Monoidal Streams and Probabilistic Dataflow with DisCoPy

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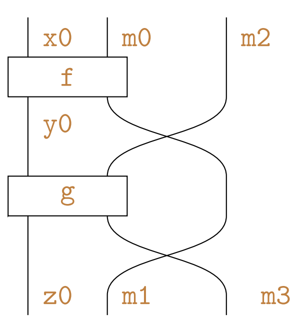
Monoidal streams give semantics to signal flow diagrams over any base monoidal category. Over the category of functions, they give rise to causal stream functions used in dataflow programming. Over categories of probabilistic functions, they give rise to different kinds of controlled stochastic processes. Here, we report on an implementation of monoidal streams in DisCoPy, the Python library for computing with string diagrams. The tool allows to specify dataflow programs in the syntax of Markov categories with delayed feedback. It then performs the semantic evaluation of these programs by unrolling their time-evolution in the chosen semantic category. Example uses include computing the Fibonacci sequence, sampling from a random walk or simulating quantum channels with memory.

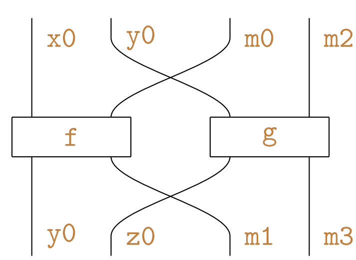
Extended Abstract

DisCoPy [3] is a Python toolkit for computing with *string diagrams*, a graphical calculus for representing sequential and parallel composition of processes. The library allows for string diagrams to be defined either as algebraic formulae or as Python programs. They can then be plotted, compared, rewritten and evaluated as code, be it for a quantum circuit, a probabilistic program or a neural network. In many cases however, we are not interested in one fixed process but in a family of processes, e.g. circuits indexed by the size of their input or communication protocols indexed by a discrete time step.

Monoidal streams [4] are infinite families of processes where each process may depend on the output of the previous one. Formally, for a symmetric monoidal category $\mathbf{C}$ and three countable sequences of objects $X, Y, Z \in \mathrm{Ob}(\mathbf{C})^{\mathbb{N}}$ we define $\mathbf{Stream}(\mathbf{C})(X,Y,Z) = \mathbf{C}(X_0 \otimes Z_0, Y_0 \otimes Z_1) \times \mathbf{Stream}(\mathbf{C})(X^+, Y^+, Z^+)$ i.e. a monoidal stream[1] $f : X \to Y$ with $\texttt{memory}(f) = Z$ is a process $\texttt{now}(f) : X_0 \times Z_0 \to Y_0 \times Z_1$ and a monoidal stream $\texttt{later}(f) : X^+ \to Y^+$ with $\texttt{memory}(\texttt{later}(f)) = Z^+ = (Z_1, Z_2, \dots)$. This gives a symmetric monoidal category $\mathbf{Stream}(\mathbf{C})(X,Y) = \coprod_{Z \in \mathrm{Ob}(C)^{\mathbb{N}}} \mathbf{Stream}(\mathbf{C})(X,Y,Z)$ where:

- $X \otimes Y = (X_0 \otimes Y_0, X_1 \otimes Y_1, \dots)$ and $\texttt{memory}(f \circ g) = \texttt{memory}(f \otimes g) = \texttt{memory}(f) \otimes \texttt{memory}(g)$
- $\texttt{now}(f \circ g)$ and $\texttt{now}(f \otimes g)$ are given by the following composition in $\mathbf{C}$:





[1] Here we define *intentional* streams which can later be quotiented by *extensional* and *observational* equivalence.

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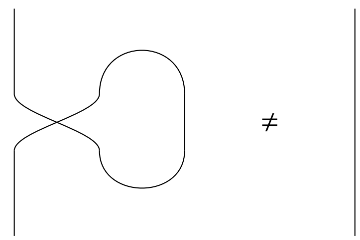
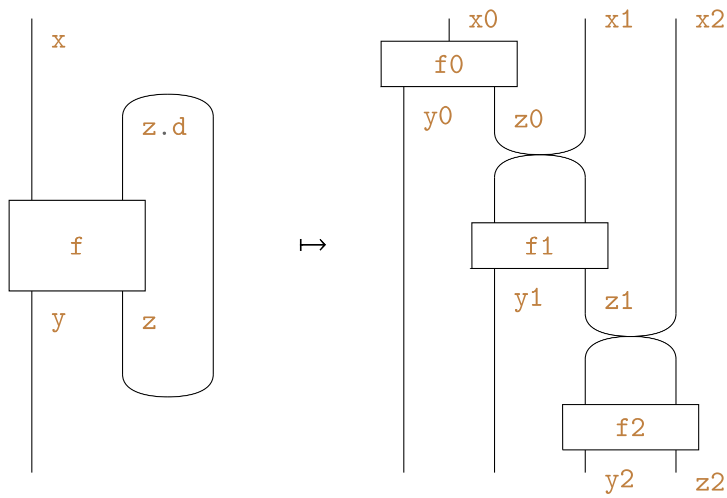
- $\texttt{later}(f \circ g) = \texttt{later}(f) \circ \texttt{later}(g)$ and $\texttt{later}(f \otimes g) = \texttt{later}(f) \otimes \texttt{later}(g)$

The category of monoidal streams comes with a *delayed feedback*, [6] i.e. an endofunctor δ called `delay` and an operation from $f : X \otimes \delta(Z) \to Y \otimes Z$ to $\texttt{feedback}_Z(f) : X \to Y$ which satisfies all the axioms of a traced monoidal category [5] except *yanking*, i.e. in general the feedback of a swap is not the identity.



Thus, we can take monoidal functors from the free category with delayed feedback over a monoidal signature (where the morphisms are string diagrams with feedback loops) to the category of monoidal streams. In effect, we are unrolling the feedback loop:



When we take $\mathbf{C} = \mathbf{Set}$ the category of sets and functions, we get an implementation of *dataflow programming*. For instance, in Appendix A we implement the Fibonacci sequence as a feedback diagram together with a functor into the category of monoidal streams of Python functions. DisCoPy implements *linear type systems* inside of Python which follow the hierarchy of graphical languages for monoidal categories [8]. That is, it can either prevent variable copying and deleting or make it explicit as structural morphisms in a *copy-discard category* [2]. In particular, this allows to implement streams of probabilistic processes e.g. in $\mathbf{C} = \mathbf{Stoch}$ the category of measurable spaces and Markov kernels. This gives an implementation of *probabilistic dataflow programming* which we showcase in Appendix B.

Another natural application would be to build upon the quantum computing features of DisCoPy [7] to implement quantum dataflow programming [1]. We also plan to develop heuristics to simplify dataflow programs via string diagram rewriting based on DisCoPy's hypergraph data structure.

References

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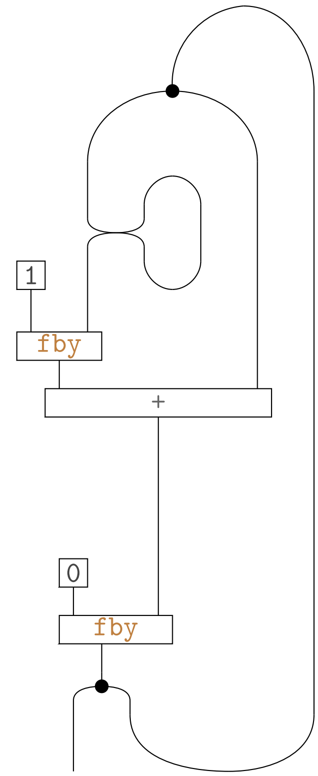
A The Fibonacci sequence as a feedback diagram

```
from discopy.feedback import *

Diagram.use_hypergraph_equality = True

X = Ty('X')
fby, wait = FollowedBy(X), Swap(X, X.d).feedback()
zero, one = Box('0', Ty(), X), Box('1', Ty(), X)
copy, plus = Copy(X), Box('+', X @ X, X)

@Diagram.feedback
@Diagram.from_callable(X.d, X @ X)
def fib(x):
    x = fby(zero.head(), plus.d(
      fby.d(one.head.d(), wait.d(x)), x))
    return (x, x)

assert fib == (copy.d >> one.head.d @ wait.d @ X.d
                      >> fby.d @ X.d
                      >> plus.d
                      >> zero.head @ X.d
                      >> fby >> copy).feedback()

F = Functor(
    ob={x: int},
    ar={zero: lambda: 0,
        one: lambda: 1,
        plus: lambda x, y: x + y},
    cod=stream.Category(python.Ty, python.Function))

assert F(fib).unroll(10).now() == (0, 1, 1, 2, 3, 5, 8, 13, 21, 34)
```



B A random walk as a feedback diagram

```
from random import choice, seed; seed(420)
from discopy import stream, python
from discopy.feedback import *

x, fby = Ty('x'), FollowedBy(Ty('x'))
zero, rand, plus = Box('0', Ty(), x), Box('rand', Ty(), x), Box('+', x @ x, x)

@Diagram.feedback
@Diagram.from_callable(x.d, x @ x)
def walk(x0):
    x1 = plus.d(rand.d(), x0)
    x2 = fby(zero.head(), x1)
    return (x2, x2)

F = Functor(
    ob={x: int},
    ar={zero: lambda: 0,
        rand: lambda: choice([-1, +1]),
        plus: lambda x, y: x + y},
    cod=stream.Category(python.Ty, python.Function))

assert F(walk).unroll(10).now() == [0, -1, 0, 1, 2, 1, 0, -1, 0, 1]
assert F(walk).unroll(10).now() == [0, -1, 0, 1, 2, 1, 2, 3, 2, 1]
assert F(walk).unroll(10).now() == [0, 1, 0, 1, 0, -1, 0, -1, 0, -1]
```