Partial Combinatory Algebras for Intensional Type Theory

Sam Speight

University of Birmingham, UK

Realizability over partial combinatory algebras

An important class of models for the meta-theoretic study of type theory comes from realizability. Not only can these models be used to show consistency of constructive principles (eg. Church's thesis, which is valid in Hyland's *effective topos* [9]), but they are also able to interpret polymorphism or impredicative universes in dependent type theory [10].

Traditionally, the starting point for a realizability interpretation is a *partial combinatory algeba* (PCA). A PCA embodies a notion of untyped (or unityped) computation (the untypedness is actually necessary for impredicativity [3, 14, 12]). Formally, a PCA consists of a set $\mathbb{A}$ and a partial "application" operation $(-) \cdot (?) : \mathbb{A} \times \mathbb{A} \rightharpoonup \mathbb{A}$. Additionally, there must exist particular elements ("combinators") obeying certain laws. Most often one sees the combinators k and s satisfying:

$$\mathsf{k}ab = a \qquad\qquad \mathsf{s}ab\downarrow \quad \text{and} \quad \mathsf{s}abc = ac(bc)$$

(we surpress the application symbol and associate to the left). The existence of these combinators is enough to guarantee *combinatorial completeness*, which means that every "polynomial" over $\mathbb{A}$ (built up from variables and elements of $\mathbb{A}$ using application) is represented by some "code" (element of $\mathbb{A}$) [6]. In this way, a PCA can mimic λ-abstraction, which, together with application, satisfies the β-law.

Among the first PCAs one encounters are the λ-calculus Λ, "Kleene's first algebra" $\mathcal{K}_1$ and categorical models of the λ-calculus (reflexive objects in cartesian closed categories). Λ is the set of λ-terms modulo β together with the application of the λ-calculus. The underlying set of $\mathcal{K}_1$ is $\mathbb{N}$ and application is: $n \cdot m := \{n\}(m)$, ie. the result of applying the n^{th} partial computable function to m.

Realizability for intensional type theory

With the advent of *homotopy type theory*, in the context of *intensional type theory* (ITT), evidence (a proof term) for an identification may be thought of as a path between points in some space [18]. Insofar as realizability interpretations formalize the BHK interpretation (in that realizers play the role of evidence for propositions), one might think that—in the context of ITT—realizers should carry higher-dimensional (categorical, homotopical) structure.

In this spirit, Angiuli and Harper have formulated a cubical generalization of *Martin-Löf's meaning explanations* [1]. Related to this is *higher-dimensional (cubical) computational type theory*, which can be seen as a realizability model of cubical type theory [2]. The starting point here is a cubical programming language that has sorts for dimensions and terms. Terms, which may contain free dimension names, can be seen as abstract cubes.

On the categorical side, [15] studies a groupoidal generalization of *partitioned assemblies*. Realizers derive from a *realizer category* $\mathbf{R}$ containing an interval (co-groupoid) $\mathbb{I} \in \mathbf{R}$. The interval furnishes a fundamental groupoid construction $\Pi : \mathbf{R} \to \mathbf{Gpd}$. A partitioned assembly has an underlying groupoid, whose objects are realized by points in the fundamental groupoid ΠA of some "realizer type" $A \in \mathbf{R}$ and whose morphisms are realized by paths ΠA. If the realizer type is always some fixed *universal* object U, the notion of realizability in untyped. An alternative approach is to consider higher-dimensional structures in traditional realizability models of *extensional* type theory, eg. *cubical assemblies* (cubical objects internal to the model of extensional type theory in assemblies over $\mathcal{K}_1$) [17, 16].

Partial combinatory algebras in groupoids

The notion of PCA makes sense in any *cartesian restriction category* (CRC; restriction categories formalize the idea of categories containing partial maps) [4]. **The goal of this work is to give examples of PCAs in CRCs of groupoids that may be used for constructing realizability models of ITT.**

The first example we give is a *higher-dimensional λ-calculus*, very much inspired by *cubical type theory* [5]. In fact, different calculi could be formulated depending on the notion of "shape" (eg. globular, cubical, etc.). For simplicity, we discuss a relatively simple 1-dimensional globular λ-calculus. Judgements in this calculus are of the form

$$\Psi \mid \Gamma \vdash t$$

where Ψ is a context of dimension variables and Γ is a context of regular variables. We have constants:

$$\cdot \mid \cdot \vdash 0 \qquad\qquad \cdot \mid \cdot \vdash 1$$

As well as the usual rules for λ-abstraction, application and β (uniform in dimension context), we have rules for composition, identities and inverses. For example:

$$\frac{i \mid \Gamma \vdash \alpha \qquad i \mid \Gamma \vdash \beta \qquad \cdot \mid \Gamma \vdash \beta[0/i] = \alpha[1/i]}{i \mid \Gamma \vdash \beta \circ \alpha}\ \text{comp}$$

Identities are obtained by weakening the dimension context. These term constructors satisfy the usual groupoid equations, ensuring that we obtain a groupoid $\Pi\Lambda$ with:

- objects: terms (in context, up to α-equivalence) of the form $\cdot \mid \Gamma \vdash t$;
- morphisms $(\cdot \mid \Gamma \vdash t) \to (\cdot \mid \Gamma \vdash u)$: terms $i \mid \Gamma \vdash \alpha$ satisfying $\cdot \mid \Gamma \vdash \alpha[0/i] = t$ and $\cdot \mid \Gamma \vdash \alpha[1/i] = u$;
- composition, identities and inverses given by the corresponding term constructors.

The groupoid $\Pi\Lambda$ is a PCA in the category of groupoids and functors (with the trivial restriction structure). The application functor is given by application of terms (given how substitution behaves and the various term constructors interact) and the combinators k and s are determined respectively by:

$$\cdot \mid \cdot \vdash \lambda xy.\ x \qquad\qquad \cdot \mid \cdot \vdash \lambda fgx.\ fx(gx)$$

Moving on, there is a class of examples coming from *2-dimensional models of the λ-calculus*, ie. cartesian closed bicategories $\mathcal{C}$ with a pseudoreflexive object U. Instances of these include *generalised species of structures* [7], *profunctorial Scott semantics* [8] and *categorified relational* ("distributors-induced") [13] and *graph* models [11]); realizer categories $(\mathbf{R}, \mathbb{I}, U)$ as discussed above also gives rise to such structures. The carrier of the (total) PCA is the groupoid $\mathcal{C}(1, U)$. This results in a "pseudo PCA", where the combinator laws hold up to isomorphism.

Further work-in-progress is to establish a groupoidal analogue of $\mathcal{K}_1$ based on a notion of *partial recursive functor over the groupoid of finite sets and bijections*. This will live in the CRC of groupoids and *partial* functors (with non-trivial restriction structure).

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