The Grothendieck construction for delta lenses

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Delta lenses are an algebraic structure used to model bidirectional transformations, and directly generalise split opfibrations. The classical Grothendieck construction establishes an equivalence between split opfibrations and functors into the category of small categories. In this talk, we introduce a variant of the Grothendieck construction which establishes an equivalence between delta lenses and certain lax double functors into the double category of sets, functions, and split multi-valued functions. The proof follows abstractly from the universal property of the left-connected completion of a double category with companions, applied to the double category of sets, functions, and spans. We demonstrate several advantages of this fibred approach to delta lenses, and provide a new characterisation of split opfibrations.

Context and Motivation Delta lenses were first introduced in 2011 by Diskin, Xiong, and Czarnecki [22] as a generalisation of the classical "state-based" lenses [24, 34]. The original motivation for delta lenses was as an algebraic framework for bidirectional transformations [1, 17], especially in model-driven engineering [21, 38], and they have since been applied to areas such as triple graph grammars [4], supervised learning [20], and optimal transport [37]. The study of delta lenses using category theory began with the work of Johnson and Rosebrugh [28], and this has led to an ongoing programme in the applied category theory community to understand their properties [8, 9, 12, 14, 18, 19].

A delta lens is a pair (f, φ) consisting of a functor $f: A \to B$ equipped with a lifting operation that provides, for each object $a \in A$ and morphism $u: fa \to b$ in B, a morphism $\varphi(a,u): a \to a'$ in A, called a chosen lift, such that $f\varphi(a,u) = u$. The lifting operation is required to preserve identities and composition of morphisms, and the intrinsic "functorial" nature of this lifting operation has resulted in numerous characterisations of delta lenses in the literature. For example, delta lenses are: compatible functor and cofunctor pairs [2, 9], coalgebras for a comonad [11], algebras for a monad [14], and the right class of an algebraic weak factorisation system [14, 15]. Each of these approaches reveals a unique perspective on the theory of delta lenses while also suggesting new ways to construct examples.

Split opfibrations are delta lenses such that the chosen lifts are opcartesian [28], and thus a delta lens may be understood capturing the underlying structure of a split opfibration. Given the close relationship between these concepts, it is not surprising that many results regarding delta lenses are drawn from the analogous results for split opfibrations. For example, split opfibrations are: algebras for a monad [29, 39], the right class of an algebraic weak factorisation system [6, 25], and coalgebras for a comonad [23]. Conversely, the development of delta lenses has led to new characterisations of split opfibrations [10].

The Grothendieck construction is widely recognised as among of the most important concepts in category theory. First introduced by Grothendieck [26], the construction describes a correspondence between functors from a category B into $\mathcal{C}\mathrm{at}$, and split opfibrations over B. There have since been numerous variants and generalisations of the Grothendieck construction introduced for different purposes [5, 7, 16, 27, 30, 31, 32, 33]. Drawing upon the connection between delta lenses and split opfibrations, it becomes natural to wonder: is there a Grothendieck construction for delta lenses?

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Outline and Contributions In this talk, we will describe an equivalence between delta lenses *into* a category B, and lax double functors *out* of a double category $\mathbb{L}\mathrm{o}(B)$.

To state the theorem, we introduce the necessary notation. Let $\mathcal{L}\mathrm{ens}_B$ denote the category whose objects are delta lenses with codomain B. Let $\mathbb{L}\mathrm{o}(B)$ denote the double category whose objects and loose morphisms come from the objects and morphisms of B, and whose tight morphisms and cells are identities. Let $\mathbb{S}\mathrm{Mult}$ denote the double category whose objects are sets, whose tight morphisms are functions, and whose loose morphisms are *split multi-valued functions*, that is, spans of sets whose left leg have a chosen section. Finally, let $[\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}$ denote the category of lax double functors from $\mathbb{L}\mathrm{o}(B)$ to $\mathbb{S}\mathrm{Mult}$ and tight natural transformations between them.

Theorem. *There is an equivalence $\mathcal{L}\mathrm{ens}_B \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}$ between the category of delta lenses with codomain B, and the category of lax double functors from $\mathbb{L}\mathrm{o}(B)$ to the double category of sets, functions, and split multi-valued functions.*

The right-to-left functor $\int : [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}} \to \mathcal{L}\mathrm{ens}_B$ of this equivalence we call the *Grothendieck construction for delta lenses*. To understand the intuition behind this equivalence, consider a lax double functor $F : \mathbb{L}\mathrm{o}(B) \to \mathbb{S}\mathrm{Mult}$ which assigns each morphism $u : x \to y$ in B to a split multi-valued function as follows.

$$\begin{tikzcd} F(x) \arrow[r, tail, shift left, "\varphi_u"] & F[u] \arrow[l, two heads, shift left, "s_u"] \arrow[r, "t_u"] & F(y) \end{tikzcd} \qquad\qquad s_u \circ \varphi_u = 1_{F(x)}$$

From this data, we may construct a category $\int F$ whose objects are pairs $(x \in B, a \in F(x))$ and whose morphisms are pairs $(u : x \to y \in B, \alpha \in F[u]) : (x, s_u(\alpha)) \to (y, t_u(\alpha))$. There is a canonical functor $\pi : \int F \to B$ given by projection in the first component, and this admits a delta lens structure, since for each object $(x,a) \in \int F$ and morphism $u : x \to y$ in B, there is a chosen lift $(u, \varphi_u(a)) : (x,a) \to (y, t_u \varphi_u(a))$. The additional data and axioms of a lax double functor with respect to identities and composition ensure that both the category $\int F$ and the delta lens structure are well-defined.

$$\begin{tikzcd} A \arrow[d, "{(f,\varphi)}"'] \\ B \end{tikzcd} \quad\simeq\quad \begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f\varphi"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \quad\simeq\quad \begin{tikzcd} & \mathbb{L}\mathrm{o}(B) \arrow[dl, "F_1"'] \arrow[dr, "F_2"] \arrow[d, phantom, "\overset{\Phi}{\Longrightarrow}"] & \\ \mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et}) \arrow[rr, "(-)_*"'] & & \mathbb{S}\mathrm{pan} \end{tikzcd} \quad\simeq\quad \begin{tikzcd} \mathbb{L}\mathrm{o}(B) \arrow[d, "F"] \\ \mathbb{S}\mathrm{Mult} \end{tikzcd} \tag{1}$$

Rather than establishing this equivalence explicitly, we provide an abstract proof via a sequence of equivalences depicted above. First, we recall the equivalence between a delta lens $(f, \varphi) : A \to B$ and a commutative diagram of functors, as shown in (1), such that φ is identity-on-objects and $f\varphi$ is a discrete opfibration [9]. Next, we recall the equivalence $\mathcal{C}\mathrm{at}/B \simeq [\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{pan}]_{\mathrm{lax}}$ between ordinary functors into B and lax double functors from $\mathbb{L}\mathrm{o}(B)$ to the double category $\mathbb{S}\mathrm{pan}$ of sets, functions, and spans [3, 35, 36], and use this to establish the equivalence, shown in the middle of (1), where $(-)_* : \mathbb{S}\mathrm{q}(\mathcal{S}\mathrm{et}) \to \mathbb{S}\mathrm{pan}$ is the strict double functor assigning each function to its *companion* span. Finally, we show that $\mathbb{S}\mathrm{Mult}$ is the *left-connected completion* of the double category $\mathbb{S}\mathrm{pan}$, and use the universal property of this completion to establish the right-most equivalence in (1).

There are numerous benefits to viewing delta lenses as lax double functors into $\mathbb{S}\mathrm{Mult}$. We demonstrate how various classes of delta lenses may be characterised by factoring a lax double functor $\mathbb{L}\mathrm{o}(B) \to \mathbb{S}\mathrm{Mult}$ through a double functor $\mathbb{D} \to \mathbb{S}\mathrm{Mult}$, for some double category $\mathbb{D}$. We provide a new perspective on split opfibrations as lax double functors $\mathbb{L}\mathrm{o}(B) \to \mathbb{S}\mathrm{Mult}$ with a certain *property*. Several monoidal structures on the category $\mathcal{L}\mathrm{ens}_B$ also shown to arise naturally from monoidal structures on the category $[\mathbb{L}\mathrm{o}(B), \mathbb{S}\mathrm{Mult}]_{\mathrm{lax}}$. In future work, we hope that our approach to delta lenses as *displayed categories* [3] with additional structure leads to new applications of delta lenses in type-theoretic settings.

Further reading The content of this extended abstract will be further developed in a forthcoming preprint with the same title, and is based on Chapter 4 of my PhD thesis [13].

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