Categorified Path Calculus

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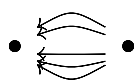
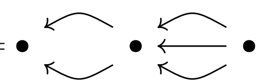
Abstract

Path calculus, or graphical linear algebra, is a string diagram calculus for the category of matrices over a base ring. It is the usual string diagram calculus for a symmetric monoidal category, where the monoidal product is the direct sum of matrices. We categorify this story to develop a surface diagram calculus for the bicategory of matrices over a base bimonoidal category. This yields a surface diagram calculus for any bimonoidal category by restricting to diagrams for 1×1 matrices. We show how additional structure on the base category, such as biproducts, duals and the dagger, adds structure to the resulting calculus. Applied to categorical quantum mechanics this yields a new graphical proof of the teleportation protocol.

Unifying the old and new quantum

What is an amplitude? Feynman's account of quantum mechanics epitomizes the particle physicist's conception of amplitudes: "counting" the paths from a source to a detector ([FLS65] §3.1). The rules for such path counting are (i) parallel paths add, and (ii) serial paths multiply:

$2 + 3 =$  $= 5,$ $\quad 2 \times 3 =$  $= 6.$

This is a cartesian, or classical, viewpoint on amplitudes, and would correspond to a classical dynamical system with uncertainty such as the Galton board, if it wasn't for the Born rule. *Path calculus* or *Graphical linear algebra* neatly encapsulates this correspondence between linear algebra and path counting as a *bimonoid in a monoidal category* [Sob, DFC22].

Around the same time Feynman was giving his lectures, John Bell was initiating the study of non-locality in quantum physics, otherwise known as entanglement [Bel64]. This also has a categorical interpretation as adjointness (duals) in a compact closed category [AC04]. Perhaps this is why particle physicists (before Bell) didn't notice entanglement: infinite dimensional Hilbert spaces don't have adjoints (duals).

There is a fundamental incompatibility here between the (bi-)cartesian monoidal structure of the *old quantum*, and the multiplicative (tensor) monoidal structure of the *new quantum*. In this work we resolve this conflict by *categorification* or *2-linear algebra* [KV94, Bae97]. We consider this work as part of 2-categorical quantum mechanics initiated by Vicary [Vic12].

Categorified rigs are also known as 2-rigs, or rig categories, or bimonoidal categories [Lap72, Kel74], or tight bimonoidal categories [JY21]. This last reference [JY21] contains much of the heavy lifting upon which this work is based.

References

[AC04] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004.*, pages 415–425. IEEE, 2004.

[Bae97] John C Baez. Higher-dimensional algebra II. 2-Hilbert spaces. *Advances in Mathematics*, 127(2):125–189, 1997.

[Bel64] On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.

[DFC22] Giovanni De Felice and Bob Coecke. Quantum linear optics via string diagrams. *arXiv preprint arXiv:2204.12985*, 2022.

[FLS65] Richard P Feynman, RB Leighton, and M Sands. Lectures on Physics, volume III. *Quantum mechanics*, 1965.

[JY21] Niles Johnson and Donald Yau. Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory. *arXiv preprint arXiv:2107.10526*, 2021.

[Kel74] G Maxwell Kelly. Coherence theorems for lax algebras and for distributive laws. In *Category seminar*, pages 281–375. Springer, 1974.

[KV94] Mikhail M Kapranov and Vladimir A Voevodsky. 2-categories and zamolodchikov tetrahedra equations. In *Proc. Symp. Pure Math*, volume 56, pages 177–260, 1994.

[Lap72] Miguel L Laplaza. Coherence for distributivity. In *Coherence in categories*, pages 29–65. Springer, 1972.

[Sob] Pawel Sobocinski. Graphical linear algebra. *https://graphicallinearalgebra.net.*

[Vic12] Jamie Vicary. Higher semantics of quantum protocols. In *2012 27th Annual IEEE Symposium on Logic in Computer Science*, pages 606–615. IEEE, 2012.