Categorical methods for a homotopical characterization of contextuality

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Bell's notion of non-locality and its generalization quantum contextuality are fundamental features of quantum theory [1, 2]. These inherently non-classical phenomena have attracted intensive study from diverse perspectives. The recent simplicial approach introduced by the author, detailed in [3], encompasses two prior significant methods: the sheaf-theoretic framework by Abramsky and Brandenburger [4] and the cohomological framework by Okay et al. [5]. This simplicial framework represents measurements and outcomes as spaces, depicting contextuality as a topological phenomenon. A measurement space or an outcome space is represented by a simplicial set X, that is, a sequence $X_0, X_1, \cdots, X_n, \cdots$ of sets representing the n-dimensional simplices together with the simplicial structure maps which tell us how these simplices are glued together. Among them face maps encode the *topological non-signaling conditions*. Our main result is a classification result for contextuality. We first describe our result and then the categorical machinery used to prove it.

A simplicial distribution on a pair (X, Y) of simplicial sets is a simplicial set map

$$p : X \to D(Y)$$

where D is the distribution monad that replaces the set of simplices with the set of probability distributions on those simplices. A simplicial distribution amounts to a collection of distributions $p_\sigma \in D(Y_n)$ for each non-degenerate simplex $\sigma \in X_n$ related by the simplicial structure maps. A simplicial distribution is called deterministic if each of the distributions p_σ's are delta distributions. Every deterministic distribution, written as δ^φ, comes from a simplicial set map $\varphi : X \to Y$. A simplicial distribution is called non-contextual if it can be expressed as a probabilistic mixture of deterministic ones. Otherwise, it is called contextual. There is a notion of support of a given simplicial distribution p that consists of those simplicial set maps $\varphi : X \to Y$ such that the distribution p_σ contains the simplex φ_σ in its support in the usual sense. A simplicial distribution is called strongly contextual if its support is empty.

In this work, we focus on measurement spaces given by the cone of a 1-dimensional simplicial set (directed graph). This topological construction starts with the 1-dimensional object, the graph, and proceeds by adding a disjoint point and connecting all the graph points to the cone point. For example, the cone of a graph in the shape of a circle would be a triangulated disk. Given a 1-dimensional simplicial set X the simplicial set model for the cone is denoted by $\Delta^0 * X$. Here, the cone point is represented by Δ^0. For the outcome space we take the nerve of a group. More specifically, for the binary outcome set $\mathbb{Z}_2 = \{0, 1\}$, regarded as the additive group of integers mod 2, our outcome space is the nerve $N\mathbb{Z}_2$. The set of simplices of this object consists of n-tuples of group elements, and the simplicial structure uses the group structure. Nerve spaces are common objects in algebraic topology and are used to classify principal bundles. Our main result is the following classification result:

Theorem. *Let X be a 1-dimensional simplicial set. A simplicial distribution*

$$p : \Delta[0] * X \to D(N\mathbb{Z}_2)$$

is strongly contextual if and only if there exists a circle $C \subset X$ such that the restriction $p|_C$ is a deterministic distribution δ^φ for some simplicial set map $\varphi : C \to N\mathbb{Z}_2$ that is not null-homotopic.

The main construction behind this proof is a category associated with a simplicial distribution p on the cone space. First, we use an adjunction known as the cone and the décalage adjunction to represent the simplicial distribution directly on the 1-dimensional simplicial set X. In this representation p consists of 2×2 matrices over $\mathbb{R}_{\geq 0}$ whose entries sum to 1:

$$p_\sigma = \begin{pmatrix} p_\sigma^{00} & p_\sigma^{01} \\ p_\sigma^{10} & p_\sigma^{11} \end{pmatrix}.$$

When there is a common vertex the matrices on the incident edges satisfy a compatibility condition induced by non-signaling. For example, consider two edges σ and τ incident to a vertex y:

$$x \xrightarrow{\sigma} y \xrightarrow{\tau} z.$$

The non-signaling condition indicates that the sum of columns of p_σ matches the sum of rows of p_τ. This marginal distribution is associated with the vertex y, denoted by p_y. Then, we construct a category whose objects are the vertices of X and whose morphisms are freely generated by the 2×2 matrices associated with each edge of X under a rescaled matrix multiplication. The category contains inverses for each edge, for which the associated matrix is transposed. For two matrices p_σ and p_τ marginalizing to p_y on the common vertex the rescaled matrix product is defined as follows:

$$(p_\sigma \circ p_\tau)^{ab} = \begin{cases} \sum_{c \in \mathbb{Z}_2} p_\sigma^{ac} p_\tau^{cb} / p_y^c & \text{if } p_y^c \neq 0 \\ 0 & \text{if } p_y^c = 0. \end{cases}$$

To prove that the resulting composition is associative, we first observe that our outcome space is a compository in the sense of [6]. Our proof of the main theorem comes from studying the endomorphisms of objects in this category.

The characterization result is new and brings forth novel features of simplicial distributions. Our categorical methods extend to an arbitrary number of outcomes. More sophisticated categorical methods may be used to extend this result on the cones to arbitrary 2-dimensional measurement spaces. Cone spaces include many interesting physical examples such as the Clauser–Horne–Shimony–Holt scenario [7] and its generalization cycle scenarios [8]. However, there are also interesting scenarios that are not in the form of a cone space. For example, the Mermin scenario [9] has a torus as its measurement space. On the other hand, characterizations of this form would be useful for studying quantum contextuality as they introduce additional constraints on the structure of the distributions, hence on the structure of quantum measurements.

References

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