{\mathscr{S}}}$.

We prove the following lemmas which connect these conditions on $\mathscr{S}$ to the Definition V.1 of n-uniform choice monad.

Lemma V.8. *Let $M = \mathcal{D}_{\mathscr{S}}$ or $M = \mathcal{M}_{\mathscr{S}}$ for some semiring $\mathscr{S}$. $\mathscr{S}$ is zero-sumfree, i.e. satisfies Condition (S1) if and only if $\mathbf{supp}^M$ is a natural transformation.*

Lemma V.9. *Let $M = \mathcal{D}_{\mathscr{S}}$ or $M = \mathcal{M}_{\mathscr{S}}$. If $\mathscr{S}$ has a natural non-trivial unit $n_{\mathscr{S}}$, i.e. satisfies Condition (S2), then $\beta\colon \mathbf{Id_{Set}} \times \cdots \times \mathbf{Id_{Set}} \to M$ defined as:*

$$\beta(x_1,\ldots,x_n) = \sum_{i\in[n]} \frac{1}{n_{\mathscr{S}}} x_i$$

is a n-ary idempotent and commutative open term. In particular, if $\mathscr{S}$ is zero-sumfree, then β is a n-uniform choice term.

Theorem V.10. *If $\mathscr{S}$ satisfies conditions (S1)- (S2), then $\mathcal{D}_{\mathscr{S}} \in \mathcal{C}_M$ and $\mathcal{M}_{\mathscr{S}} \in \mathcal{C}_M$. If $\mathscr{S}$ is also complete, then $\mathcal{D}^{\infty}_{\mathscr{S}} \in \mathcal{C}_M$ and $\mathcal{M}^{\infty}_{\mathscr{S}} \in \mathcal{C}_M$.*

As a corollary, we obtain an instance of the V.4 and for multiset and distribution monads.

Corollary V.11. *If $(W, \varepsilon, \delta) \in \mathcal{C}_W$ and $\mathscr{S}$ satisfies conditions (S1)-(S2), then there is no distributive law $\rho\colon WM \to MW$ where $M = \mathcal{D}_{\mathscr{S}}$ or $M = \mathcal{M}_{\mathscr{S}}$.*

Example V.12. *The multiset $\mathcal{M}_{\mathscr{S}}$ and distribution $\mathcal{D}_{\mathscr{S}}$ monads over the (sub)-semiring of non-negative rationals of the form $\frac{n}{3^k}$ for $n, k \in \mathbb{N}$ are in $\mathcal{C}_M$. In this case, since 2 is not a unit, there are no uniform distributions $\varphi \in \mathcal{D}_{\mathscr{S}}(X)$ such that $|\mathbf{supp}(\varphi)| = 2$ However, for every $k \in \mathbb{N}$, 3^k is a unit and so $\mathcal{D}_{\mathscr{S}}(X)$ has uniform distributions φ such that $|\mathbf{supp}(\varphi)| = 3^k$.*

Example V.13 (Non-Example)**.** *For every ring R, the free R-module monad $\mathcal{M}_R$ and distribution monad $\mathcal{D}_R$ of Example II.3 are not zero-sum-free and therefore not in $\mathcal{C}_M$.*

Example V.14 (Non-Example)**.** *There are semirings which are zero-sum-free, but fail to satisfy the condition (S2). For instance, the semiring of naturals $(\mathbb{N}, 0, 1, +, *)$ does not satisfy condition (S2). Therefore, although $\mathcal{C}_M$ has many multiset monads over other semirings $\mathscr{S}$, $\mathcal{C}_M$ does not contain the ordinary multiset/bag monad $\mathcal{M} = \mathcal{M}_{\mathbb{N}}$. Moreover, $\mathcal{D}_{\mathbb{N}}$ over the semiring $(\mathbb{N}, 0, 1, +, *)$ of natural numbers only has distributions which are singletons and is, in fact, isomorphic to the identity monad.*

Example V.15 (Non-Example)**.** *For a example of a zero-sum-free semiring $\mathscr{S}$ which does not satisfy condition (S2), but where $\mathcal{D}_{\mathscr{S}}$ is not the identity monad, consider $\mathscr{S} = \mathbb{N}[x, y]/(x + y = 1)$. This semiring is the quotient of the free commutative semiring on the set $\{x, y\}$ by the equation $x + y = 1_{\mathscr{S}}$. The additional equation $x + y = 1_{\mathscr{S}}$ ensures that $\mathcal{D}_{\mathscr{S}}$ is not the identity monad by allowing for non-singleton distributions, i.e. $ax + by \in \mathcal{D}_{\mathscr{S}}(\{a, b\})$. However, this is the only non-singleton distribution in $\mathcal{D}_{\mathscr{S}}(\{a, b\})$ and neither x nor y are inverses to $\top^{\mathscr{S}}(n)$ for some $n \geq 2$. Thus, $\mathscr{S}$ does not satisfy condition (S2) and $\mathcal{D}_{\mathscr{S}} \notin \mathcal{C}_M$.*

VI. No-go Theorems in $\mathcal{R}(\sigma)$

In this section we extend our results to the category of relational structures, $\mathcal{R}(\sigma)$. In order to achieve this, we prove a two part *transfer theorem* which dictates conditions under which the existence of a mixed distributive law in $\mathcal{R}(\sigma)$ implies the existence of a mixed distributive law in $\mathbf{Set}$.

We recall that a relational vocabulary σ is a finite set of relation symbols R, each with a specified positive integer arity. A σ-structure $\mathscr{A}$ is given by a set A, the universe of the structure, and for each R in σ with arity n, a relation $R^{\mathscr{A}} \subseteq A^n$. A homomorphism $h\colon \mathscr{A} \to \mathscr{B}$ is a function $h\colon A \to B$ such that, for each relation symbol R of arity n in σ, for all $a_1, \ldots, a_n$ in A: $R^{\mathscr{A}}(a_1, \ldots, a_n) \Rightarrow R^{\mathscr{B}}(h(a_1), \ldots, h(a_n))$. We write $\mathcal{R}(\sigma)$ for the category of σ-structures and homomorphisms. For ease of presentation, we focus on the case where σ contains only a single relation R.

The class of comonads we are interested in are the game comonads whose study aims to apply ideas from category theory to (finite) model theory [2].

Example VI.1. *The Ehrenfeucht-Fraïssé comonad $(\mathbb{E}_k, \varepsilon, \delta)$ sends a relational structure $\mathscr{A}$ to a new structure $\mathbb{E}_k\mathscr{A}$ with universe $\mathbb{E}_k A = L^+_k A$. The components of the counit and comultiplication have the same elementwise definition as the prefix list comonad i.e. $\varepsilon^{\mathbb{E}_k}_{\mathscr{A}} = \varepsilon^{L^+_k}_A, \delta^{\mathbb{E}_k}_{\mathscr{A}} = \delta^{L^+_k}_A$. $R^{\mathbb{E}_k\mathscr{A}}$ is the set of tuples $(l_1, \ldots, l_n)$ which are pairwise comparable by prefix order, and such that $R^{\mathscr{A}}(\varepsilon^{\mathbb{E}_k}_{\mathscr{A}} l_1, \ldots, \varepsilon^{\mathbb{E}_k}_{\mathscr{A}} l_n)$.*

Example VI.2. *The pebbling comonad $(\mathbb{P}_k, \varepsilon, \delta)$ sends a relational structure $\mathscr{A}$ to a new structure $\mathbb{P}_k(\mathscr{A})$ with universe $L^+[k](A)$. Counit and comultiplication have the same elementwise definition as the pebble list comonad. $R^{\mathbb{P}_k\mathscr{A}}$ is the set of tuples $(l_1, \ldots, l_n)$ such that (1) The l_i are pairwise comparable by prefix order, (2) The pebble index of the last move in each l_i does not appear in the suffix of l_i in l_j for any l_j, and (3) $R^{\mathscr{A}}(\varepsilon^{\mathbb{P}_k}_{\mathscr{A}} l_1, \ldots, \varepsilon^{\mathbb{P}_k}_{\mathscr{A}} l_n)$.*

Note that if we consider the action of $\mathbb{P}_k$ on the underlying universe of relational structures, we recover the pebble list comonad of Example II.14.

The class of monads we consider are those whose actions on the universe is dictated by one of the monads $\mathcal{C}_M$ defined earlier.

Example VI.3. *We refer to the monad $(\mathbb{T}, \eta, \mu)$ taken from [36, Chapter 9] as the tree duality monad since $\mathbb{T}$-algebras correspond to tree dual structures. The functor $\mathbb{T}$ sends a relational structure $\mathscr{A}$ to a new structure $\mathbb{T}\mathscr{A}$ with universe $\mathcal{P}A$. Counit and comultiplication have the same elementwise definition as the powerset monad, i.e. $\eta^{\mathbb{T}}_{\mathscr{A}} = \eta^{\mathcal{P}}_A, \mu^{\mathbb{T}}_{\mathscr{A}} = \mu^{\mathcal{P}}_A$. $R^{\mathbb{T}\mathscr{A}}$ is the set of tuples of subsets $(X_1, ..., X_n)$ such that for all $j \in [n]$ and for every choice $x_j \in X_j$ there exists $x_k \in X_k$ for all $k \in [n]\backslash\{j\}$ such that $(x_1, ..., x_n) \in R^{\mathcal{A}}$.*

Our next example is a monad whose Kleisli morphisms capture the basic linear programming relaxation of the constraint satisfaction problem [13].

Example VI.4. *The BLP monad $(\mathbb{D}, \eta, \mu)$ has functor $\mathbb{D}$ which sends a relational structure $\mathscr{A}$ to a new structure $\mathbb{D}\mathscr{A}$ with universe $\mathcal{D}A$. Counit and comultiplication have the same elementwise definition as the distribution monad, i.e. $\eta^{\mathbb{D}}_{\mathscr{A}} = \eta^{\mathcal{D}}_A, \mu^{\mathbb{D}}_{\mathscr{A}} = \mu^{\mathcal{D}}_A$. Finally we have $R^{\mathbb{D}\mathscr{A}}$ is interpreted as the set $\{(\sum_{\boldsymbol{a} \in R^{\mathscr{A}}} \gamma(\boldsymbol{a}).\boldsymbol{a}[1] \ldots, \sum_{\boldsymbol{a} \in R^{\mathscr{A}}} \gamma(\boldsymbol{a}).\boldsymbol{a}[m]) \mid \gamma\colon R^{\mathscr{A}} \to [0, 1], \sum_{\boldsymbol{a} \in R^{\mathscr{A}}} \gamma(\boldsymbol{a}) = 1\}$*

For the remainder of this section let us assume we are working in the following setup:

1) There exists categories $\mathfrak{C}, \mathfrak{D}$ with a coreflective adjunction $L : \mathfrak{C} \to \mathfrak{D} \dashv U : \mathfrak{D} \to C$ between them. We write α, β for the unit and counit of this adjunction.
2) $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$, $(W, \varepsilon^{W}, \delta^{W})$ are comonads over $\mathfrak{D}, \mathfrak{C}$ respectively.
3) $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$, (M, η^{M}, μ^{M}) are monads over $\mathfrak{D}, \mathfrak{C}$ respectively.

We are now ready to prove our transfer theorems.

Theorem VI.5. *Assume the following:*

1) There exists a coKleisli law $w : WU \to U\mathbb{W}$.
2) There exists a Kleisli law $m : U\mathbb{M} \to MU$.
3) $\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ and $\rho'\colon WM \to MW$ are natural transformations satisfying the following "Yang-Baxter" condition:

$$Mw \circ \rho' U \circ Wm = m\mathbb{W} \circ U\rho \circ w\mathbb{M}$$

Then we have:

1) If ρ is a distributive law, m is epic, and w is monic, ρ' is a distributive law.
2) If ρ' is a distributive law, m is monic, and w is epic, ρ is a distributive law.

Remark VI.6. *This theorem can be seen as a generalised comonad-monad variant of [33, Theorem 3.1.3], which considers transfer* *theorems for monad-monad distributive laws defined on the same category. We can recover the theorem of [33] by simply considering the case where the functor U above is the identity.*

Theorem VI.7. *Assume the following:*

1) *There exists a split epic natural transformation $w : WU \to U\mathbb{W}$. We write w^- for the section of w.*
2) *There exists a split monic natural transformation $m : U\mathbb{M} \to MU$. We write m^- for the retraction of m.*
3) *$\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ is a natural transformation.*

Then, ρ together with the natural transformation $\rho' : WM \to MW$ defined as $\rho' = MW\alpha^{-1} \circ Mw^- L \circ m\mathbb{W}L \circ U\rho L \circ w\mathbb{M}L \circ Wm^- L \circ WM\alpha$, satisfy the following "Yang-Baxter" equation:

$$Mw \circ \rho' U \circ Wm = m\mathbb{W} \circ U\rho \circ w\mathbb{M}$$

A proof of both theorems using string diagrammatic techniques akin to [24] is provided in the appendix.

Let us now consider what happens when we enforce the condition $\mathfrak{C} = \mathbf{Set}$. We show that we can combine the two theorems above with our earlier results to obtain sufficient conditions under which no distributive law exists between a class of comonads $\mathcal{C}_{\mathbb{W}}(\mathfrak{D})$ and a class of monads $\mathcal{C}_{\mathbb{M}}(\mathfrak{D})$ defined on $\mathfrak{D}$.

$\mathcal{C}_{\mathbb{W}}(\mathfrak{D})$ denotes the class of comonads $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$ on $\mathfrak{D}$ with an isomorphic coKleisli law $w : WU \to U\mathbb{W}$ for some $W \in \mathcal{C}_W$.

$\mathcal{C}_{\mathbb{M}}(\mathfrak{D})$ denotes the class of monads $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$ on $\mathfrak{D}$ with an isomorphic Kleisli law $m : U\mathbb{M} \to MU$ for some $M \in \mathcal{C}_M$.

Theorem VI.8. *If $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}}) \in \mathcal{C}_{\mathbb{W}}\mathfrak{D}$ and $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}}) \in \mathcal{C}_{\mathbb{M}}\mathfrak{D}$, then there is no distributive law $\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ of (W, ε, δ) over (M, η, μ).*

Proof. Assume for the sake of contradiction that a distributive law ρ does exist. We can use Theorem VI.7 to construct a natural transformation $\rho'\colon WM \to MW$ which satisfies the Yang-Baxter condition. It follows from Theorem VI.5 that ρ is a distributive law of (W, ε, δ) over (M, η, μ). However, we know by Theorem V.4 that no such law exists. ☐

Note that it is necessary for w and m to be isomorphisms in order for the above argument to work. This is because any monic and split epic morphism must be an isomorphism (similarly for any epic and split monic morphism).

As before, our techniques have actually proven a stronger statement.

Theorem VI.9. *If $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}}) \in \mathcal{C}_{\mathbb{W}}\mathfrak{D}$ and $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}}) \in \mathcal{C}_{\mathbb{M}}\mathfrak{D}$, then there is no natural transformation $\rho\colon \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ which simultaneously satisfies the unit and comultiplication axioms.*

The above result is the most general no-go theorem in this paper. Our earlier no-go results are all covered by the special case where $\mathfrak{D} = \mathbf{Set}$. By varying $\mathfrak{D}$ we can observe the non-existence of distributive laws between further comonads, as exemplified below.

Example VI.10. *Let us consider the case where $\mathfrak{D} = \mathcal{R}(\sigma)$. Here there exists a forgetful functor $U\colon \mathcal{R}(\sigma) \to \mathbf{Set}$ which sends a relational structure to its underlying universe. This functor has a left adjoint $L\colon \mathbf{Set} \to \mathcal{R}(\sigma)$ which sends a set to a relational structure with empty relations. The adjunction $L \dashv U$ is colelective. All of the comonads over $\mathcal{R}(\sigma)$ that we introduced earlier in this section belong to the class $\mathcal{C}_{\mathbb{W}}(\mathcal{R}(\sigma))$. Likewise, all of the monads belong to the class $\mathcal{C}_{\mathbb{M}}(\mathcal{R}(\sigma))$. To see this in the case of $(\mathbb{P}_k, \varepsilon, \delta)$ for instance, consider the pebble list comonad $(L^+[k], \varepsilon, \delta) \in \mathcal{C}_W$, together with the natural transformation $w\colon U\mathbb{P}_k \to L^+[k]U$ whose components, as set functions, are the identity. It is easy to verify that w is an isomorphic coKleisli law.*

VII. Conclusions and Future Work

We have proven that there can be no comonad-monad distributive law $WM \to MW$ for a suitable class of comonads $\mathcal{C}_W \subseteq \mathbf{Com}(\mathbf{Set})$ and a suitable class of monads $\mathcal{C}_M \subseteq \mathbf{Mon}(\mathbf{Set})$. The class of comonads $\mathcal{C}_W$ is all directed containers except the coreader comonad. The class of monads $\mathcal{C}_M$ is any monad which admits a sensible notion of "uniform distribution". We then proved a transfer theorem which allowed us to extend our results to classes of (co)monads defined over any category $\mathfrak{D}$ which admits $\mathbf{Set}$ as a coreflective subcategory. This transfer theorem may be of independent interest to researchers working on (co)monad theory. Overall, our results show the non-existence of mixed distributive laws between a large number of (co)monads used in areas such as probability theory, programming languages, and finite model theory. As such, we hope that they will be of relevance to researchers working in these areas.

There are many avenues for future work. We list some of them below:

- Axiomatic account: [45] determined axioms for when two algebraic theories do not yield a composite algebraic theory. Since algebraic theories correspond to finitary monads over $\mathbf{Set}$, the axioms provide a framework for determining the non-existence of monad-monad distributive laws. Do similar (co)alegebraic axioms exist for determining the non-existence of comonad-monad distributive laws? This would involve formulating and working with coalgebraic presentations of directed containers and algebraic presentations of monads.
- Extension to other categories: We believe that the transfer theorems derived in section VI have applications beyond what we have considered in this paper. Take for instance the Vietoris monad $\mathcal{V}$, and Radon monad $\mathcal{R}$ [18] defined on the category of compact Hausdorff spaces. These can be seen as topological analogues of $\mathcal{P}$, and $\mathcal{D}_{\mathbb{R}_{\geq 0}}$ respectively. Can we use a monad-monad variant of our transfer theorem to prove a no-go theorem between these monads? Such a result would make use of the well-known fact that there is no distributive law $\lambda\colon \mathcal{P}\mathcal{D}_{\mathbb{R}_{\geq 0}} \to \mathcal{D}_{\mathbb{R}_{\geq 0}}\mathcal{P}$ [45]. Another interesting direction to pursue is the extension of our comonad-monad no-go results to other categories. For instance, quasi-Borel spaces $\mathbf{QBS}$ introduced in [22] have recently gained much attention in the context of probabilistic programming. The monad of probability measures on $\mathbf{QBS}$ generalises the well-known Giry monad [17], and acts as an analogue of $\mathcal{D}_{\mathscr{S}}$ over this category. Comonads in $\mathbf{QBS}$ are not well-studied, thus it would be interesting to see if any of the directed containers we have considered admit analogues in $\mathbf{QBS}$ and whether our no-go theorems extend to this category.
- *Weak Distributive Laws*: There is substantive literature [16], [19], [20], [18] showing that one can recover many of the desirable properties of a composite monad by constructing a natural transformation which only satisfies a subset of the monad-monad distributive law axioms. In the case of comonad-monad laws we ask if similar relaxations can be used to recover features of a biKleisli category.
- Monad-Comonad Distributive Laws: Unlike the comonad-monad case we can construct valid monad-comonad distributive laws for many of the (co)monads we considered. For instance, even though there is no distributive law of the form $\kappa\colon L^\infty\mathcal{P}^+ \to \mathcal{P}^+L^\infty$,

it is easy to check that $\gamma\colon \mathcal{P}^+ L^\infty \to L^\infty \mathcal{P}^+$ as defined below is a distributive law.

$$\begin{aligned}\gamma_X\{(a_1,a_2,\ldots),(b_1,b_2,\ldots),\ldots,(z_1,z_2,\ldots)\}\\ =(\{a_1,b_1,\ldots,z_1\},\{a_2,b_2,\ldots,z_2\},\ldots)\end{aligned}$$

Such laws give rise to categories of bialgebras, a construction which has applications in mathematics and computer science (see e.g. [32], [28]). The following is an open question:

Question VII.1. *For which $M \in \mathcal{C}_M$ and $W \in \mathcal{C}_W$ does there exist a distributive law $\gamma\colon MW \to WM$?*

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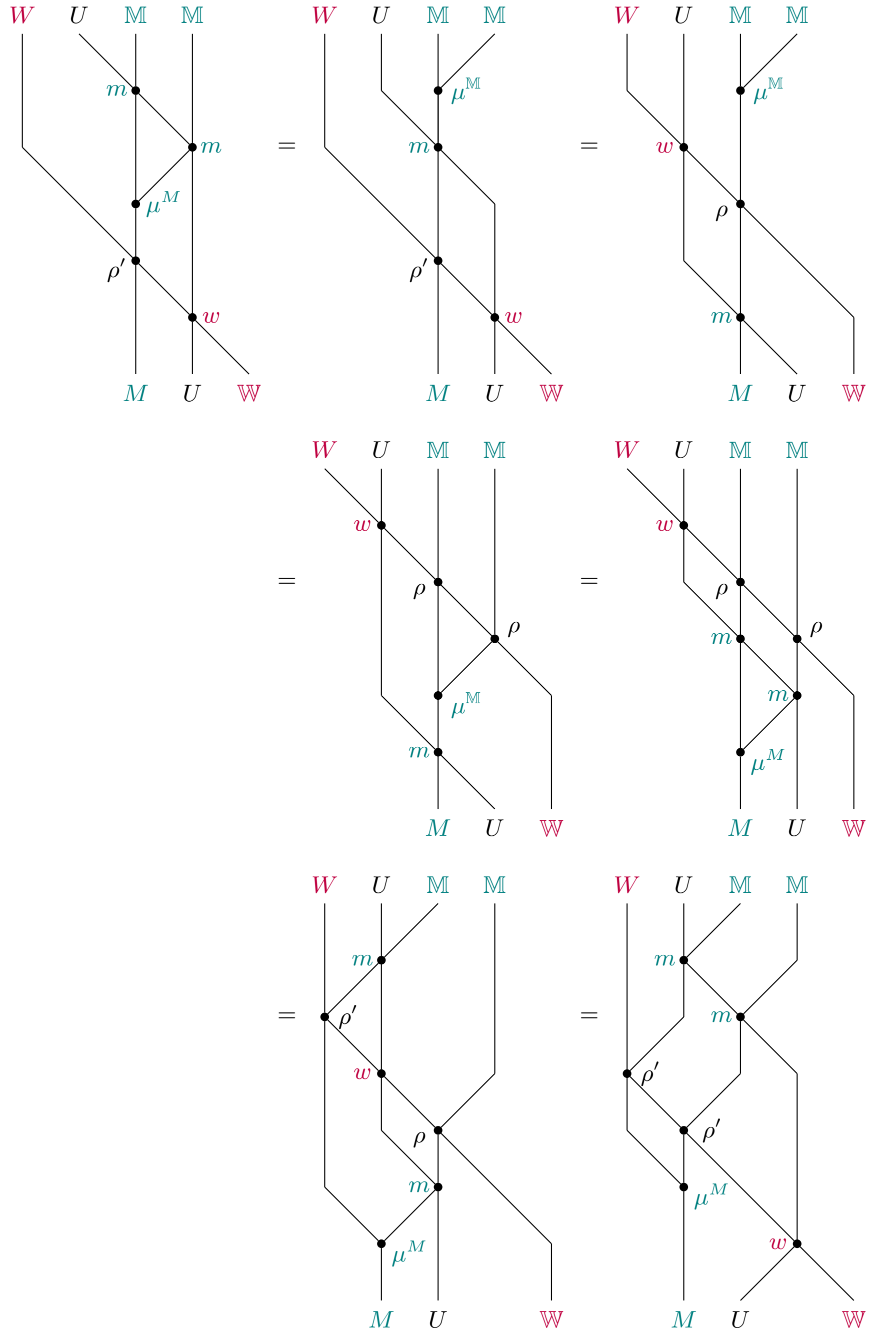
A. *Proofs of Transfer Theorems*

In this appendix we shall prove theorems VI.5 and VI.7. We will make use of string diagrams in these proofs. We shall assume basic familiarity with string diagrams, at the level of [25]. We begin with a proof of theorem VI.5:

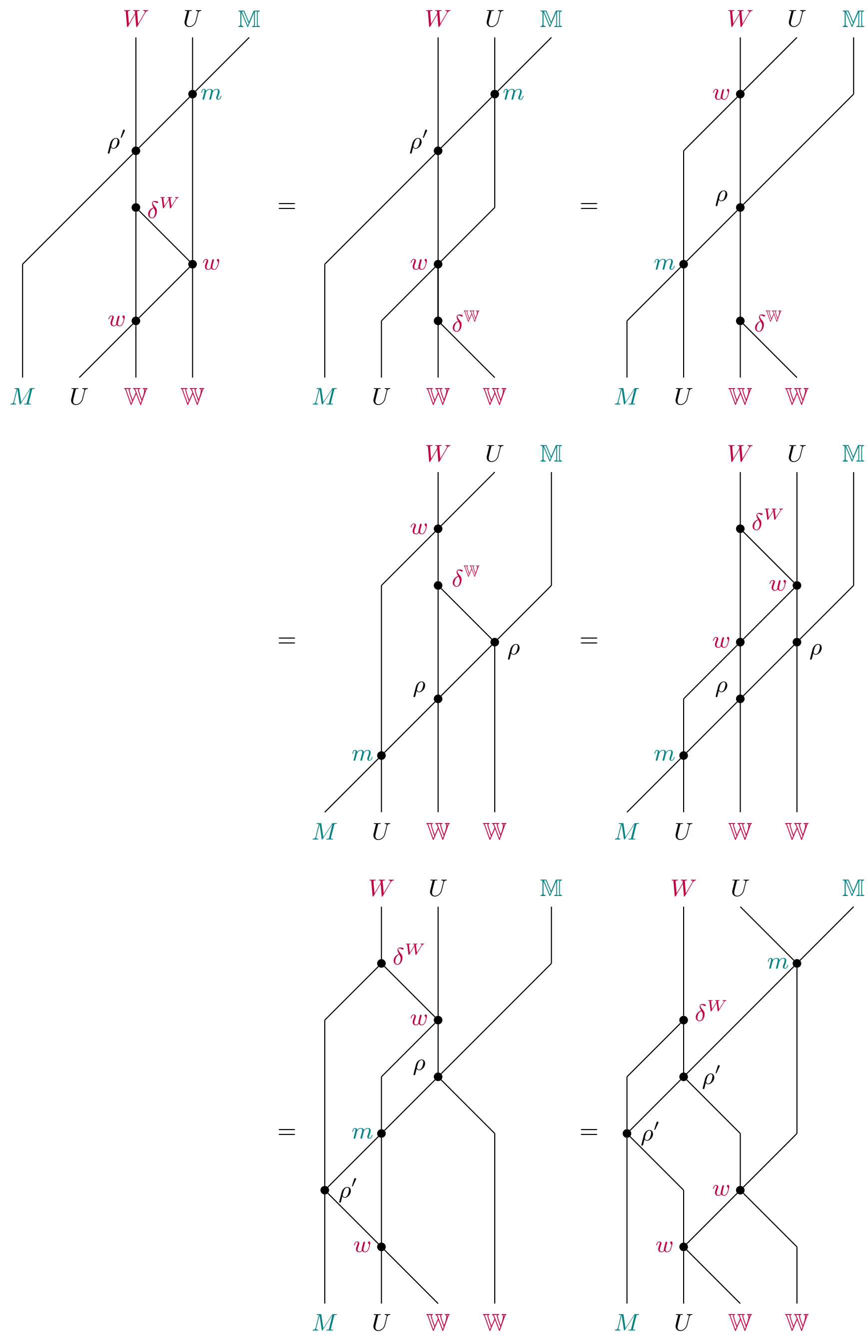
Proof. We prove the (co)multiplication, and (co)unit axioms for the first part of the theorem. The proof for the second part the theorem is analogous.

- Multiplication:



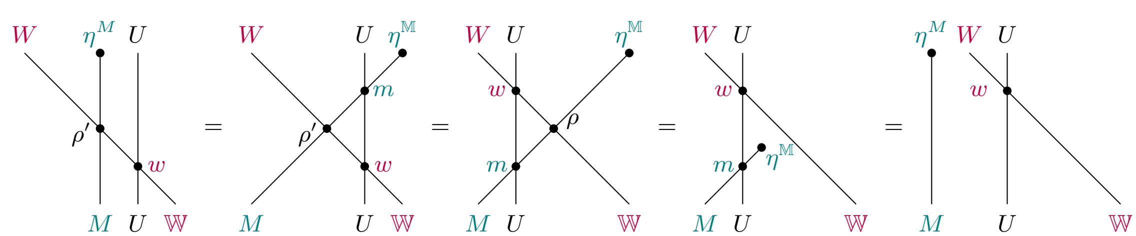
The result follows from m being epic and w being monic.

- Comultiplication:



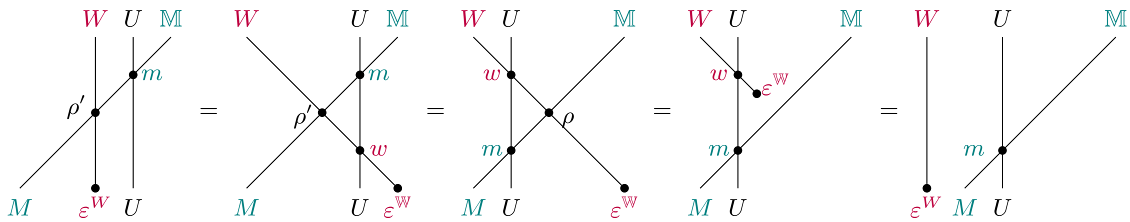
The result follows from m being epic and w being monic.

- Unit:



The result follows from w being monic.

- Counit:

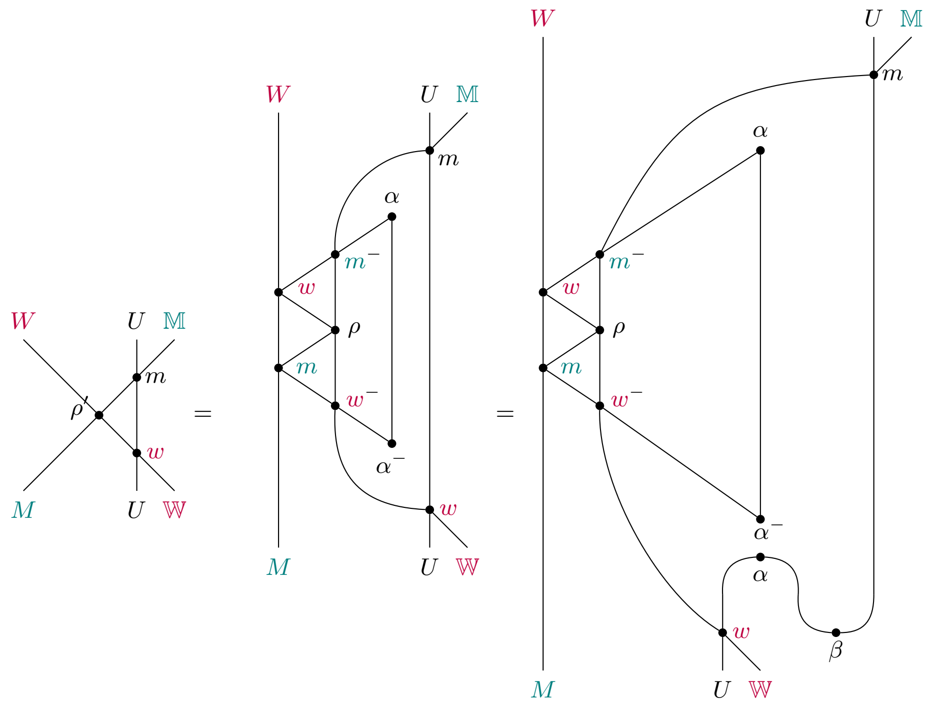


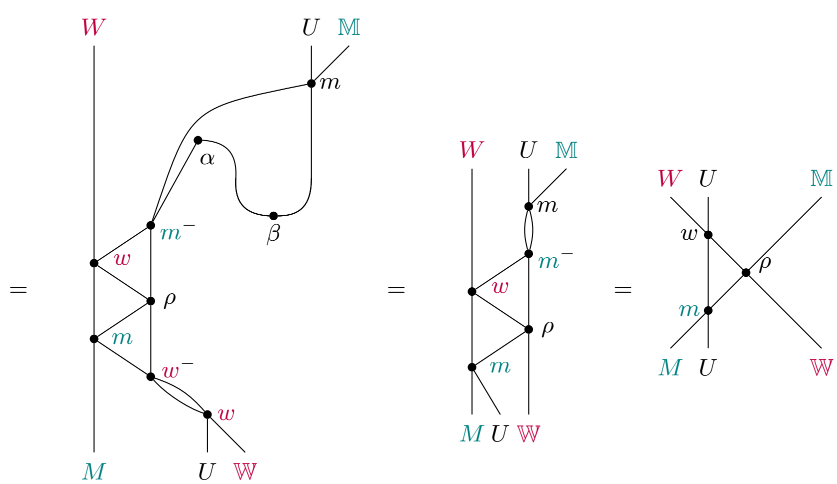
The result follows from m being epic.

□

Next we prove theorem VI.7.

Proof.





□

APPENDIX

PROOFS FOR SECTION V (DIRECTED CONTAINERS OVER UNIFORM CHOICE MONADS)

Theorem V.2. *If $F = [S \triangleleft P]$ is a container, (M, η, μ) is a n-uniform choice monad, and there exists a pointed law $\rho\colon FM \to MF$, then for all sets X and elements $(s, l\colon P(s) \to U_\beta(X)) \in F(U_\beta(X)) \subseteq F(M(X))$,*

$$\begin{aligned}&\mathbf{supp}(\rho_X(s,l))\\ &= \{(s, j\colon P(s) \to X) \mid \forall p \in P(s), j(p) \in \mathbf{supp}(l(p))\}.\end{aligned} \tag{10}$$

The proof mirrors collapse-swap-relabel argument of the uniqueness proof of Theorem III.1. Thus, we prove analogues of Lemmas III.2 and III.3. Condition 1 that β is idempotent is key to demonstrating the 'collapse' argument in Lemma A.1. Condition 2 that β is commutative is key to demonstrating that the 'swap' argument in Lemma A.2.

Similar to Theorem III.1, we first consider elements $(s, l) \in FM(X)$ where the following pairwise disjoint condition holds:

(PDS) For all $p, q \in P(s)$, if $\mathbf{supp}(l(p)) \cap \mathbf{supp}(l(q)) \neq \varnothing$, then $p = q$.

Lemma A.1 (Collapse)**.** *If F, M, and ρ satisfy the hypotheses of Theorem V.2 and $(s, l) \in FU(X) \subseteq FM(X)$ satisfies condition (PDS), then*

$$\mathbf{supp}(\rho(s,l)) \subseteq \{(s, j) \mid \forall p \in P(s) \in \mathbf{supp}(l(p))\}.$$

Proof. The argument proceeds in two steps.

1) Unpacking the unit axiom, we will show that equation (10) holds if (s, l) is such that for all $p \in P(s)$, there exists a $x_p \in X$, $l(p) = \eta(x_p)$. There is function $j = \lambda p.x_p\colon P(s) \to X$. Consider the singleton $\{(s, \lambda p.x_p)\} \in \mathcal{P}(F(X))$. For clarity of notation, we use η for the unit of M and $\eta^{\mathcal{P}}$ for the unit of $\mathcal{P}$.

$$\begin{aligned}
\{(s, \lambda p.x_p)\} &= \eta^{\mathcal{P}}(s, \lambda p.x_p) && \text{definition of } \eta^{\mathcal{P}}\\
&= \mathbf{supp}(\eta(s, \lambda p.x_p)) && \text{equation (9)}\\
&= \mathbf{supp}(\rho(F(\eta)(s, \lambda p.x_p))) && \text{unit axiom}\\
&= \mathbf{supp}(\rho(s, \lambda p.\eta(x_p))) && F \text{ functorial action}\\
&= \mathbf{supp}(\rho(s, l)) && \text{definition of } l
\end{aligned}$$

From the above equation and $\{(s, \lambda p.x_p)\} = \{(s, j) \mid \forall p \in P(s), j(p) \in \mathbf{supp}(l(p))\}$, we obtain that the desired result.

2) For all $(s, l) \in F(U(X))$ and $p \in P(s)$, by the definition of $U(X)$, $l(p) = \beta(x_1^p, \dots, x_n^p)$ for some $x_1^p, \dots, x_n^p \in X$, and $\mathbf{supp}(l(p)) = \{x_1^p, \dots, x_n^p\} \neq \varnothing$. Hence, for every $p \in P(s)$, we can choose a $y^p = x_1^p \in \mathbf{supp}(l(p))$ in order to construct a 'collapse the $\mathbf{supp}(l(p))$' function $c\colon X \to X$. The definition of $c\colon X \to X$ is

$$c(x) = \begin{cases} y^p & \text{if } x \in \mathbf{supp}(l(p)) \\ x & \text{otherwise} \end{cases}.$$

By the (PDS) assumption, the function c is well-defined. Next, we prove that $M(c) \circ l = \lambda p.\eta(y^p)$. Suppose $p \in P(s)$:

$$\begin{aligned}
M(c) \circ l(p) &= M(c)(\beta(x_1^p, \dots, x_n^p)) && l(p) \in U(X)\\
&= \beta(c(x_1^p), \dots, c(x_n^p)) && \beta\text{-naturality square of } c\\
&= \beta(y^p, \dots, y^p) && \text{definition of } c\\
&= \eta(y^p) && \beta \text{ idempotent}
\end{aligned}$$

Chasing (s, l) along the ρ-naturality square of c composed with the $\mathbf{supp}$-naturality square of $F(c)$, we obtain

$$\begin{tikzcd}
(s,l) \arrow[r, mapsto, "\rho"] \arrow[d, mapsto, "F(M(c))"'] & \rho(s,l) \arrow[r, mapsto, "\mathbf{supp}"] \arrow[d, mapsto, "M(F(c))"] & \mathbf{supp}(\rho(s,l)) \arrow[d, mapsto, "\wp(F(c))"] \\
(s, \lambda p.\eta(y^p)) \arrow[r, mapsto, "\rho"] & \eta(s, \lambda p.y^p) \arrow[r, mapsto, "\mathbf{supp}"] & \{(s, \lambda p.y^p)\}
\end{tikzcd}$$

where the bottom left arrow follows from the first step. Since $\wp(F(c))$ maps the set $\mathbf{supp}(\rho(s,l))$ to the singleton $\{(s, \lambda p.y_p)\}$, we can conclude that for every $(t, j) \in \mathbf{supp}(\rho(s,l))$, $F(c)(t, j) = (s, \lambda p.y^p)$. Since the function $F(c)$ preserves shape, for every $(t, j) \in \mathbf{supp}(\rho(s,l))$, $t = s$. Moreover, from the definition of $\wp(F(c))$, we conclude that

$$\mathbf{supp}(\rho(s,l)) \subseteq \{(s, j) \mid \forall p \in P(s), j(p) \in c^{-1}(y_p) = \mathbf{supp}(l(p))\}.$$

$\square$

Lemma A.2 (Swap)**.** *If F, M, and ρ satisfy the hypotheses of Theorem V.2 and $(s, l) \in FU(X) \subseteq FM(X)$ satisfies condition (PDS), then*

$$\mathbf{supp}(\rho(s,l)) \supseteq \{(s, j) \mid \forall p \in P(s) \in \mathbf{supp}(l(p))\}.$$

Proof. For all $(s, l) \in F(U(X))$ satisfying condition (PDS), we construct a 'swap' bijection $b\colon X \to X$. By the assumption that $(s, l) \in F(U(X))$, for all $p \in P(s)$, there exists $x_1^p, \dots, x_n^p \in X$ such that $l(p) = \beta(x_1^p, \dots, x_n^p)$ and $\mathbf{supp}(l(p)) = \{x_1^p, \dots, x_n^p\}$. Suppose $(s, j) \in F(X)$ such that $j(p) = x_z^p \in \mathbf{supp}(l(p))$, we need to show that $(s, j) \in \mathbf{supp}(\rho(s,l))$. By the previous case and the fact that $\mathbf{supp}(\rho(s,l)) \in \wp(F(X))$ is a non-empty subset of $F(X)$, we know there exists at least one $(s,j') \in \mathbf{supp}(\rho(s,l))$ such that $j'(p) = x^p_{z'} \in \mathbf{supp}(l(p))$. For every $p \in P(s)$, let $s_p \colon X \to X$ be the permutation which swaps $x^p_z, x^p_{z'} \in \mathbf{supp}(l(p))$ and $(z\, z') \colon [\mathsf{n}] \to [\mathsf{n}]$ be the permutation which swaps $z, z' \in [\mathsf{n}]$. By construction, for all $j \in [\mathsf{n}]$, $s_p(x^p_j) = x^p_{(z\, z')(j)}$. Next, we prove that $M(s_p) \circ l = l$. Suppose $p \in P(s)$:

$$\begin{aligned} M(s_p) \circ l(p) &= M(s_p)(\beta(x^p_1, \ldots, x^p_n)) && l(p) \in U(X) \\ &= \beta(s_p(x^p_1), \ldots, s_p(x^p_n)) && \beta\text{-naturality square of } s_p \\ &= \beta(x^p_{(z\, z')(1)}, \ldots, x^p_{(z\, z')(n)}) && \text{definition of } s_p \\ &= \beta(x^p_1, \ldots, x^p_n) && \beta \text{ commutative} \\ &= l(p) && l(p) \in U(X) \end{aligned}$$

Thus, the function s_p fixes $l(p)$ and by the (PDS) condition leaves $l(q)$ unchanged for all $q \neq p$, so $F(M(s_p))(s,l) = (s,l)$. Hence, we can conclude from the ρ-naturality square of s_p:

$$\begin{aligned} \rho(F(M(s_p))(s,l)) &= M(F(s_p))(\rho(s,l)) \\ \rho(s,l) &= M(F(s_p))(\rho(s,l)) \end{aligned}$$

Therefore, we compose all these swapping bijections $\{s_p\}_{p \in P(s)}$ (in any order) to obtain a bijection $b \colon X \to X$ such that $M(F(b))(\rho(s,l)) = \rho(s,l)$ and $b \circ j' = j$. By the $\mathbf{supp}$-naturality square of $F(b)$, we obtain that:

$$\begin{aligned} \mathbf{supp}(\rho(s,l)) &= \mathbf{supp}(M(F(b))(\rho(s,l))) \\ &= \wp(F(b))(\mathbf{supp}(\rho(s,l))) \end{aligned}$$

Unpacking the definition of $\wp(F(b))$ and using the above equality, we obtain that

$$\begin{aligned} (s,j') \in \mathbf{supp}(\rho(s,l)) &\Rightarrow F(b)(s,j') \in \mathbf{supp}(\rho(s,l)) \\ &\Rightarrow (s, b \circ j') \in \mathbf{supp}(\rho(s,l)) \\ &\Rightarrow (s,j) \in \mathbf{supp}(\rho(s,l)) \end{aligned}$$

Since $(s,j') \in \mathbf{supp}(\rho(s,l))$, we can conclude that $(s,j) \in \mathbf{supp}(\rho(s,l))$ as desired. □

Finally, we complete the proof of Theorem V.2 by repeating a version of the 'relabel' argument of Theorem III.1 which demonstrates that assuming (PDS) does not constitute a loss of generality.

Proof of Theorem V.2. Suppose $(s,l) \in FU(X)$ where for all $p \in P(s)$, there exists $x^p_1, \ldots, x^p_n \in X$ such that $l(p) = \beta(x^p_1, \ldots, x^p_n)$. We factor $l = M(t) \circ z$ where $z \colon P(s) \to M(P(s) \times X)$ is defined as

$$z(p) = \beta((p, x^p_1), \ldots, (p, x^p_n))$$

and $t \colon P(s) \times X \to X$ is the projection onto the second component. By construction, $(s,z) \in FU(P(s) \times X)$ satisfies condition (PDS), so applying Lemmas A.1 and A.2, we obtain the equation

$$\mathbf{supp}(\rho(s,z)) = \{(s,m) \mid \forall p \in P(s), m(p) \in \mathbf{supp}(z(p))\} \tag{11}$$

Through diagram chasing, we obtain the desired equality:

$$\begin{aligned} &\mathbf{supp}(\rho(s,l)) \\ &= \mathbf{supp}(\rho(s, M(t) \circ z)) && l = M(t) \circ z \\ &= \mathbf{supp}(\rho(F(M(t))(s,z))) && \text{definition of } F \\ &= \mathbf{supp}(M(F(t))(\rho(s,z))) && \rho\text{-naturality square of } t \\ &= \wp(F(t))(\mathbf{supp}(\rho(s,z))) && \mathbf{supp}\text{-naturality square of } F(t) \\ &= \wp(F(t))(\{(s,m) \mid \forall p \in P(s), m(p) \in \mathbf{supp}(z(p))\}) && \text{equation (11)} \\ &= \{F(t)(s,m) \mid \forall p \in P(s), m(p) \in \mathbf{supp}(z(p))\} && \text{definition of } \wp \\ &= \{(s, t \circ m) \mid \forall p \in P(s), m(p) \in \mathbf{supp}(z(p))\} && \text{definition of } F \\ &= \{(s, t \circ m) \mid \forall p \in P(s), m(p) \in \{(p, x^p_1), \ldots, (p, x^p_n)\}\} && z(p) = \beta((p, x^p_1), \ldots, (p, x^p_n)) \\ &= \{(s,j) \mid \forall p \in P(s), j(p) \in \{t(p, x^p_1), \ldots, t(p, x^p_n)\}\} && j = t \circ m \\ &= \{(s,j) \mid \forall p \in P(s), j(p) \in \{x^p_1, \ldots, x^p_n\}\} && \text{definition of } t \\ &= \{(s,j) \mid \forall p \in P(s), j(p) \in \mathbf{supp}(\beta(x^p_1, \ldots, x^p_n))\} && \mathbf{supp} \text{ preserves } \beta \\ &= \{(s,j) \mid \forall p \in P(s), j(p) \in \mathbf{supp}(l(p))\} && \text{definition of } l(p) \end{aligned}$$

□

Lemma V.8. *Let $M = \mathcal{D}_{\mathscr{S}}$ or $M = \mathcal{M}_{\mathscr{S}}$ for some semiring $\mathscr{S}$. $\mathscr{S}$ is zero-sumfree, i.e. satisfies Condition (S1) if and only if $\mathbf{supp}^M$ is a natural transformation.*

Proof. By contrapositive, assume that $\mathscr{S}$ is not zero-sumfree, then there exists $r, t \in \mathscr{S}$ such that $r + t = 0_{\mathscr{S}}$, but $r \neq 0_{\mathscr{S}}$ or $t \neq 0_{\mathscr{S}}$. Note that it follows from the semiring axiom $0_{\mathscr{S}} + a = a = a + 0_{\mathscr{S}}$, that $r \neq 0_{\mathscr{S}} \Leftrightarrow t \neq 0_{\mathscr{S}}$. Thus, both $r \neq 0_{\mathscr{S}}$ and $t \neq 0_{\mathscr{S}}$. Let $X = \{x, y, z\}$, $A = \{a, b\}$ and $f\colon X \to A$ defined as $x, y \mapsto a$ and $z \mapsto b$. Consider the distribution $\varphi = r \cdot x + t \cdot y + 1_{\mathscr{S}} \cdot z \in \mathcal{D}_{\mathscr{S}}(X)$. As $r + t = 0_{\mathscr{S}}$, φ satisfies the normalisation condition and is indeed a distribution in $\mathcal{D}_{\mathscr{S}}(X)$. We obtain the following inequality:

$$\begin{aligned}
&(\mathbf{supp}_A \circ \mathcal{D}_{\mathscr{S}}(f))(r \cdot x + t \cdot y + 1_{\mathscr{S}} z) \\
&\quad = \mathbf{supp}_A((r + t) \cdot a + 1_{\mathscr{S}} \cdot b) && \text{definition of } \mathcal{D}_{\mathscr{S}}(f) \\
&\quad = \mathbf{supp}_A(1_{\mathscr{S}} \cdot b) && r + t = 0_{\mathscr{S}} \\
&\quad = \{b\} && \text{definition of } \mathbf{supp} \\
&\quad \neq \{a, b\} && a \neq b \\
&\quad = \mathcal{P}(f)(\{x, y, z\}) && \text{definition of } \mathcal{P}(f) \\
&\quad = (\mathcal{P}(f) \circ \mathbf{supp}_X)(r \cdot x + t \cdot y + 1_{\mathscr{S}} \cdot z) && \text{definition of } \mathbf{supp},\ r \neq 0_{\mathscr{S}}
\end{aligned}$$

Therefore, $\mathbf{supp}_A \circ \mathcal{D}_{\mathscr{S}}(f) \neq \mathcal{P}(f) \circ \mathbf{supp}_X$, the $\mathbf{supp}$-naturality square of f does not commute. Hence, the family of maps $\mathbf{supp}_X$ is not a natural transformation.

Conversely, suppose $\mathscr{S}$ is zero-sumfree and that $f\colon X \to Y \in \mathbf{Set}_1$. We must show that $\mathbf{supp}_Y(\mathcal{D}_{\mathscr{S}}(f)(\varphi)) = \mathcal{P}(f)(\mathbf{supp}_X(\varphi))$ for all $\varphi \in \mathcal{D}_{\mathscr{S}}(X)$. Suppose $\varphi = \sum_{i \in I} s_i x_i \in \mathcal{D}_{\mathscr{S}}(X)$ for indexing set I. By definition of $\mathcal{D}_{\mathscr{S}}(f)$, $\mathcal{D}_{\mathscr{S}}(f)(\varphi) = \sum_{y \in f(X)} (\sum_{j \in J_y} s_j) y$ where $J_y = \{i \in I \mid x_i \in f^{-1}(y)\} \subseteq I$ and $f(X)$ is the image of f. Since $\mathscr{S}$ is zero-sumfree, the sums $\sum_{j \in J_y} s_j$ are non-zero whenever $y = f(x_i)$. Hence, we can compute $\mathbf{supp}_Y(\mathcal{D}_{\mathscr{S}}(f)(\varphi)) = \{f(x_i) \mid \forall i \in I\}$. By definition of $\mathcal{P}(f)$, $\{f(x_i) \mid \forall i \in I\} = \mathcal{P}(f)(\{x_i \mid \forall i \in I\}) = \mathcal{P}(f)(\mathbf{supp}_X(\varphi))$ as desired. ☐

Lemma V.9. *Let $M = \mathcal{D}_{\mathscr{S}}$ or $M = \mathcal{M}_{\mathscr{S}}$. If $\mathscr{S}$ has a natural non-trivial unit $n_{\mathscr{S}}$, i.e. satisfies Condition (S2), then $\beta\colon \mathbf{Id_{Set}} \times \cdots \times \mathbf{Id_{Set}} \to M$ defined as:*

$$\beta(x_1, \ldots, x_n) = \sum_{i \in [\mathsf{n}]} \frac{1}{n_{\mathscr{S}}} x_i$$

is a n-ary idempotent and commutative open term. In particular, if $\mathscr{S}$ is zero-sumfree, then β is a n-uniform choice term.

Proof. To verify that β is idempotent, suppose $X \in \mathbf{Set}_0$ and $x \in X$, then:

$$\begin{aligned}
\beta(x, \ldots, x) &= \sum_{i \in [\mathsf{n}]} \frac{1}{n_{\mathscr{S}}} x && \text{definition of } \beta \\
&= \frac{1}{n_{\mathscr{S}}} (\sum_{i \in [\mathsf{n}]} 1_{\mathscr{S}} x) && \text{distribution axiom of } \mathscr{S} \\
&= \frac{1}{n_{\mathscr{S}}} (\sum_{i \in [\mathsf{n}]} 1_{\mathscr{S}}) x && \text{collect coefficents of } x \\
&= \frac{1}{n_{\mathscr{S}}} n_{\mathscr{S}} x && \text{definition of } n_{\mathscr{S}} \\
&= 1_{\mathscr{S}} x && n_{\mathscr{S}} \text{ is a unit} \\
&= \eta(x) && \text{definition of } \eta
\end{aligned}$$

To verify β is commutative, suppose $X \in \mathbf{Set}_0$ and $x_1, \ldots, x_n \in X$, then:

$$\begin{aligned}
\beta(x_{\pi(1)}, \ldots, x_{\pi(n)}) &= \sum_{i \in [\mathsf{n}]} \frac{1}{n_{\mathscr{S}}} x_{\pi(i)} \\
&= \frac{1}{n_{\mathscr{S}}} \sum_{i \in [\mathsf{n}]} x_{\pi(i)} \\
&= \frac{1}{n_{\mathscr{S}}} \sum_{i \in [\mathsf{n}]} x_i \\
&= \sum_{i \in [\mathsf{n}]} \frac{1}{n_{\mathscr{S}}} x_i \\
&= \beta(x_1, \ldots, x_n)
\end{aligned}$$

Finally, if we assume that $\mathscr{S}$ is zero-sumfree, then $n_{\mathscr{S}} \neq 0_{\mathscr{S}}$ and $\frac{1}{n_{\mathscr{S}}} \neq 0_{\mathscr{S}}$. Thus,

$$\mathbf{supp}(\beta(x_1, \ldots, x_n)) = \mathbf{supp}(\sum_{i \in [\mathsf{n}]} \frac{1}{n_{\mathscr{S}}} x_i) = \{x_1, \ldots, x_n\}.$$

Hence, in the case where $\mathscr{S}$ is zero-sumfree, β is a n-ary uniform choice term. ☐