Learners are Almost Free Compact Closed

Mitchell Riley
Mathematics, Division of Science
New York University Abu Dhabi
mitchell.v.riley@nyu.edu

I expand on an observation made at the end of [15] and further discussed online [16]—that the category of learners [7] has a pleasant symmetric formulation when the morphisms are considered up to a certain coarser equivalence than the one originally used. A quotient of this modified category gives a construction of the free compact closed category on a symmetric monoidal category.

1 Extensional Learners

Learners are a categorical construction used in a compositional approach to supervised learning and backpropagation [7, 2]. First, we recall the their definition.

Definition 1.1 ([7, Definition 2.1])**.** For A and B sets, a *learner* $A \nrightarrow B$ is a set P together with functions $I : P \times A \to B$ and $U : P \times A \times B \to P$ and $r : P \times A \times B \to A$, considered up to isomorphism of parameter sets: two learners (P, I, U, r) and (P', I', U', r') are identified whenever there is a bijection $f : P \to P'$ with

$$\begin{aligned} I'(f(p), a) &= I(p, a) \\ U'(f(p), a, b) &= f(U(p, a, b)) \\ r'(f(p), a, b) &= r(p, a, b) \end{aligned}$$

Learners expose their parameter sets P to the outside world, and two learners with non-isomorphic parameter sets are considered different even if they 'behave the same' on all input data A and B. Cribbing some terminology from type theory, we will label these learners *intensional*: their identity depends on the specific choice of parameter set even if it is not observable through their input-output behaviour. (Unfortunately this clashes with the meaning of intensional/extensional used in [10].)

Proposition 1.2 ([7, Proposition 2.4])**.** *Intensional learners in any category with finite products form a symmetric monoidal category* **IntLearn**$_{\mathcal{C}}$. □

The underlying data of a learner $A \nrightarrow B$ with fixed parameter set P has an appealing symmetric description, via the chain of isomorphisms

$$\begin{aligned} &\mathcal{C}(P \times A, B) \times \mathcal{C}(P \times A \times B, P) \times \mathcal{C}(P \times A \times B, A) \\ \cong\ &\mathcal{C}(P \times A, B) \times \mathcal{C}(P \times A \times B, P \times A) \\ \cong\ &\int^{Q:\mathcal{C}} \mathcal{C}(P \times A, Q) \times \mathcal{C}(P \times A, B) \times \mathcal{C}(Q \times B, P \times A) \\ \cong\ &\int^{Q:\mathcal{C}} \mathcal{C}(P \times A, Q \times B) \times \mathcal{C}(Q \times B, P \times A) \end{aligned}$$

The author acknowledges support by *Tamkeen* under *NYUAD Research Institute grant* CG008.

Submitted to:
ACT 2024

The quotient up to isomorphism of P can then be expressed as an additional coend over the core of $\mathcal{C}$:

$$\mathbf{IntLearn}(A,B) = \int^{P:\mathrm{Core}(\mathcal{C})} \int^{Q:\mathcal{C}} \mathcal{C}(P\times A, Q\times B)\times \mathcal{C}(Q\times B, P\times A)$$

Three generalisations immediately suggest themselves:

- Replace the former coend with one over all of $\mathcal{C}$ rather than only the isomorphisms;
- Replace $\times$ with an arbitrary symmetric monoidal product $\otimes$; and,
- Allow the occurrences of A and B to be different objects, as in the category of optics [15, 1].

We arrive at our generalised definition of learner.

Definition 1.3. For objects A, A', B, B' of a symmetric monoidal category $\mathcal{C}$, an *extensional learner* $(A,A') \nrightarrow (B,B')$ is an element of

$$\mathbf{Learn}_{\mathcal{C}}((A,A'),(B,B')) := \int^{P,Q:\mathcal{C}} \mathcal{C}(P\otimes A, Q\otimes B)\times \mathcal{C}(Q\otimes B', P\otimes A')$$

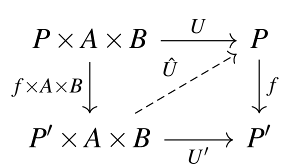
The extensional learner represented by a pair of maps $f : P\otimes A \to Q\otimes B$ and $g : Q\otimes B' \to P\otimes A'$ will be written $(f \mid g) : (A,A') \nrightarrow (B,B')$.

Because these are our focus, we omit 'extensional' and call these simply *learners*.

The definition unwinds to a coarser notion of equivalence on the data of intensional learners. Two learners (P,I,U,r) and (P',I',U',r') are identified by this new definition when there is a (not-necessarily-bijective) function $f : P \to P'$ and an intermediate function $\hat{U} : P'\times A\times B \to P$ such that

$$\begin{aligned} I'(f(p),a) &= I(p,a) \\ r'(f(p),a,b) &= r(p,a,b) \\ \hat{U}(f(p),a,b) &= U(p,a,b) \\ U'(p',a,b) &= f(\hat{U}(p',a,b)) \end{aligned}$$

That is, the relationship of f to I and r is the same as before, but for U we require a diagonal filler in the diagram



Lemma 1.4. *Intensionally equivalent learners are extensionally equivalent.*

Proof. The bijection $f : P \to P'$ satisfies the first two equations of extensional equivalence by assumption. For the intermediate function $\hat{U} : P'\times A\times B \to P$, take $\hat{U}(p',a,b) := U(f^{-1}(p'),a,b)$, then

$$\begin{aligned} \hat{U}(f(p),a,b) &= U(f^{-1}(f(p)),a,b) = U(p,a,b) \\ f(\hat{U}(p',a,b)) &= f(U(f^{-1}(p'),a,b)) = U'(f(f^{-1}(p')),a,b) = U'(p',a,b) \end{aligned}$$

□

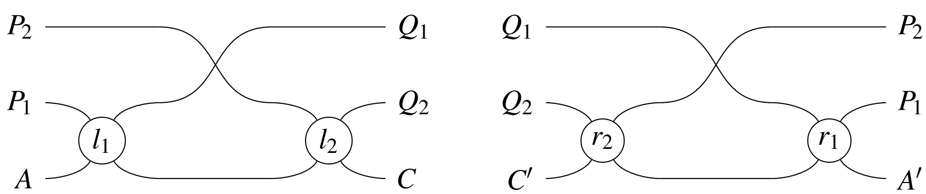
Remark 1.5. The extensional relation is similar the notion of 2-morphism between learners given in [7, Definition 7.1], which only requires that the outer square in the above diagram commutes. The intensional relation identifies two learners when there is a 2-isomorphism between them.

In [6], Fong and Johnson consider two intensional learners identical when there is a 2-morphism such that f is only *surjective*, and not necessarily a bijection. This is a coarser notion of equivalence still: for example, it identifies all intensional learners $1 \nrightarrow 1$.

Proposition 1.6. *Learners form a category* $\mathbf{Learn}_{\mathcal{C}}$*, where the identity learner* $(A, A') \nrightarrow (A, A')$ *is* $(\mathsf{id}_{I\otimes A} \mid \mathsf{id}_{I\otimes A'})$*, and the composite of* $(l_1 \mid r_1) : (A, A') \nrightarrow (B, B')$ *and* $(l_2 \mid r_2) : (B, B') \nrightarrow (C, C')$ *with representatives*

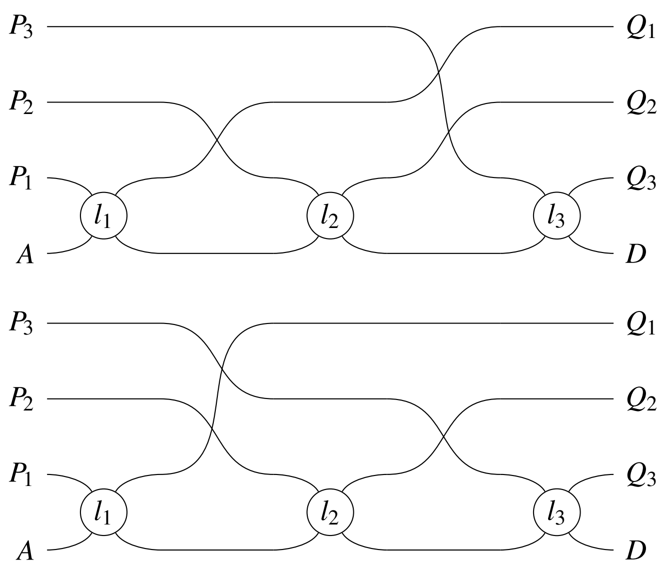
$$\begin{aligned} l_1 &: P_1 \otimes A \to Q_1 \otimes B \\ r_1 &: Q_1 \otimes B' \to P_1 \otimes A' \\ l_2 &: P_2 \otimes B \to Q_2 \otimes C \\ r_2 &: Q_2 \otimes C' \to P_2 \otimes B' \end{aligned}$$

is given by



.

Proof. Similar to [15, Proposition 2.0.3]. The unit laws for composition follow quickly from the unit laws in $\mathcal{C}$. Associativity of composition follows (for the left component) from the isotopy of the following string diagrams:



The right component is symmetric. □

Remark 1.7. Intensional learners manifestly do not form a locally small category: any choice of parameter set P determines a learner $(1,1) \nrightarrow (1,1)$ with $U := \mathsf{pr}_1$, and any two sets with different cardinalities determine two intensionally inequivalent learners. The situation is not improved with the extensional relation: these learners remain unequal and $\mathbf{Learn}_{\mathcal{C}}$ is not locally small. (Compare ordinary lenses, which can be described in a similar coend style but always form a locally small category.) For a symmetric monoidal category, the collection of learners $(I,I) \nrightarrow (I,I)$ is the *trace* of $\mathcal{C}$ in the sense of [4], which can be difficult to calculate even in simple cases.

The category $\mathbf{Learn}_{\mathcal{C}}$ accepts an obvious functor $\iota : \mathcal{C} \to \mathbf{Learn}_{\mathcal{C}}$, sending $f : A \to B$ to $(I \otimes f \mid \mathsf{id}_{I \otimes I}) : (A,I) \nrightarrow (B,I)$. And, it also accepts a functor from $\mathbf{Optic}_{\mathcal{C}}$, which sends an optic $\langle l \mid r \rangle : (A,A') \nrightarrow (B,B')$ to the learner with components

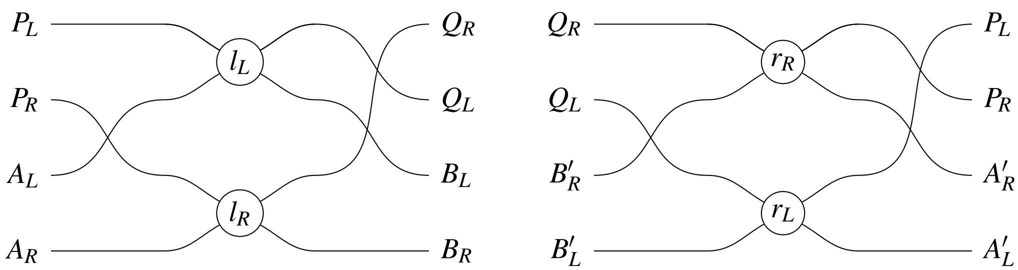
$$I \otimes A \to A \xrightarrow{l} M \otimes B$$
$$M \otimes B' \xrightarrow{r} A' \to I \otimes A'$$

The observation that lenses include into learners as those with trivial parameter set was made in [7], and this fact is made especially clear by the characterisation $\mathbf{IntLearn}_{\mathcal{C}} \cong \mathbf{Para}(\mathbf{Lens}_{\mathcal{C}})$ given in [2].

Proposition 1.8. $\mathbf{Learn}_{\mathcal{C}}$ *is symmetric monoidal, with action on objects given by* $(A,A') \otimes (B,B') := (A \otimes B, B' \otimes A')$ *and action on morphisms*

$$(l_L \mid r_L) \otimes (l_R \mid r_R) : (A_L, A'_L) \otimes (A_R, A'_R) \to (B_L, B'_L) \otimes (B_R, B'_R)$$

given by



Proof. Similar to [15, Theorem 2.0.12], but using a version of the 'switched' tensor from [15, Definition 2.1.1]. The key facts to check are that the action on morphisms does not depend on the choice of representative and that $\otimes$ is functorial. The first is clear from the above diagrams: for example, a morphism on the Q_L string between l_L and r_L can be slid from l_L to r_L in the string diagram (using the coend relation to hop the gap), and similarly for the three strings for Q_R, P_L and P_R.

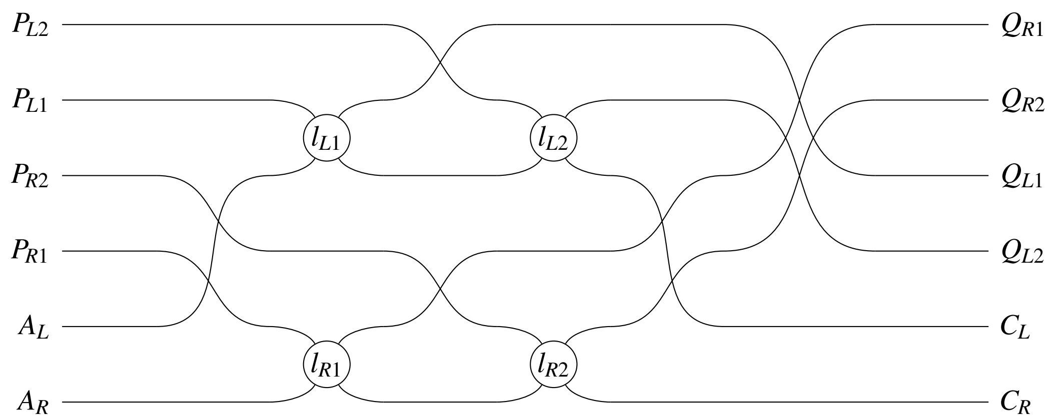
Functoriality of $\otimes$ follows from the equality of the following diagrams, depicting the left component of the morphisms

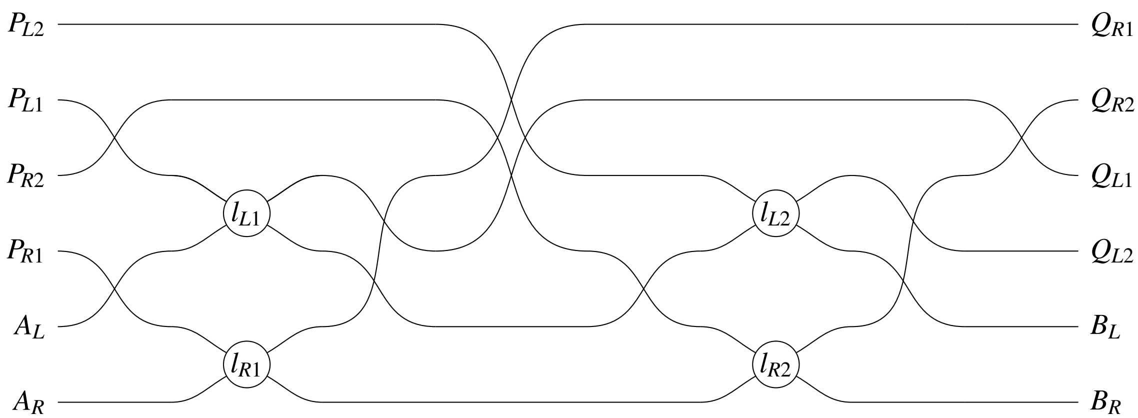
$$((l_{L2} \mid r_{L2}) \circ (l_{L1} \mid r_{L1})) \otimes ((l_{R2} \mid r_{R2}) \circ (l_{R1} \mid r_{R1}))$$

and

$$((l_{L2} \mid r_{L2}) \otimes (l_{R2} \mid r_{R2})) \circ ((l_{L1} \mid r_{L1}) \otimes (l_{R1} \mid r_{R1}))$$

respectively. (Imagine dragging the l_{L1} node below the P_{R2} string to the left, and the l_{L2} node below the Q_{R1} string to the right.)





The latter diagram is almost exactly the correct one, other than the outer twists on the pairs (P_{L1}, P_{R2}) and (Q_{R2}, Q_{L1}). These are cancelled with similar twists in the right component using the coend relations.

As in the **Optic** setting, the structure morphisms (including the symmetry isomorphism) are constructed as the image of the structure morphisms of $\mathcal{C} \times \mathcal{C}^{\mathsf{op}}$, and functoriality of the inclusion guarantees that the equations of a symmetric monoidal category hold. □

2 Duality

For extensional learners, there is an obvious involution $(-)^* : \mathbf{Learn} \to \mathbf{Learn}^{\mathsf{op}}$ that comes into view by virtue of the new symmetrical formulation: just switch the two components!

Proposition 2.1. *There is a (strict) involutive symmetric monoidal functor* $(-)^* : \mathbf{Learn} \to \mathbf{Learn}^{\mathsf{op}}$ *given by* $(A, A')^* := (A', A)$ *and* $(L \mid R)^* := (R \mid L)$.

Proof. On objects, we have

$$\big((A,A') \otimes (B,B')\big)^* = (A \otimes B, B' \otimes A')^* = (B' \otimes A', A \otimes B) = (B', B) \otimes (A', A) = (B, B')^* \otimes (A, A')^*$$

Preservation of identity, composition and tensor of morphisms is clear by observing that their definitions are exactly symmetric between each component. □

The resulting strictness of this functor is why it is convenient to switch the objects in the right component of $(A,A') \otimes (B,B') :\equiv (A \otimes B, B' \otimes A')$.

Remark 2.2. Let us calculate how this duality acts on intensional learners. Stepping through the equivalence between intensional learners and their coend formulation, the dual of a learner $(P, I, U, r) : (A,A') \nrightarrow (B,B')$ is a learner $(P^*, I^*, U^*, r^*) : (B',B) \nrightarrow (A',A)$ with parameter set $P^* := P \times A$, and

$$\begin{aligned}
&I^* : (P \times A) \times B' && \to A' \\
&I^*((p, p_a), b') && := r(p, p_a, b') \\
&U^* : (P \times A) \times B' \times A && \to (P \times A) \\
&U^*((p, p_a), b', a) && := (U(p, p_a, b'), a) \\
&r^* : (P \times A) \times B' \times A && \to B \\
&r^*((p, p_a), b', a) && := I(U(p, p_a, b'), a)
\end{aligned}$$

If we do this twice, we return to a learner $(A,A') \nrightarrow (B,B')$ with parameter set $P^{**} := P \times A \times B'$, and

$$\begin{aligned}
&I^{**} : (P \times A \times B') \times A && \to B \\
&I^{**}((p, p_a, p_{b'}), a) && := I(U(p, p_a, p_{b'}), a) \\
&U^{**} : (P \times A \times B') \times A \times B' && \to (P \times A \times B') \\
&U^{**}((p, p_a, p_{b'}), a, b') && := (U(p, p_a, p_{b'}), a, b') \\
&r^{**} : (P \times A \times B') \times A \times B' && \to A' \\
&r^{**}((p, p_a, p_{b'}), a, b') && := r(U(p, p_a, p_{b'}), a, b')
\end{aligned}$$

Under intensional equivalence this is not equal to the original learner, but under extensional equivalence it is: use $U : P \times A \times B' \to P$ as the function f between parameter sets, and the identity $\mathrm{id} : P \times A \times B' \to P \times A \times B'$ as the diagonal filler $\hat{U}$. This double-dual learner is extensionally identical to the original learner, but intensionally it is 'running one training datum behind': when fed a new element of $A \times B'$, it updates P using the pair remembered in the parameter set and stores the provided element of $A \times B'$ for next time.

An object (A,A') and its counterpart (A',A) are related by cup and cap morphisms:

Definition 2.3. For any object (A,A'), define the cup

$$\eta_{(A,A')} : (I,I) \nrightarrow (A,A') \otimes (A,A')^* = (A \otimes A', A \otimes A')$$

as the learner with $P := A \otimes A'$ and $Q := I$ and the obvious maps

$$\begin{gathered}
(A \otimes A') \otimes I \to I \otimes (A \otimes A') \\
I \otimes (A \otimes A') \to (A \otimes A') \otimes I
\end{gathered}$$

and define the cap

$$\varepsilon_{(A,A')} : (A' \otimes A, A' \otimes A) = (A,A')^* \otimes (A,A') \nrightarrow (I,I)$$

as the learner with $P := I$ and $Q := A' \otimes A$ and the obvious maps

$$\begin{gathered}
I \otimes (A' \otimes A) \to (A' \otimes A) \otimes I \\
(A' \otimes A) \otimes I \to I \otimes (A' \otimes A).
\end{gathered}$$

These maps are dual, in that:

$$(\eta_{(A,A')})^* = \varepsilon_{(A,A')^*} \qquad (\varepsilon_{(A,A')})^* = \eta_{(A,A')^*}$$

All signs are pointing to $(A,A')^*$ being the actual monoidal dual of (A,A'). Tragically,

Proposition 2.4. $\mathbf{Learn}_{\mathcal{C}}$ *is typically not compact closed.*

Proof. Consider an object of the form (A,I). After cancelling some monoidal units, the composite learner

$$(A,I) \nrightarrow (A,I) \otimes ((A,I)^* \otimes (A,I)) \nrightarrow ((A,I) \otimes (A,I)^*) \otimes (A,I) \nrightarrow (A,I)$$

corresponding to the snake equation for (A,I) has $P := A$ and $Q := A$, and components

$$A \otimes A \xrightarrow{s_{A,A}} A \otimes A$$
$$A \otimes I \xrightarrow{\mathsf{id}} A \otimes I.$$

This is typically not equal to the identity learner; a priori there are no non-trivial maps involving A whatsoever, as can be seen by setting $\mathcal{C}$ to be a commutative monoid considered as a discrete category.

As an intensional learner, the above learner is similar to the identity map but with its output "delayed one step", and has $P := A$ with maps

$$\begin{aligned}
I : A \times A &\to A \\
I(p,a) &:= p \\
U : P \times 1 \times A &\to P \\
U(p,b,a) &:= a \\
r : P \times 1 \times A &\to 1
\end{aligned}$$

□

Learners do have the following weaker structure, much like the teleological categories discussed in [9, Section 5] and [15, Section 2.1] but with cups as well as caps.

Proposition 2.5. *The families of morphisms*

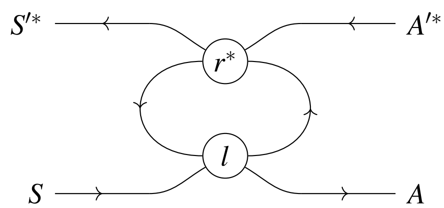
$$\eta_{(A,A')} : (I,I) \nrightarrow (A,A') \otimes (A,A')^*$$
$$\varepsilon_{(A,A')} : (A,A')^* \otimes (A,A') \nrightarrow (I,I)$$

are monoidal, and are extranatural with respect to morphisms in the image of $\iota : \mathcal{C} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Learn}_{\mathcal{C}}$. *Furthermore,* $s_{(A,A'),(A,A')^*}\eta_{(A,A')} = \eta_{(A,A')^*}$ *and* $\varepsilon_{(A,A')}s_{(A,A'),(A,A')^*} = \varepsilon_{(A,A')^*}$, *where* $s_{(A,A'),(A,A')^*}$ *denotes the symmetry morphism.*

Proof. The definitions of η and ε are so simple that these are all easily checked by direct calculation. □

We have a canonical decomposition of any learner using these morphisms, in the same style as [15, Proposition 2.1.10].

Lemma 2.6. *Every learner* $(l \mid r) : (S,S') \nrightarrow (A,A')$ *is equal to the learner described by the diagram*



recalling that we silently include objects and morphsisms of $\mathcal{C}$ in $\mathbf{Learn}_{\mathcal{C}}$ *using* ι.

Proof. This can be checked by direct calculation, cancelling all the occurrences of the monoidal unit created by the use of ι. □

The snake equations are precisely what is missing for $\mathbf{Learn}_{\mathcal{C}}$ to be compact closed.

Definition 2.7. The category $\mathbf{Atemp}_{\mathcal{C}}$ of *atemporal learners* is the quotient of $\mathbf{Learn}_{\mathcal{C}}$ obtained by identifying the snake diagram

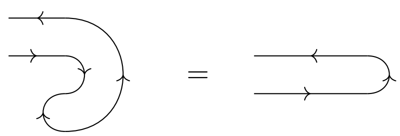
$$(A,A') \nrightarrow ((A,A') \otimes (A,A')^*) \otimes (A,A') \nrightarrow (A,A') \otimes ((A,A')^* \otimes (A,A')) \nrightarrow (A,A')$$

with the identity morphism for every object (A,A').

Trivialising just one of the snake diagrams also trivialises the other: the dual snake diagram is the image of this one under the strictly monoidal functor $-^*$.

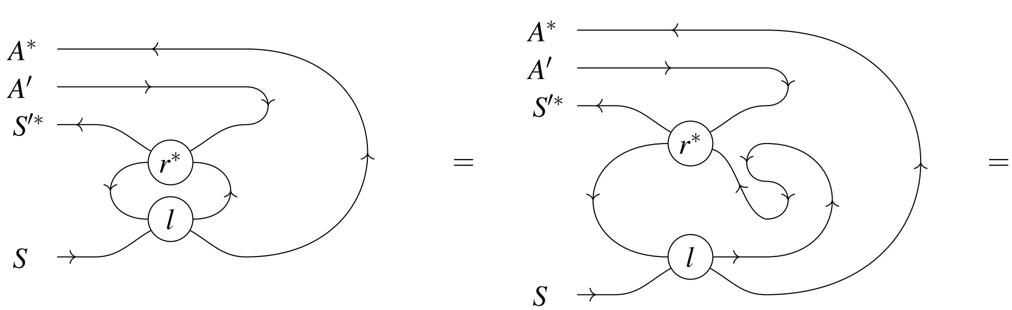
Proposition 2.8. $\mathbf{Atemp}_{\mathcal{C}}$ *is compact closed.*

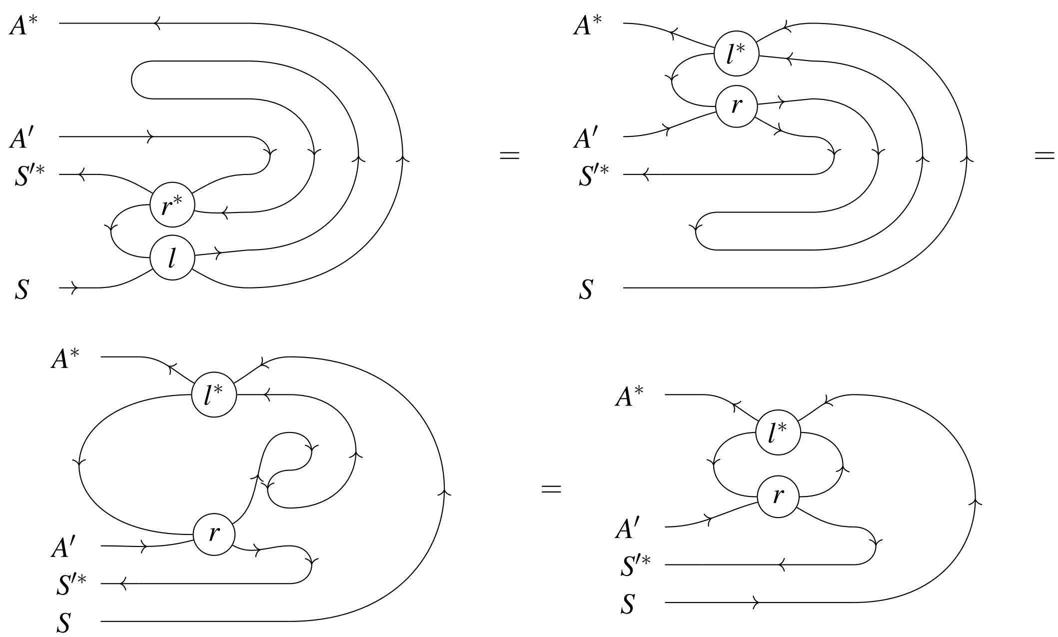
Proof. The only missing property is extranaturality of the cup and cap with respect to *all* morphisms in $\mathbf{Atemp}_{\mathcal{C}}$. The snake equations holding implies that η and ε are extranatural with respect to each other:



and similarly with the roles reversed.

From Lemma 2.6, it follows that η and ε are natural with respect to *every* map in $\mathbf{Atemp}_{\mathcal{C}}$. Decomposing a map as via the Lemma, we can then see:





The unit is similar. □

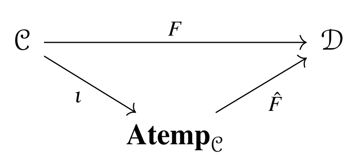
3 Free Compact Closure

We conclude by showing that this $\mathbf{Atemp}_{\mathcal{C}}$ is the free compact closed category on a symmetric monoidal category $\mathcal{C}$.

There is an existing construction of the free compact closed category given by composing three simpler constructions: [13] constructs the free *feedback* monoidal category on a monoidal category before quotienting it to form the free *traced* monoidal category. Then [12] constructs the free compact closed category on a traced monoidal category. In the literature one also finds the free compact closed category on a bare category [14], and the free compact closed category on a *closed* monoidal category [3], but these are less comparable to the construction given here.

As a first step, any functor from $\mathcal{C}$ into a compact closed category factors through $\mathbf{Atemp}_{\mathcal{C}}$.

Theorem 3.1. *Suppose $\mathcal{C}$ is a symmetric monoidal category an $\mathcal{D}$ is a compact closed category. For any symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$, there is a factorisation*

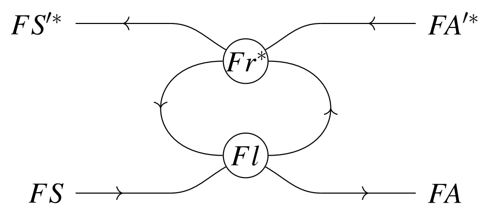


Here we follow the shape of the argument in [12, Section 5].

Proof. Any object (A,A') of $\mathbf{Atemp}_{\mathcal{C}}$ is isomorphic to

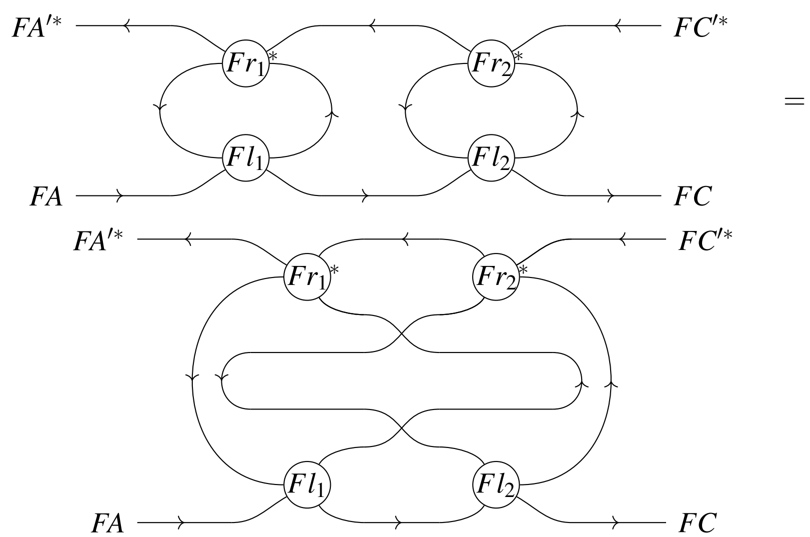
$$(A,A') \cong (A',I)^* \otimes (A,I) = (\iota A')^* \otimes \iota A$$

which forces us to define $\hat{F}(A,A') := (FA')^* \otimes FA$. On a morphism $(l \mid r)$, define $\hat{F}(l \mid r)$ as indicated in the diagram



This is well defined with respect to the quotient on $\mathbf{Learn}_{\mathcal{C}}$, because the snake equations already hold in $\mathcal{D}$. The functor $\hat{F}$ behaves nicely with the inclusion $\iota : \mathcal{C} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Atemp}_{\mathcal{C}}$, because $\hat{F}(\iota(f,g)) = g^* \otimes f$ for any two morphisms of $\mathcal{C}$.

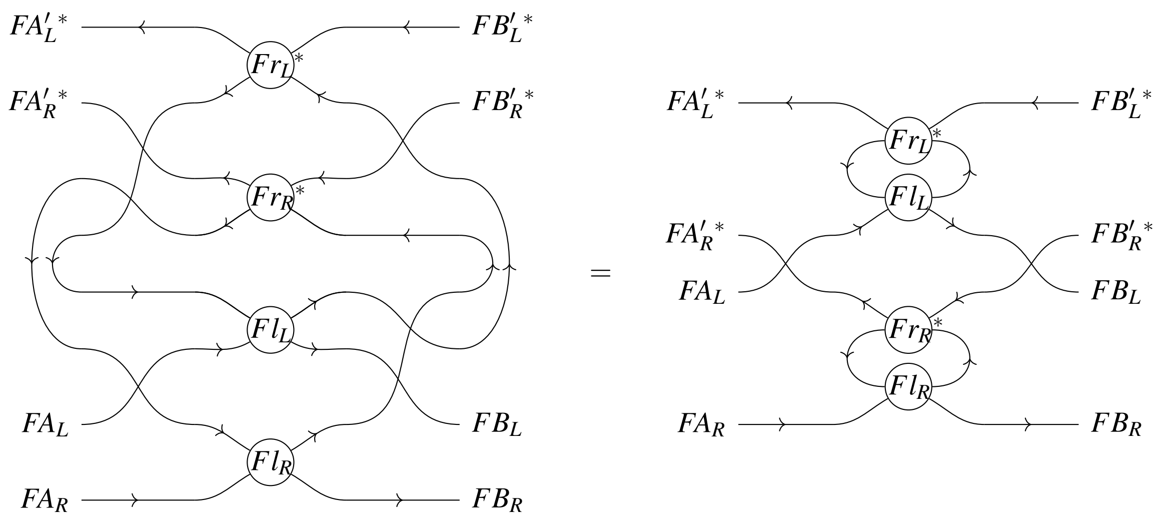
There are now several things to check. First, that $\hat{F}$ is a functor. Preservation of identity follows because the cup and cap for the unit object in $\mathcal{D}$ are the unitors. Preservation of composition is the equivalence of the following diagrams:



We must equip $\hat{F}$ with the structure of a monoidal functor. For the structure isomorphism ϕ : $\hat{F}(A_L, A'_L) \otimes \hat{F}(A_R, A'_R) \to \hat{F}(A_L \otimes A_R, A'_R \otimes A'_L)$, we take the composite

$$
\begin{aligned}
&\hat{F}(A_L, A'_L) \otimes \hat{F}(A_R, A'_R) \\
&= (FA_L \otimes FA'_L{}^*) \otimes (FA_R \otimes FA'_R{}^*) \\
&\to (FA_L \otimes FA_R) \otimes (FA'_L{}^* \otimes FA'_R{}^*) \\
&\to (FA_L \otimes FA_R) \otimes (FA'_R \otimes FA'_L)^* \\
&\to F(A_L \otimes FA_R) \otimes (F(A'_R \otimes A'_L))^* \\
&= \hat{F}(A_L \otimes A_R, A'_R \otimes A'_L) \\
&= \hat{F}((A_L, A'_L) \otimes (A_R, A'_R))
\end{aligned}
$$

That this gives a monoidal functor follows from the equivalence of the diagrams



The functor is also symmetric monoidal, as the symmetry morphisms of $\mathbf{Atemp}_{\mathcal{C}}$ are inherited from the symmetry morphisms of $\mathcal{C} \times \mathcal{C}^{\mathsf{op}}$ via ι. □

Let $\mathbf{SymMon}_g$ denote the 2-category of symmetric monoidal categories, monoidal functors and monoidal natural *isomorphisms*, and let **Comp** denote the 2-category of compact closed categories, monoidal functors and monoidal natural transformations. Recall that no condition on monoidal functors between compact closed categories is necessary: duals are always preserved up to canonical isomorphism. We also do not need to explicitly restrict the 2-cells in **Comp** to be invertible, because *any* such monoidal natural transformation is invertible:

Proposition 3.2. *[11, Proposition 7.1] Any monoidal natural transformation $\alpha : F \to G$ in* **Comp** *has an inverse given by*

$$GA \to (G(A^*))^* \xrightarrow{(\alpha_{A^*})^*} (F(A^*))^* \to FA$$

at each object A. □

Theorem 3.3. $\mathbf{Atemp}_{\mathcal{C}}$ *is the free compact closed category on a symmetric monoidal category $\mathcal{C}$. That is,* $\mathbf{Atemp} : \mathbf{SymMon}_g \to \mathbf{Comp}$ *assembles into a 2-functor that is left biadjoint to the inclusion* $\mathbf{Comp} \to \mathbf{SymMon}_g$*, with the unit of the biadjunction having component $\iota : \mathcal{C} \to \mathbf{Atemp}_{\mathcal{C}}$ at $\mathcal{C}$.*

Proof. Suppose $\mathcal{D}$ is compact closed. For two symmetric monoidal functors $F, G : \mathbf{Atemp}_{\mathcal{C}} \to \mathcal{D}$, the restriction map along $\iota_{\mathcal{C}}$ gives a map

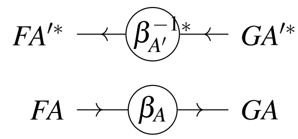
$$\mathbf{Comp}(\mathbf{Atemp}_{\mathcal{C}}, \mathcal{D})(F,G) \to \mathbf{SymMon}_g(\mathcal{C}, \mathcal{D})(F \circ \iota, G \circ \iota)$$

We claim that this restriction map is a bijection.

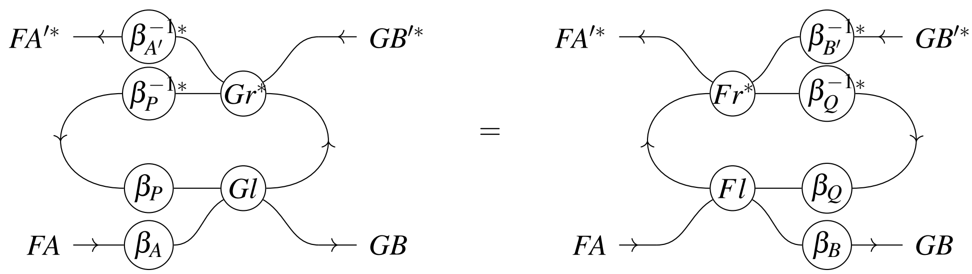
Given a monoidal isomorphism $\beta : F \circ \iota \to G \circ \iota$, define the monoidal isomorphism $\overline{\beta} : F \to G$ whose component at (A, A') is

$$\begin{aligned} F(A,A') = F(\iota A \otimes (\iota A')^*) &\to F\iota A \otimes F(\iota A'^*) \\ &\to F\iota A \otimes (F\iota A')^* \xrightarrow{\beta_A \otimes (\beta_{A'}^{-1})^*} G\iota A \otimes (G\iota A')^* \\ &\to G\iota A \otimes G(\iota A'^*) \to G(\iota A \otimes \iota A'^*) = G(A,A') \end{aligned}$$

Eliding the structural morphisms, this corresponds simply to the following diagram in $\mathcal{D}$:



Monoidalness of $\overline{\beta}$ follows immediately from the monoidalness of β. For naturality, suppose we have a morphism $(l \mid r) : (A,A') \to (B,B')$ in $\mathbf{Atemp}_{\mathcal{C}}$. Naturality of $\overline{\beta}$ with respect to $(l \mid r)$ corresponds to the equivalence of the following diagrams:



using monoidalness and naturality of β to pass it through l and r. The extraneous β_P and β_Q are cancelled with their inverses by passing them around the cup and cap.

Finally, we must show that this process is inverse to restriction along ι. It is clear from the definition that $\overline{\beta}\iota = \beta$. The interesting case is the converse, that $\overline{\alpha\iota} = \alpha$ for a monoidal isomorphism $\alpha : F \to G$. We verify

$$
\begin{aligned}
&F(A,A') \xrightarrow{\overline{\alpha\iota}_{(A,A')}} G(A,A')\\
&= F(A,A') \xrightarrow{\sim} F\iota A \otimes (F\iota A')^* \xrightarrow{\alpha\iota_A \otimes (\alpha\iota_{A'}{}^{-1})^*} G\iota A \otimes (G\iota A')^* \xrightarrow{\sim} G(A,A')\\
&= F(A,A') \xrightarrow{\sim} F\iota A \otimes (F\iota A')^* \xrightarrow{\alpha_{\iota A} \otimes (\alpha_{\iota A'}{}^{-1})^*} G\iota A \otimes (G\iota A')^* \xrightarrow{\sim} G(A,A')\\
&= F(A,A') \xrightarrow{\sim} F\iota A \otimes (F\iota A')^* \xrightarrow{\sim} F\iota A \otimes (F(\iota A')^*)^{**} \xrightarrow{\alpha_{\iota A} \otimes (\alpha_{(\iota A')^*})^{**}} G\iota A \otimes (G(\iota A')^*)^{**}\\
&\qquad \xrightarrow{\sim} G\iota A \otimes (G\iota A')^* \xrightarrow{\sim} G(A,A')\\
&= F(A,A') \xrightarrow{\sim} F\iota A \otimes (F(\iota A')^*) \xrightarrow{\alpha_{\iota A} \otimes (\alpha_{(\iota A')^*})} G\iota A \otimes (G\iota A')^* \xrightarrow{\sim} G(A,A')\\
&= F(A,A') \xrightarrow{\alpha_{(A,A')}} G(A,A')
\end{aligned}
$$

by expanding the explicit description of the inverse of $\alpha_{\iota A'}$ given by Proposition 3.2, and using the monoidalness of α at the last step.

From this we conclude that restriction along ι is an equivalence of categories between $\mathbf{Comp}(\mathbf{Atemp}_{\mathcal{C}}, \mathcal{D})$ and $\mathbf{SymMon}_g(\mathcal{C}, \mathcal{D})$, making ι a *biuniversal arrow*. It follows abstractly [5, Theorem 9.17] that **Atemp** assembles into a bifunctor $\mathbf{Atemp} : \mathbf{SymMon}_g \to \mathbf{Comp}$, left biadjoint to the inclusion. □

Remark 3.4. The restriction to $\mathbf{SymMon}_g$ with natural isomorphisms as 2-cells is necessary. As pointed out by [8], this assumption is erroneously missing from the similar theorem given in [12].

References

[1] Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore & Marin Román (2024): *Profunctor optics: a categorical update*. *Compositionality* 6(1), pp. 1–39, doi:10.32408/compositionality-6-1.

[2] Geoffrey S. H. Cruttwell, Bruno Gavranović, Neil Ghani, Paul Wilson & Fabio Zanasi (2022): *Categorical Foundations of Gradient-Based Learning*. In Ilya Sergey, editor: *Programming Languages and Systems*, Springer International Publishing, Cham, pp. 1–28, doi:10.1007/978-3-030-99336-8_1.

[3] B. J. Day (1977): *Note on compact closed categories*. *J. Austral. Math. Soc. Ser. A* 24(3), pp. 309–311, doi:10.1017/s1446788700020334.

[4] Emilio Faro (2008): *On the trace of an endofunctor of a small category*. Technical Report. Available at `https://ddd.uab.cat/record/44451`.

[5] Thomas M. Fiore (2006): *Pseudo limits, biadjoints, and pseudo algebras: categorical foundations of conformal field theory*. *Mem. Amer. Math. Soc.* 182(860), pp. x+171, doi:10.1090/memo/0860.

[6] Brendan Fong & Michael Johnson (2019): *Lenses and Learners*. In: *Proceedings of the 8th International Workshop on Bidirectional Transformations (Bx 2019)*, pp. 16–29. Available at `https://ceur-ws.org/Vol-2355/paper2.pdf`.

[7] Brendan Fong, David Spivak & Rémy Tuyéras (2019): *Backprop as functor: a compositional perspective on supervised learning*, doi:10.1109/LICS.2019.8785665.

[8] Masahito Hasegawa & Shin-ya Katsumata (2010): *A note on the biadjunction between 2-categories of traced monoidal categories and tortile monoidal categories*. *Math. Proc. Cambridge Philos. Soc.* 148(1), pp. 107–109, doi:10.1017/S0305004109002606.

[9] J. Hedges (2017): *Coherence for lenses and open games*. arXiv:1704.02230.

[10] James Hefford & Cole Comfort (2023): *Coend Optics for Quantum Combs*. In: *Proceedings of the Fifth International Conference on Applied Category Theory (ACT2022)*, *Electron. Proc. Theor. Comput. Sci. (EPTCS)* 380, EPTCS, pp. 63–76, doi:10.4204/EPTCS.380.4.

[11] André Joyal & Ross Street (1993): *Braided tensor categories*. *Adv. Math.* 102(1), pp. 20–78, doi:10.1006/aima.1993.1055.

[12] André Joyal, Ross Street & Dominic Verity (1996): *Traced monoidal categories*. *Math. Proc. Cambridge Philos. Soc.* 119(3), pp. 447–468, doi:10.1017/S0305004100074338.

[13] P. Katis, Nicoletta Sabadini & Robert F. C. Walters (2002): *Feedback, trace and fixed-point semantics*. 36, pp. 181–194, doi:10.1051/ita:2002009. Fixed points in computer science (Florence, 2001).

[14] G. M. Kelly & M. L. Laplaza (1980): *Coherence for compact closed categories*. *J. Pure Appl. Algebra* 19, pp. 193–213, doi:10.1016/0022-4049(80)90101-2.

[15] Mitchell Riley (2018): *Categories of Optics*. arXiv:1809.00738.

[16] Mitchell Riley & The CT Zulip Community (2021): *An involution on 'Learn'*. Available at `https://mattecapu.github.io/ct-zulip-archive/stream/229156-practice:-applied-ct/topic/An.20involution.20on.20'Learn'.html#235034768`.