A Coalgebraic Model of Quantum Bisimulation

Lorenzo Ceragioli
IMT School for Advanced Studies, Lucca, Italy
lorenzo.ceragioli@imtlucca.it

Elena Di Lavore
University of Pisa, Pisa, Italy
elena.dilavore@di.unipi.it

Giuseppe Lomurno
University of Pisa, Pisa, Italy
giuseppe.lomurno@phd.unipi.it

Gabriele Tedeschi
University of Pisa, Pisa, Italy
gabriele.tedeschi@phd.unipi.it

Recent works have shown that defining a bisimilarity that matches the observational properties of a quantum-capable, concurrent, non-deterministic system is a surprisingly difficult task. We explore coalgebras over distributions taking weights from a generic effect algebra, subsuming probabilistic transition systems. To abide by the properties of quantum theory, we introduce monads graded on a partial commutative monoid, intuitively allowing composition of two processes only if they use different quantum resources, as prescribed by the *no-cloning theorem*. By taking *quantum effects* we therefore characterize the evolution of quantum processes and, implicitly, their probabilistic behaviour. We investigate the relation between an open quantum system and its probabilistic counterparts obtained when instantiating the input with a specific quantum state. We consider Aczel-Mendler and kernel bisimilarities, advocating for the latter as it characterizes quantum systems that exhibit the same behaviour for any input state. Finally, we propose operators on quantum effect labelled transition systems, paving the way for a process calculi semantics that is parametric over the quantum input.

1 Introduction

The recent development of quantum technologies calls for grounded methods for modelling and verifying computing systems that exploit quantum phenomena like superposition and entanglement. In particular, concurrent processes are of main interest, like communication protocols [31, 3, 28, 14] and distributed implementations of algorithms via the *Quantum Internet* [6, 35]. However, there is still no clear notion of behavioural equivalence for such systems, and most of the proposals lack a decision procedure [8].

In [7], some of the authors have proposed a new semantic model for quantum protocols, namely quantum effect labelled transition systems (qLTSs). These use *quantum effects* as weights, in the same way probabilities are used in a probabilistic labelled transition system (pLTS). Coming from quantum measurement theory, quantum effects represent the observable probabilistic properties that may be expressed by a parametric input quantum state. Therefore, qLTSs model quantum processes that are parametric over their quantum input.

In this paper we build upon this concept, characterizing quantum effects and probabilities as effect algebras, of which we investigate their (sub)distributions, and we give effect labelled transition systems ($\mathbb{E}$LTS), a uniform coalgebraic framework encompassing non-deterministic, probabilistic and quantum concurrent systems. The coalgebraic language is well suited to treat dynamical systems in their essential features, and allows us to extend properties and constructs of probabilistic systems to quantum ones. We introduce monads graded on a partial commutative monoid (PCM), allowing us to grade effect distributions over their quantum resources, the copy of which is forbidden by the no cloning theorem [30]. Transition systems built over this PCM intuitively represent quantum computations that consume some quantum resources: only computations using disjoint resources can be composed. Besides, thanks to our peculiar

Submitted to:
ACT 2024

grading, we define a commutative Kronecker product of effects, and thus a commutative multiplication of effect distributions, generalizing the joint distribution operator of the probabilistic case [2].

We investigate kernel and Aczel-Mendler (AM) bisimilarities for $\mathbb{E}$LTSs, and relate them by generalizing previous results. By applying our findings to the quantum setting, we prove that each quantum state ρ defines a functor from qLTSs to pLTSs that "instantiate" a quantum process to the probabilistic behaviour it exhibits when the measured quantum state is ρ. We study which equivalences are preserved and/or reflected by these functors, giving us a categorical way to pinpoint the correct notion of equivalence for quantum systems, i.e. the one that equates all and only the processes that exhibit the same observable probabilistic behaviour. Both AM and kernel bisimilarities correctly relates indistinguishable quantum processes only. However, only the latter is complete and relates all the indistinguishable processes, given that the weights of the qLTS are taken from a finite effect algebra of quantum effects.

Finally, we investigate operators over $\mathbb{E}$LTSs, paving the way for an $\mathbb{E}$LTS semantics of quantum process calculi. We provide a generalized, compositional parallel operator which can model different notions of synchronization (CCS, CSP, ACT) and different kinds of "weights" (nondeterministic, probabilistic or quantum systems). In addition, we introduce a purely quantum operator of "partial evaluation" of qLTSs that instantiates the value of some of the qubits. These operators are defined as functors between $\mathbb{E}$LTSs, thus they preserve bisimilarity. While operators typically act on the final coalgebra, treating them as functors allows for "multi sorted" operators: partial evaluation reduces the resources needed for the computation, while parallel composition joins them by increasing the number of qubits (only if they are compatible, i.e. of different quantum systems). A qLTS semantics for quantum processes would allow to algorithmically decide bisimilarity for any possible input quantum state via standard techniques [24].

Related Works This work builds over [7], generalizing the results in a categorical, coalgebraic setting. Our approach extends the quantum monad of [1], which is based on projectors, a subset of quantum effects. The author in [22] proposes effects monoids, i.e. effect algebras with multiplication, and use them as weights of distributions. Our effects do have tensoring as a multiplication operator, but it does not form a proper effect monoid since it changes the effects dimensions. Our qLTS can be seen as a labelled, non-deterministic version of the effect-valued Quantum Markov Chain of [15], where tensor product is used instead of sequential effect composition. The most general model of "quantum transition system" is the one of [18, 32, 27], where the weights are superoperators instead of effects, so to capture also non-destructive measurements. The author of [32] introduces two different notions of bisimilarity, that we recover in our minimal, effect-based setting as AM and kernel bisimilarity. However, none of these works feature nondeterminism, nor do they investigate operators over labelled transition systems, suitable for modelling quantum protocols, e.g. via process calculi. Usually, works from the process algebra literature [26, 12, 10, 11, 8] define the semantics of quantum processes via pLTSs, and strive to adapt probabilistic bisimilarity to capture the peculiar observable properties of quantum values. The defined pLTSs are made of configurations, i.e. pairs of quantum values and syntactic processes, impeding algorithmic verification of processes when the quantum input is not given (the only symbolic approach [13] has been proved too strict [8]). We instead introduce a purely quantum transition system: we do not represent directly quantum values but only their observable probabilistic features in the form of effects.

2 Background

We recall the definitions of partial commutative monoid and effect algebra. Then, we give some background on quantum computing, and on quantum effects.

2.1 Partial Commutative Monoid and Effect Algebra

Partial commutative monoids obey the properties of commutative monoids, but $+$ is not always defined.

Definition 1. *A partial commutative monoid (PCM) is a tuple $\langle M, 0, +\rangle$ (often referred as M) with $0 \in M$ and $+ : M \times M \to M$ a partial binary operation on M such that for all $a, b, c \in M$ the following hold:*

- *(Commutativity) $a \perp b$ implies $b \perp a$ and $a + b = b + a$;*
- *(Associativity) $b \perp c$ and $a \perp (b + c)$ implies $a \perp b$ and $(a + b) \perp c$ and also $(a + b) + c = a + (b + c)$;*
- *(Zero) $0 \perp a$ and $0 + a = a$.*

Where $a \perp b$ means that a and b are orthogonal, i.e. $a + b$ is defined. A PCM homomorphism is a function $f : M \to N$ on the underlying carrier sets such that $f(0) = 0$, and $a \perp b$ implies $f(a) \perp f(b)$ and $f(a + b) = f(a) + f(b)$. PCMs and their homomorphisms form the category **PCM**.

Every PCM induces a preorder on the carrier set M where $a \preceq b$ if and only if $\exists c \in M. a + c = b$.

Effect algebras are a special kind of PCMs for which an inverse operation is defined.

Definition 2. *An effect algebra [21] is a tuple $\langle \mathbb{E}, 0, +, \cdot' \rangle$ (often referred as $\mathbb{E}$) with $\langle \mathbb{E}, 0, +\rangle$ a PCM and $\cdot' : E \to E$ a unary operation such that, for all $e \in E$:*

- *$e' \in E$ is the unique element in E such that $e + e' = 1$ with $1 = 0'$;*
- *$e \perp 1$ implies $e = 0$.*

An effect homomorphism is a PCM homomorphism that also preserves $\cdot'$. The category **EA** *of effect algebras and effect homomorphisms is the full subcategory of* **PCM** *whose objects are effect algebras.*

Effect algebras have a partial order $\sqsubseteq$ (defined as $\preceq$) and a partial operation $e_1 - e_2$ returning e_3 such that $e_2 + e_3 = e_1$. **EA** is a symmetric monoidal category with $2 = \{0, 1\}$ its unit object. Moreover, there is a bijective correspondence between morphisms $\mathbb{E}_A \otimes \mathbb{E}_B \to \mathbb{E}_C$ and bihomomorphisms $\mathbb{E}_A \times \mathbb{E}_B \to \mathbb{E}_C$ (a bihomomorphism is such that the morphisms obtained by fixing either objects are homomorphisms).

The most common example of effect algebras are probabilities, i.e. real numbers in the unit interval $[0, 1]$ with $+$ the arithmetic sum in $[0, 1]$ and e' defined as $1 - e$. Probabilities allow for defining *probability (sub)distributions* over a given set S, i.e. functions $\Delta : S \to [0, 1]$ such that $\sum_{s \in S} \Delta(s) \leq 1$. For each $s \in S$, we let $\overline{s}$ be the *point distribution* that assigns 1 to s. Given a finite set of non-negatives reals $\{p_i\}_{i \in I}$ such that $\sum_{i \in I} p_i \leq 1$, the weighted sum $\sum_{i \in I} p_i \cdot \Delta_i$ defines a distribution such that $(\sum_{i \in I} p_i \cdot \Delta_i)(s) = \sum_{i \in I} p_i \Delta_i(s)$.

Probability distributions form a *convex set* [5], meaning that for any two distributions Δ, Θ and any real $p \in [0, 1]$ there exists a distribution $\Delta \,{}_p\!\oplus \Theta$ defined as $p \cdot \Delta_1 + (1 - p) \cdot \Delta_2$. Given a function f between convex sets X and Y, we call f convex if it preserves the ${}_p\!\oplus$ operator, i.e. if $f(x_1 \,{}_p\!\oplus x_2) = f(x_1) \,{}_p\!\oplus f(x_2)$. We denote as $\mathbf{Conv}(X, Y)$ the set of convex functions between X and Y.

2.2 Quantum Computing

A (finite-dimensional) *Hilbert space*, denoted as $\mathscr{H}$, is a complex vector space equipped with a binary operator $\langle \cdot | \cdot \rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ called *inner product*, defined as $\langle \psi | \phi \rangle = \sum_i \alpha_i^* \beta_i$, where $|\psi\rangle = (\alpha_1, \ldots, \alpha_i)^T$ and $|\phi\rangle = (\beta_1, \ldots, \beta_i)^T$, with T the transpose. We indicate column vectors as $|\psi\rangle$ and their conjugate transpose as $\langle \psi | = |\psi\rangle^\dagger$. The state of an isolated physical system is represented as a *unit vector* $|\psi\rangle$ (called *state vector*), i.e. a vector such that $\langle \psi | \psi \rangle = 1$. The two-dimensional Hilbert space $\mathbb{C}^2$ is called a *qubit*. The vectors $\{|0\rangle = (1, 0)^T, |1\rangle = (0, 1)^T\}$ form an orthonormal basis of $\mathbb{C}^2$, called the *computational basis*. Other important vectors in $\mathbb{C}^2$ are $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, which form the *Hadamard basis*. Intuitively, different bases represent different observable properties of a quantum system. Note that $|+\rangle$ and $|-\rangle$ are non-trivial linear combinations of $|0\rangle$ and $|1\rangle$, roughly meaning that the property associated with the computational basis is undetermined in $|+\rangle$ and $|-\rangle$. In the quantum jargon, $|+\rangle$ and $|-\rangle$ are *superpositions* with respect to the computational basis. Symmetrically, $|0\rangle$ and $|1\rangle$ are superpositions with respect to the Hadamard one.

We represent the state space of a composite physical system as the *tensor product* of the state spaces of its components. Let $\mathscr{H}$ and $\mathscr{H}'$ be n and m-dimensional Hilbert spaces: their tensor product $\mathscr{H} \otimes \mathscr{H}'$ is an $n \cdot m$ Hilbert space. Moreover, if $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$ and $\{|\phi_1\rangle, \dots, |\phi_m\rangle\}$ are bases of respectively $\mathscr{H}$ and $\mathscr{H}'$, then $\{|\psi_i\rangle \otimes |\phi_j\rangle \mid i = 1, \dots, n, j = 1, \dots, m\}$ is a basis of $\mathscr{H} \otimes \mathscr{H}'$, where $|\psi\rangle \otimes |\phi\rangle$ is the Kronecker product. We often omit the tensor product and write $|\psi\rangle|\phi\rangle$ or $|\psi\phi\rangle$. Note that such product is not commutative. Categorically, finite-dimensional Hilbert spaces with the Kronecker product and the conjugate transpose form the dagger compact category *FDHilb*. Further references are available in [9].

The density operator formalism puts together quantum systems and probability by considering mixed states, i.e. *probability distributions of quantum states*. A point distribution $\overline{|\psi\rangle}$ (called a pure state) is represented by the matrix $|\psi\rangle\langle\psi|$. In general, a probability distribution Δ of n-dimensional states is represented as the matrix $\rho \in \mathbb{C}^{n\times n}$, known as its *density operator*, with $\rho = \sum_i \Delta(\psi_i)\,|\psi_i\rangle\langle\psi_i|$. Recall that a complex matrix N is called *positive semi-definite*, shortly positive, when $\langle\psi|N|\psi\rangle \geq 0$ for any $|\psi\rangle$. The *Löwner order* is the partial order defined by $L \sqsubseteq L'$ whenever $L' - L$ is positive. Given a d-dimensional Hilbert space $\mathscr{H}$, density operators coincide with the positive matrices in $\mathbb{C}^{d\times d}$ of trace one, we denote them as $DM_{\mathscr{H}} = \left\{ \rho \in \mathbb{C}^{d\times d} \mid \rho \sqsupseteq 0_d,\ \mathrm{tr}(\rho) = 1 \right\}$. Density operators form a convex set, where the convex combination operator is defined by $\rho\ {}_p\!\oplus\ \sigma = p\rho + (1-p)\sigma$.

Density operators can describe the state of a subsystem of a composite quantum system. Let $\mathscr{H}_S$ denote the Hilbert space of a physical system S, then $\mathscr{H}_{S_1} \otimes \mathscr{H}_{S_2}$ is the Hilbert space of a composite system with subsystems S_1 and S_2. Given a (not necessarily separable) $\rho \in \mathscr{H}_{S_1} \otimes \mathscr{H}_{S_2}$, the *reduced density operator* of system S_1, $\rho_1 = \mathrm{tr}_{S_2}(\rho)$, describes the state of S_1, with tr_{S_2} the *partial trace over* S_2, defined as the linear transformation such that $\mathrm{tr}_{S_2}(|\psi\rangle\langle\psi'| \otimes |\phi\rangle\langle\phi'|) = |\psi\rangle\langle\psi'|\,\mathrm{tr}(|\phi\rangle\langle\phi'|)$ for each $|\psi\rangle\langle\psi'| \in DM_{\mathscr{H}_{S_1}}$ and $|\phi\rangle\langle\phi'| \in DM_{\mathscr{H}_{S_2}}$. We refer to [30] for further reading on quantum computing.

2.3 Quantum Effects

Quantum measurements are needed for describing systems that exchange information with the environment. Performing a measurement on a quantum state returns a probabilistic classical result and either destroys or otherwise changes the quantum system. We focus in this paper on destructive measurements.

The simplest kind of measurements are *quantum effects* (simply called effects in quantum textbooks [19]), i.e. yes-no tests over quantum systems. Each effect can be represented as a positive matrix smaller than the identity in the Löwner order. We denote the set of effects on a d-dimensional Hilbert space $\mathscr{H}$ as $\mathbb{Q}_{\mathscr{H}} = \left\{ L \in \mathbb{C}^{d\times d} \mid 0_d \sqsubseteq L \sqsubseteq \mathbb{I}_d \right\}$, where $\mathbb{I}_d$ is the $d \times d$ identity matrix. The probability of getting a "yes" outcome when measuring an effect L on a state $|\psi\rangle$ is given by the *Born rule* $tr(L\rho)$. Effects of dimension d form an effect algebra with the matrix sum, $\mathbb{I}_d$ as 1 and $L' = \mathbb{I}_d - L$. Furthermore, the induced partial order $\sqsubseteq$ is exactly the Löwner order.

In general, a measurement with n different outcomes is a set $\{L_1, \dots L_n\}$ of effects, such that the *completeness* equation $\sum_{i=1}^{n} L_i = \mathbb{I}$ holds. If the state of the system is $|\psi\rangle$ before the measurement, then the probability of the i outcome occurring is $p_i = tr(L_i\rho)$.

As examples of measurements, consider M_{01} and $M_{\pm}$ that project a state into the elements of the computational and Hadamard basis of $\mathbb{C}^2$ respectively, with M_{01} defined as $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and $M_{\pm}$ as $\{|+\rangle\langle +|, |-\rangle\langle -|\}$. Applying the measurement M_{01} on $|0\rangle$ returns the outcome associated with $|0\rangle\langle 0|$ with probability 1. When measuring $|+\rangle$, instead, the same result occurs with probability $\frac{1}{2}$. Notice that a measurement for a composite system may measure only some of the qubits, e.g. $\{|0\rangle\langle 0| \otimes \mathbb{I}, |1\rangle\langle 1| \otimes \mathbb{I}\}$ measures (in the computational basis) the first qubit of a pair.

Density operators and effects are dual, as effects are isomorphic to the convex functions from the set of density operators to the probability interval. The isomorphism is given by the Born rule.

Theorem 1. *It holds that $\mathbb{Q}_{\mathscr{H}} \cong \mathbf{Conv}(DM_{\mathscr{H}}, [0,1])$ through the isomorphism $L \mapsto \lambda\rho.tr(L\rho)$ [19].*

Roughly, effects can be considered as probabilities *parametrized* on an unknown quantum state.

3 Effect Distributions

Following the work of [21], for each effect algebra $\mathbb{E}$ we build a functor of effect distributions.

Definition 3 (Effect Distributions)**.** *Given an effect algebra $\langle \mathbb{E}, 0, +, \cdot' \rangle$ we define the functor of effect (sub)distributions $D_{\mathbb{E}} : \mathbf{Set} \to \mathbf{Set}$ by*

$$D_{\mathbb{E}}X = \left\{ \Delta \in \mathbb{E}^X \;\middle|\; \text{supp}(\Delta) \textit{ is finite, } \sum_{x \in \text{supp}(\Delta)} \Delta(x) \sqsubseteq 1_{\mathbb{E}} \right\} \qquad (D_{\mathbb{E}}f)(\Delta) = \lambda y \in Y. \sum_{x \in f^{-1}(y)} \Delta(x)$$

where supp(Δ) is the set $\{ x \in X \mid \Delta(x) \neq 0 \}$, and $\sum$ is the n-ary sum in $\mathbb{E}$.

We will often write an effect distribution $\Delta = \{x_i \mapsto e_i\}_{x_i \in X}$ in a compact form, as $\Delta = \sum_{x_i \in X} e_i \bullet x_i$.

Our running examples will be probability and quantum distributions over some Hilbert space $\mathscr{H}$, i.e. effect distributions associated with the $[0,1]$ and $\mathbb{Q}_{\mathscr{H}}$ effect algebras respectively. If $\mathscr{H}$ is 1 dimensional, the effect algebra $\mathbb{Q}_{\mathscr{H}}$ is isomorphic to $[0,1]$. Moreover, for each $\rho \in DM_{\mathscr{H}}$, $L \mapsto tr(L\rho)$ is an effect algebra homomorphism from $\mathbb{Q}_{\mathscr{H}}$ to $[0,1]$. Intuitively, this homomorphism applies a unique initial input density operator ρ that transforms each effect L to the probability of observing the "yes" outcome when performing the L measurement a quantum system in state ρ.

The following theorem, rephrasing Proposition 21 of [21], guarantees that $L \mapsto tr(L\rho)$ yields a natural transformation from $D_{\mathbb{Q}_{\mathscr{H}}}$ to $D_{[0,1]}$.

Theorem 2. *Each effect morphism $m : \mathbb{E} \to \mathbb{F}$ yields a natural transformation $m \circ \cdot : D_{\mathbb{E}} \Rightarrow D_{\mathbb{F}}$. Moreover, if m is injective then the components of $m \circ \cdot$ are injective.*

We generalize the notion of effect monoid in [22] to the one of graded effect monoid. Instead of grading on a monoid or a monoidal category, we use a partial commutative monoid (PCM) as a grade. This is useful when working with quantum effects, because the no-cloning theorem imposes that quantum data cannot be copied. Since measurements alter the quantum system, it is impossible to perform multiple measurements on the same quantum state. Our grading keeps track of the qubits measured by the effects, and the partial sum reflects that the composition of effects is defined over disjoint systems only.

Definition 4 (Graded Effect Monoid)**.** *Given a partial commutative monoid $\langle M, 0, + \rangle$, an M-graded effect monoid is a graded monoid object in **EA**, that is*

- *for each $m \in M$, an effect algebra $\mathbb{E}_m$*
- *for each $m \perp n \in M$, an effect morphism $\nabla_{m,n} : \mathbb{E}_m \otimes \mathbb{E}_n \to \mathbb{E}_{m+n}$*
- *an effect morphism $\eta : 2 \to \mathbb{E}_0$*

such that the two following diagrams on the left commute

$$\begin{array}{ccc} (\mathbb{E}_m \otimes \mathbb{E}_n) \otimes \mathbb{E}_o & \xrightarrow{\alpha} & \mathbb{E}_m \otimes (\mathbb{E}_n \otimes \mathbb{E}_o) \\ \downarrow{\scriptstyle \nabla_{m,n} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes \nabla_{n,o}} \\ \mathbb{E}_{m+n} \otimes \mathbb{E}_o & & \mathbb{E}_m \otimes \mathbb{E}_{n+o} \\ \downarrow{\scriptstyle \nabla_{m+n,o}} & \swarrow{\scriptstyle \nabla_{m,n+o}} & \\ \mathbb{E}_{m+n+o} & & \end{array} \qquad \begin{array}{ccc} 2 \otimes \mathbb{E}_m & \xrightarrow{\eta \otimes id} & \mathbb{E}_0 \otimes \mathbb{E}_m \\ \downarrow{\scriptstyle \lambda} & \swarrow{\scriptstyle \nabla_{0,m}} & \\ \mathbb{E}_m & & \\ \uparrow{\scriptstyle \rho} & \nwarrow{\scriptstyle \nabla_{m,0}} & \\ \mathbb{E}_m \otimes 2 & \xrightarrow[id \otimes \eta]{} & \mathbb{E}_m \otimes \mathbb{E}_0 \end{array} \qquad \begin{array}{ccc} \mathbb{E}_m \otimes \mathbb{E}_n & \xrightarrow{B_{\mathbb{E}_m,\mathbb{E}_n}} & \mathbb{E}_n \otimes \mathbb{E}_m \\ \downarrow{\scriptstyle \nabla_{m,n}} & \swarrow{\scriptstyle \nabla_{n,m}} & \\ \mathbb{E}_{m+n} & & \end{array}$$

If also the diagram on the right commutes, where B is the braiding natural transformation, then we say that the graded effect monoid is commutative.

Each monoid object can be considered as being graded on the degenerate 1-element monoid $\{0\}$, thus all effect monoids described in [22] are an example of trivially graded effect monoids. Among them, the most typical example is the probability interval $[0,1]$, which constitutes a monoid in **EA** with the usual multiplication and unit.

We grade our effect monoid over the PCM of disjoint sets of quantum systems.

Definition 5 (PCM of Quantum Systems)**.** *Assume a finite set of quantum systems $Sys = \{S_i\}$, each associated with the Hilbert space $\mathscr{H}_{S_i}$ of finite dimension d_i. Let $\mathscr{S} = \langle \mathscr{P}(Sys), \emptyset, \uplus \rangle$ be the PCM where $\mathscr{P}(Sys)$ is the powerset of Sys and $\uplus$ is the partial disjoint union, i.e. $C_i \uplus C_j$ is defined only if $C_i \cap C_j = \emptyset$, and in that case $C_i \uplus C_j = C_i \cup C_j$. We associate each collection of systems $C \in \mathscr{P}(Sys)$ with a Hilbert space defined as $\mathscr{H}_\emptyset = \mathbb{C}$ and $\mathscr{H}_C = \bigotimes_{S \in C} \mathscr{H}_S$, where we impose that the arguments of the (non-commutative) Kronecker product are ordered according to their indices.*

The PCM above grades an effect monoid of quantum effects.

Theorem 3 (Effect Monoid of Quantum Effect)**.** *Quantum effects carry a commutative $\mathscr{S}$-graded effect monoid structure, given by:*

- *The effect algebra $\mathbb{E}_C$ is $\mathbb{Q}_{\mathscr{H}_C}$ for any collection of systems $C \in \mathscr{P}(Sys)$*
- *The operator $\nabla_{C,D} : \mathbb{E}_C \otimes \mathbb{E}_D \to \mathbb{E}_{C \uplus D}$ is denoted as $\boxtimes$ and defined by $L_1 \boxtimes L_2 = Sort_{C,D}(L_1 \otimes_k L_2)$, where $\otimes_k$ is the Kronecker product between $L_1 \in \mathbb{Q}_{\mathscr{H}_C}$ and $L_2 \in \mathbb{Q}_{\mathscr{H}_D}$, and $Sort_{C,D}$ is the unitary transformation which "sorts" the Hilbert Space $\mathscr{H}_C \otimes \mathscr{H}_D$ into $\mathscr{H}_{C \uplus D}$.*
- *The effect morphism $\eta : 2 \to \mathbb{E}_\emptyset$ defined by $\{0 \mapsto 0, 1 \mapsto 1\}$.*

We now introduce *PCM-graded monads*, generalizing the notion of monoid-graded monad of [25].

Definition 6 (Graded monad)**.** *Given a partial commutative monoid $\langle M, 0, + \rangle$, an M-graded monad on* **Set** *is a graded monoid in the category of* **Set***-endofunctors, that is*

- *for each $m \in M$, an endofunctor T_m :* **Set** $\to$ **Set**
- *a natural transformation $\eta : Id \to T_0$ called* unit
- *for each $m \perp n \in M$, a natural transformation $\mu_{m,n} : T_m T_n \to T_{m+n}$ called* multiplication

such that the following diagrams commute

$$\begin{array}{ccc} T_m T_n T_o & \xrightarrow{T_m \mu_{n,o}} & T_m T_{n+o} \\ \downarrow{\scriptstyle \mu_{m,n} T_o} & & \downarrow{\scriptstyle \mu_{m,n+o}} \\ T_{m+n} T_o & \xrightarrow[\mu_{m+n,o}]{} & T_{m+n+o} \end{array} \qquad \begin{array}{ccc} T_m & \xrightarrow{T_m \eta} & T_m T_0 \\ \downarrow{\scriptstyle \eta T_m} & \searrow{=} & \downarrow{\scriptstyle \mu_{m,0}} \\ T_0 T_m & \xrightarrow[\mu_{0,m}]{} & T_m \end{array}$$

We now show that, whenever $\{\mathbb{E}_m\}$ is a graded effect monoid, $\{D_{\mathbb{E}_m}\}$ has a graded monad structure.

Theorem 4 (Graded monads of graded effect monoids)**.** *If $\{\mathbb{E}_m\}$ is an M-graded effect monoid, there is a graded monad $\{D_{\mathbb{E}_m}\}$ with unit $\eta : Id \to D_{\mathbb{E}_0}$ and multiplication $\mu_{m,n} : D_{\mathbb{E}_m} D_{\mathbb{E}_n} \to D_{\mathbb{E}_{m+n}}$ given by*

$$\eta(x) = 1_{\mathbb{E}_0} \bullet x \qquad \mu_{m,n}(\textstyle\sum_i e_i \bullet \Delta_i)x = \textstyle\sum_i \nabla_{m,n}(e_i, \Delta_i(x))$$

Note that in the probabilistic case μ corresponds to the weighted sum of probability distributions [20]. For quantum effects instead we are taking the (sorted) Kronecker product of the effects: given a set of quantum effect distributions Δ_i in $D_{\mathscr{H}_C} X$ and a quantum distribution Θ in $D_{\mathscr{H}_{C'}}(D_{\mathscr{H}_C} X)$ associating each distribution Δ_i with a quantum effect, the multiplication returns a distribution in $D_{\mathscr{H}_{C' \uplus C}} X$ associating each $x \in X$ with $\sum_i \Theta(\Delta_i) \boxtimes \Delta_i(x)$ (if C and C' are disjoint). This coincides with the intuition of measuring first the qubits in C, and then, based on the outcome, performing a second measurement over C', i.e. Δ_i.

For later use, we finally define commutative graded monads. Commutativity allows us to reduce the pairing of effect distributions to a distribution of pairs, permitting a well-behaved definition of the parallel composition of effect distributions of which the probabilistic case à la [34] is a (trivially graded) example.

Definition 7 (Commutative Graded Monad)**.** *An M-graded monad $\{T_m\}$ on* **Set** *is strong if it has* left strength $\sigma_{m,X,Y} : X \times T_m Y \to T_m(X \times Y)$ *and a* right strength $\tau_{m,X,Y} : T_m X \times Y \to T_m(X \times Y)$ *which respect the monoidal structure $\times$ on* **Set** *and the graded multiplication and unit of T (the diagrams it must satisfy are just the graded version of the usual ones for strong monads). A strong graded monad is* commutative *if for any $m \perp n \in M$ there is a canonical natural transformation $\alpha : T_m X \times T_n Y \to T_{m+n}(X \times Y)$ defined by any of the two compositions $\mu_{m,n} \circ T_m \sigma_n \circ \tau_m = \mu_{n,m} \circ T_n \tau_m \circ \sigma_n$.*

Theorem 5. *If $\{\mathbb{E}_m\}$ is a commutative graded effect monoid, then $\{D_{\mathbb{E}_m}\}$ is a commutative graded monad with its canonical strength.*

4 eLTS and Bisimilarity

We first investigate coalgebras defined on effect distributions and their bisimilarities. Then, we focus on the specific case of quantum systems, discussing the correct notion of behavioural equivalence.

4.1 Coalgebra over Effect Distributions

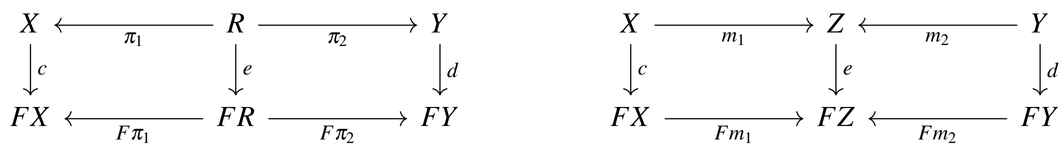
We recall the definition of Aczel-Mendler and kernel bisimilarities for coalgebras. By generalizing previous results to our graded monads, we compare them and investigate their preservation and reflection.

Definition 8 (Coalgebra)**.** *Let F :* **Set** $\to$ **Set** *be an endofunctor on the* **Set** *category. An F-*coalgebra *is a pair (X, c), with X an object of* **Set**, *and a morphism $c : X \to FX$ (also written $X \xrightarrow{c} FX$).*

*Given two F-coalgebras (X, c) and (Y, d), A morphism $f : X \to Y$ is an F-*coalgebra homomorphism, *written $f : (X, c) \to (Y, d)$, if $Ff \circ c = d \circ f$. F-coalgebra homomorphisms include the identity and are closed for composition, and thus F-coalgebras constitute a category, denoted as* $\mathbf{Set}_F$.

We are interested in two kinds of bisimilarity on F-coalgebras: AM-bisimilarity and kernel bisimilarity.

Definition 9 (Aczel-Mendler bisimilarity)**.** *A relation $R \subseteq X \times Y$ is an* AM-*bisimulation between the F-coalgebras $X \xrightarrow{c} FX$ and $Y \xrightarrow{d} FY$ if there exists an F-coalgebra $R \xrightarrow{e} FR$ such that the projections $\pi_1 : R \to X$ and $\pi_2 : R \to Y$ are coalgebra homomorphisms, i.e. the diagram in Figure 1a commutes. Two states $x \in X$ and $y \in Y$ are* AM bisimilar, *written $x \sim_{AM} y$, if $x R y$ for some AM-bisimulation R.*



(a) Aczel-Mendler bisimulation (b) Cocongruence

Figure 1: The two commutative diagrams defining AM and Kernel bisimulations, respectively

Definition 10 (Kernel bisimilarity)**.** *A* cocongruence *between two F-coalgebras $X \xrightarrow{c} FX$ and $Y \xrightarrow{d} FY$ is a cospan $X \xrightarrow{m_1} Z \xleftarrow{m_2} Y$ such that there exists an F-coalgebra $Z \xrightarrow{e} FZ$ making m_1 and m_2 coalgebra homomorphisms, i.e. the diagram in Figure 1b commutes. We call* kernel bisimulation *the pullback in* **Set** *of a cocongruence $X \xrightarrow{m_1} Z \xleftarrow{m_2} Y$, that is the relation R such that xRy if and only if $m_1(x) = m_2(y)$.* Kernel bisimilarity *$\sim_k$ is the largest kernel bisimulation.*

In line with the results in [16, 32], we present the conditions under which the two relations coincide.

Definition 11 (Decomposable Effect Algebra)**.** *We say that an effect algebra $\mathbb{E}$ is decomposable if for all $a,b,c,d \in \mathbb{E}$ such that $a \perp b$, $c \perp d$ and $a+b = c+d$, there exists $e_{11}, e_{12}, e_{21}, e_{22} \in \mathbb{E}$ such that $a = e_{11}+e_{12}$, $b = e_{21}+e_{22}$, $c = e_{11}+e_{21}$ and $d = e_{12}+e_{22}$.*

Theorem 6. *Let $\mathbb{E}$ be an effect algebra. Let $X \xrightarrow{c} D_{\mathbb{E}}X$ and $Y \xrightarrow{d} D_{\mathbb{E}}Y$ be two $D_{\mathbb{E}}$-coalgebras. For any $x \in X$ and $y \in Y$: (i) $x \sim_{AM} y \Longrightarrow x \sim_k y$, and (ii) $x \sim_k y \Longrightarrow x \sim_{AM} y$ if and only if $\mathbb{E}$ is decomposable.*

Natural transformations between the functors that define a coalgebra have the important property of preserving bisimilarities, as demonstrated in [2].

Theorem 7. *Let $\alpha : F \Rightarrow G$ be a natural transformation between two functors F and $G : \mathbf{Set} \to \mathbf{Set}$. The natural transformation α induces a functor, denoted $\alpha \circ \cdot : \mathbf{Set}_F \to \mathbf{Set}_G$, which maps objects (X,c) to $(X, \alpha_X \circ c)$ and homomorphisms $f : (X,c) \to (Y,d)$ to homomorphisms $f : (X, \alpha_X \circ c) \to (Y, \alpha_Y \circ d)$. Furthermore, bisimilarities are preserved by this functor.*

Note that, functors induced by natural transformations do not reflect bisimilarities, in general. Nonetheless, in [2] further conditions are identified in order for functors to reflect bisimulations.

Theorem 8. *Let $\alpha : F \Rightarrow G$ be a natural transformation between two functors F and $G : \mathbf{Set} \to \mathbf{Set}$. If all components α_X of α are injective then the induced functor $\alpha \circ \cdot$ reflects the kernel bisimilarity.*

4.2 The Quantum Case

Hereafter, we apply the results above to the specific case of probability and quantum effect distributions and their monads. For brevity, we will write Q_C for $D_{\mathbb{Q}_{\mathscr{H}_C}}$. Note that coalgebras $(X, c : X \to D_{\mathbb{E}}X)$ are essentially a generalization of Markov Chains, coinciding with the usual definition but with partial distributions if $\mathbb{E} = [0,1]$. Q_C-coalgebras can be seen instead as a quantum version of Markov Chains.

We start by showing that kernel and AM-bisimilarities do not coincide in the quantum case, in contrast with the probabilistic case on which the equality is known to hold, since $[0,1]$ is decomposable (as demonstrated implicitly in [29]). Thanks to Theorem 6 of the previous section, the following suffices.

Proposition 1. *If the dimension of $\mathscr{H}$ is grater or equal than 2, then $\mathbb{Q}_{\mathscr{H}}$ is not decomposable.*

Recall the effect homomorphism $L \mapsto tr(L\rho)$ that for each $\rho \in DM_{\mathscr{H}_C}$ maps a quantum effect to a probability, and the induced natural transformation from Q_C to $D_{[0,1]}$ by Theorem 2. Thanks to Theorem 7, a functor $\downarrow_\rho$ is defined for each $\rho \in DM_{\mathscr{H}_C}$ that maps a Q_C-coalgebra in the $D_{[0,1]}$-coalgebra characterizing the behaviour of the Quantum Markov Chain when the input state is ρ. The functor $\downarrow_\rho$ can be seen as a function that given a Quantum Markov Chain simply updates its weights by computing the Born rule.

Example 1. *Take* $X \xrightarrow{c} Q_C$*, with* $X = \{x_1, x_2, x_3, x_4\}$ *and* c *such that* $c(x_1) = |0\rangle\langle 0| \bullet x_3 + |1\rangle\langle 1| \bullet x_4$*,* $c(x_2) = |+\rangle\langle +| \bullet x_3 + |-\rangle\langle -| \bullet x_4$ *and* $c(x_3) = c(x_4) = \mathbb{I} \bullet x_3$*, then the* $D_{[0,1]}$*-coalgebra,* $\downarrow_\rho |0\rangle\langle 0|(X,c)$ *is* (X,c') *with* $c'(x_1) = 1 \bullet x_3$*,* $c'(x_2) = \frac{1}{2} \bullet x_3 + \frac{1}{2} \bullet x_4$ *and* $c'(x_3) = c'(x_4) = 1 \bullet x_3$*.*

Since their probabilistic behaviour is the only observable property of quantum systems, our (first) correctness principle for a bisimilarity over quantum systems is that two elements of the Q_C-coalgebras (X,c) and (Y,d) shall be bisimilar if and only if they are indistinguishable in the probabilistic systems obtained by instantiating their quantum input, i.e. in the $D_{[0,1]}$-coalgebras $\downarrow_\rho(X,c)$ and $\downarrow_\rho(Y,d)$ for any $\rho \in DM_{\mathscr{H}_C}$. By Theorem 7, all the functors $\downarrow_\rho$ preserve both bisimilarities, hence one side of the implication holds for both of them. However,Theorem 6 and Proposition 1 implies that AM-bisimilarity does not satisfy the other direction.

Example 2. *Take the* Q_C*-coalgebra* (X,c) *of the previous example. We cannot build an AM-bisimulation such that* $x_1 \sim_{AM} x_2$*. However, if we take* $Z = \{z_1, z_2\}$ *and* m *such that* $m(x_1) = m(x_2) = z_1$*,* $m(x_3) = m(x_4) = z_2$*, then* $(X,c) \xrightarrow{m} (Z,d) \xleftarrow{m} (X,c)$ *is a cocongruence for* d *defined as* $d(z_1) = d(z_2) = \mathbb{I}_2 \bullet z_2$*.*

We obtain that kernel bisimilarity fully satisfies our correctness principle when the quantum effects used in (X,c) and (Y,d) are contained in a finite effect algebra $\mathbb{L} \subsetneq \mathbb{Q}_{\mathscr{H}}$. Since $\mathbb{L}$ is finite, a density operator exists that distinguish every pair of distinct effects in $\mathbb{L}$, as we proved in Lemma 4 of [7].

Lemma 1. *Let* $\mathbb{L}$ *be a finite set of quantum effects in* $\mathbb{Q}_{\mathscr{H}}$*, then a density operator* $\widehat{\rho} \in DM_{\mathscr{H}}$ *exists such that for each* $L, L' \in \mathbb{L}$*,* $tr(L\widehat{\rho}) = tr(L'\widehat{\rho})$ *if and only if* $L = L'$*.*

This means that among the natural transformations induced by $L \mapsto tr(L\rho)$ there is at least one of them with injective components, and by Theorem 7 the resulting functor $\downarrow_{\widehat{\rho}}$ reflects kernel bisimilarity.

Theorem 9. *Let* (X,c) *and* (Y,d) *be two* $D_{\mathbb{L}}$*-coalgebras, and let* $x \in X$ *and* $y \in Y$*. If* x *and* y *are kernel bisimilar in* $\downarrow_\rho(X,c)$ *and* $\downarrow_\rho(Y,d)$ *for any* ρ*, then* x *and* y *are kernel bisimilar in* (X,c) *and* (Y,d)*.*

We investigate now a different correctness principle, called locally parameterized probabilistic bisimilarity (lpp) in [7]. In the previous approach, two quantum systems are equated if they behave the same when given any possible quantum state. Nevertheless, the input is given globally for the whole Markov Chain. This means we assume that an adversary trying to disprove equivalence can only choose the state once at the beginning. In lpp instead, the adversary can give a different quantum state at each step of the two Markov Chains. In the following, we model this feature coalgebraically. Recall that $L \mapsto \lambda\rho.tr(L\rho) : \mathbb{Q}_{\mathscr{H}} \to \mathbf{Conv}(DM_{\mathscr{H}}, [0,1])$ is an isomorphism from quantum effects to convex parameterized probabilities. Indeed, $\mathbf{Conv}(DM_{\mathscr{H}_C}, [0,1])$ is itself an effect algebra, that we name f_C for convenience, and $L \mapsto \lambda\rho.tr(L\rho)$ is an effect algebra isomorphism. We can therefore define D_{f_C} and a natural transformation $\alpha_{lpp} : Q_C \to D_{f_C}$ transforming a quantum distribution over X into a distribution that associates each element $x \in X$ with a function that given a quantum input returns a probability. Intuitively, while $\downarrow_\rho$ applies a unique initial input density operator that transforms a Quantum Markov Chain into a probabilistic system, $\alpha_{lpp} \circ \cdot$ allow us to change ρ at every step of our bisimilarity. Kernel bisimilarity over Q_C-coalgebras exactly coincides with this notion of locally parameterized probabilistic bisimilarity.

Theorem 10. *Let* (X,c) *and* (Y,d) *be* Q_C*-coalgebras, and let* $x \in X$ *and* $y \in Y$*:* $x \sim_k y$ *in* (X,c) *and* (Y,d) *holds if and only if* $x \sim_k y$ *holds in* $(\alpha_{lpp} \circ \cdot)(X,c)$ *and* $(\alpha_{lpp} \circ \cdot)(Y,d)$*.*

An interesting corollary is that, when coalgebras are build over a finite quantum effect algebra $\mathbb{L}$, the stronger adversary that can choose a different quantum input at each step is not capable of discriminating more than the weaker one that only chose once at the beginning.

5 Effect Labelled Transition System

We can now define a generalization of probabilistic transition systems to effect distributions, for which we extend the properties of the previous section and introduce both a generic parallel composition and a specifically quantum operator of partial evaluation.

Definition 12 ($\mathbb{E}$-Labelled Transition System)**.** *Given an effect algebra $\mathbb{E}$, a* $\mathbb{E}$-labelled transition system *($\mathbb{E}$LTS) is a coalgebra $X \xrightarrow{c} \mathscr{P}(D_{\mathbb{E}}X)^L$, where X is a set of states, L is a fixed set of labels, $\mathscr{P}$ is the finitary powerset endofunctor and $D_{\mathbb{E}}$ is the $\mathbb{E}$-distribution endofunctor on* **Set**.

As is typical for process calculi, we will assume that there exist a special symbol $\tau \in L$ and an involutive unary operation $\overline{\cdot}$ for all labels different from τ. Intuitively, τ represents a silent, invisible action, and whenever μ represents a visible action (e.g. outputting a value), $\overline{\mu}$ represents its "dual" action (e.g. receiving that value). Moreover, we will write $x \xrightarrow{\mu} \Delta$ for $\Delta \in c(x)(\mu)$.

In the following we will deal with $D_{[0,1]}$LTS and Q_CLTS (called pLTS on qLTS hereafter), i.e. LTS over distributions of probabilities and of quantum effect in some $\mathscr{H}_C$ respectively. For simplicity, all results given in the previous section are defined on $D_{\mathbb{E}}$-coalgebras, but we can easily extend them to more complex coalgebras through *whiskering*. For example, we can whisker the natural transformation $\downarrow_\rho \colon Q_C \to D_{[0,1]}$ and the finite powerset functor $\mathscr{P}$, obtaining a natural transformation $\mathscr{P}\downarrow_\rho \colon \mathscr{P}Q_C \to \mathscr{P}D_{[0,1]}$ and recovering the previous results also for systems that transition non-deterministically to effect distributions.

Corollary 1. *Let (X,c) and (Y,d) be qLTSs, with $x \in X$ and $y \in Y$.*

- *$x \sim_{AM} y \implies x \sim_k y$ in general for qLTSs;*
- *$x \sim_k y \not\Longrightarrow x \sim_{AM} y$ if the dimension of the Hilbert space is at least two;*
- *for each ρ over the adequate Hilbert space, there is a functor transforming c and d into the pLTSs c_ρ and d_ρ representing the probabilistic behaviour of the system when the quantum state is ρ;*
- *if x and y are kernel bisimilar in c and d, then they are bisimilar also in c_ρ and d_ρ for every ρ;*
- *if c and d use a finite effect algebra of weights, and $\forall \rho . x \sim_k y$ in c_ρ and d_ρ, then $x \sim_k y$ in c and d;*
- *x and y are kernel bisimilar in c and d if and only if they are bisimilar in the D_{f_C}LTSs where effects are considered as convex functions from density operators to probabilities, i.e. when the adversary can choose a different density operator at each step of the bisimilarity.*

In the rest of the section we describe some operators on $\mathbb{E}$LTSs. When dealing with coalgebraic LTSs, it is typical to define operators like nondeterministic sum, restriction or parallel composition as acting on *processes*, i.e. states of the final coalgebra of the functor under consideration [23]. We instead define operators directly between coalgebras as in [33], allowing us to compose coalgebras of different functors. We postpone to future work how these "macroscopic" operators relates to the "microscopic" ones in the final coalgebra, following the approach of [17]. We present two operators: the first generalizes parallel composition to all $\mathbb{E}$LTSs, the other is specific to qLTSs and specifies how to perform partial evaluation of the input quantum state. We first introduce the notion of *extensible* graded monad.

In [25], the author proposes a monad graded on a preordered monoid, intended to model the composition of computational side effects and their scope. We replicate this notion in our PCM setting, employing the fact that each PCM automatically carries a preorder structure.

Definition 13 (Extensible Graded Monad)**.** *An M-graded monad is* extensible *if for each $m \preceq n \in M$, a natural transformation $\xi_{m\preceq n} : T_m \to T_n$ called* extension *exists, with $\xi_{m\preceq m} = Id$ and $\xi_{n\preceq o} \circ \xi_{m\preceq n} = \xi_{m\preceq o}$.*

Extensible monads give us a way to canonically extend a computation to a greater grade, which we will need to define parallel composition. All non graded monad are trivially extensible with the identity transformation. We provide an extension for monads built on effect monoids graded on an effect algebra.

Theorem 11. *Suppose $\{\mathbb{E}_m\}$ is an M-graded effect monoid and M an effect algebra, with $-$ its induced difference operator. There is an extensible graded monad $\{D_{\mathbb{E}_m}\}$ with unit $\eta : Id \to D_{\mathbb{E}_0}$ and multiplication $\mu_{m,n} : D_{\mathbb{E}_m} D_{\mathbb{E}_n} \to D_{\mathbb{E}_{m+n}}$ defined as in Theorem 4 and extension $\xi_{m\preceq n}$*

$$\xi_{m\preceq n}(\sum_i e_i \bullet x_i) = \sum_i \nabla_{m,n-m}(e_i, 1_{\mathbb{E}_{n-m}}) \bullet x_i.$$

Indeed, Q_C is extensible, since the partial monoid of quantum systems is indeed an effect algebra.

Lemma 2. *$\mathscr{S} = \langle \mathscr{P}(Sys), \emptyset, \uplus, Sys \setminus \cdot \rangle$ is an effect algebra, with $\setminus$ the usual set difference.*

We now focus on the parallel composition of $\mathbb{E}$LTSs, in which two subsystems can move on their own or exchange messages. In the quantum case, we will be able to compose qLTSs only when their grades are summable, ensuring that they do not perform measurements on the same quantum systems.

In the process-calculi literature, there are different notions of parallel composition, corresponding to different "synchronization styles": à la CCS, CSP or ACP. Moreover, different extensions have been considered from the original non deterministic setting to the probabilistic [20, 2] and quantum [7] one. We introduce a *generic parallel operator*, which is parametric both with respect to the "synchronization style" and the commutative graded monad chosen for the weights (Id for LTSs, $D_{[0,1]}$ for pLTSs, Q_C for qLTSs). Our parallel operator is defined in two steps: the first specifies how to compose the "non-deterministic structure" of the systems (the $\mathscr{P}^L$ functor), the second how to combine a couple of distributions into a distribution of couples. For the first step, we focus on a CCS-style interleaving composition, formalized as a binary *synchronization operator* $\cdot|\cdot$ on transition functions.

Definition 14 (CCS-Style Synchronization)**.** *Consider the extensible graded monad $\{T_m\}$. Given two coalgebras $X \xrightarrow{c} \mathscr{P}(T_m X)^L$ and $Y \xrightarrow{d} \mathscr{P}(T_n Y)^L$ we define $c|d : X \times Y \to \mathscr{P}(T_m X \times T_n Y)^L$, the "CCS-style" synchronization of c and d, as*

$$\frac{s \xrightarrow{\mu} \Delta}{s \parallel t \xrightarrow{\mu} \langle \Delta,\ \xi_{0\preceq n}(\eta(t))\rangle} \qquad \frac{t \xrightarrow{\mu} \Theta}{s \parallel t \xrightarrow{\mu} \langle \xi_{0\preceq m}(\eta(s)),\ \Theta\rangle} \qquad \frac{s \parallel t \xrightarrow{\mu} \Delta \quad s \parallel t \xrightarrow{\mu} \Delta}{s \parallel t \xrightarrow{\tau} \langle \Delta,\ \Theta\rangle}$$

where $\times$ is the cartesian product on **Set***, $s \parallel t$ is an element of $X \times Y$, Δ (resp. Θ) is an element of $T_m X$ (resp. $T_n Y$), and $s \xrightarrow{\mu} \Delta$ is the usual SOS-style notation for $\Delta \in c(s)(\mu)$.*

To model CCS-style synchronization we represent the "idle behaviour" of a state s, obtained by taking the monad unit and extending it to the desired grade, i.e. $\xi_{0\preceq m}(\eta(s))$. In the pLTSs this coincides with the unit $1 \bullet s$, while in qLTSs we get the point distribution "scaled" for the given dimension $\mathbb{I}_d \bullet s$. For pure non deterministic systems, where T is the identity monad, we get the usual parallel composition of CCS.

For the second step, we define a parallel operator $\cdot \parallel \cdot$ on $\mathbb{E}$LTSs, in the form of a functor $\mathbf{Set}_{\mathscr{P}(T_m)^L} \times \mathbf{Set}_{\mathscr{P}(T_n)^L} \to \mathbf{Set}_{\mathscr{P}(T_{m+n})^L}$. The definition is parametric with respect to a natural transformation $\alpha : T_m X \times T_n Y \to T_{m+n}(X \times Y)$ which specifies how to combine two distributions. For classical, probabilistic and quantum systems, we have commutative monads, which have a canonical transformation α.

Theorem 12. *Given a transformation $\alpha : T_m X \times T_n Y \to T_{m+n}(X \times Y)$, we can construct a functor $\parallel$: $\mathbf{Set}_{\mathscr{P}(T_m)^L} \times \mathbf{Set}_{\mathscr{P}(T_n)^L} \to \mathbf{Set}_{\mathscr{P}(T_{m+n})^L}$ defined by*

$$(X, c) \parallel (Y, d) = (X \times Y, \mathscr{P}(\alpha)^L \circ (c|d)) \qquad f \parallel g = f \times g$$

We now consider a special operator acting over qLTS, i.e. the *partial evaluation*.

Definition 15 (Partial Evaluation of Quantum Effects)**.** *Consider a quantum effect $L \in \mathbb{Q}_{\mathscr{H}_C}$. For any quantum state $\rho \in DM_{\mathscr{H}_{C'}}$ with $C' \subseteq C$, we define the partial evaluation of L with state ρ as $tr_{C'}(L(\rho \boxtimes \mathbb{I}))$, where $\mathbb{I}$ is the identity operator on $\mathscr{H}_{C \setminus C'}$ and $\boxtimes$ is the "commutative" Kronecker product as in Theorem 3.*

Our previously defined total evaluation $\downarrow_\rho$ is of course a specific case of partial evaluation, when $C' = C$. For any ρ, partial evaluation $tr_{C'}(\cdot(\rho \boxtimes \mathbb{I}))$ is an effect morphism, and thus yields a functor from $\mathbf{Set}_{Q_C}$ to $\mathbf{Set}_{Q_{C \setminus C'}}$, and can be extended also to qLTSs via whiskering.

Definition 16 (Partial Evaluation of qLTS)**.** *Let C, C' be collections of quantum systems such that $C' \subseteq C$. For each $\rho \in DM_{\mathscr{H}_{C'}}$ we define the partial evaluation of $\mathbf{Set}_{\mathscr{P}(Q_C)^L}$ with input ρ as the functor $\downarrow_\rho : \mathbf{Set}_{\mathscr{P}(Q_C)^L} \to \mathbf{Set}_{\mathscr{P}(Q_{C \setminus C'})^L}$ induced by the effect morphism $L \mapsto tr_{C'}(L(\rho \boxtimes \mathbb{I}))$.*

Thanks to functoriality, we prove that parallel composition and partial evaluation preserve bisimilarity.

Theorem 13. *If $s \sim_k t$ then, for any ρ, $s \downarrow_\rho \sim t \downarrow_\rho$. Moreover, if $s' \sim t'$ then $s \parallel s' \sim t \parallel t'$.*

6 Conclusions

We have characterized distributions with weights from a generic effect algebra, subsuming probability and quantum effect distributions. We have introduced monads graded on a partial commutative monoid (PCM) that allow us to grade quantum distributions over their resources, which must be treated linearly, as prescribed by the no-cloning theorem. We have studied effect weighted Markov Chains and labelled transition systems ($\mathbb{E}$LTS) in a coalgebraic framework, extending previous results about kernel and Aczel-Mendler bisimilarities. We have applied our findings to the quantum setting, proving that each quantum state ρ defines a functor from qLTSs to pLTSs that “instantiate” a quantum process to the probabilistic behaviour it exhibits when the quantum state is ρ. We have compared the two notions of bisimilarity in hand with the desired properties of a behavioural equivalence for quantum systems, proving that the kernel bisimilarity is the only one of the two that captures theirs observable features, i.e. the induced probabilistic behaviour. Finally, we have defined parallel composition of $\mathbb{E}$LTSs in a functorial, compositional way, and a partial evaluation operator that given a qLTS instantiates some of its input qubits, paving the way for an $\mathbb{E}$LTS semantics of quantum process calculi.

Future work We will study the final coalgebra of our functors, investigating the definition of graded GSOS operators for our graded $\mathbb{E}$LTSs, and their relation with the functorial ones defined in this work. Our framework allows us to compose effects in parallel via the tensor product. We will consider also sequental composition, extending our graded approach to the case of superoperators, which model more general quantum operations. We expect such an extension to come naturally and to preserve our constructions and results. Finally, we will explore the reflection of the kernel bisimilarity in broader cases, i.e. when the quantum effect algebra is finitely generated, but possibly infinite, and when it is numerable in general. Recent work [4] defines bisimulation for Mealy machines with generic effects. We intend to look at the relationship between their effectful bisimilarity and our notion of kernel bisimilarity.

References

[1] Samson Abramsky, Rui Soares Barbosa, Nadish de Silva & Octavio Zapata (2017): *The Quantum Monad on Relational Structures*, p. 19 pages. doi:10.4230/LIPIcs.MFCS.2017.35. arXiv:1705.07310.

[2] Falk Bartels, Ana Sokolova & Erik de Vink (2004): *A Hierarchy of Probabilistic System Types*. *Theoretical Computer Science* 327(1-2), pp. 3–22, doi:10.1016/j.tcs.2004.07.019.

[3] Charles H. Bennett & Gilles Brassard (2014): *Quantum Cryptography: Public Key Distribution and Coin Tossing*. *Theoretical Computer Science* 560, pp. 7–11, doi:10.1016/j.tcs.2014.05.025.

[4] Filippo Bonchi, Elena Di Lavore & Mario Román (2024): *Effectful Trace Semantics via Effectful Streams*. Preprint.

[5] Filippo Bonchi, Alexandra Silva & Ana Sokolova (2017): *The Power of Convex Algebras*. In: *28th International Conference on Concurrency Theory (CONCUR 2017)*, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

[6] Marcello Caleffi, Angela Sara Cacciapuoti & Giuseppe Bianchi (2018): *Quantum Internet: From Communication to Distributed Computing!* In: *Proceedings of the 5th ACM International Conference on Nanoscale Computing and Communication*, ACM, Reykjavik Iceland, pp. 1–4, doi:10.1145/3233188.3233224.

[7] Lorenzo Ceragioli, Fabio Gadducci, Giuseppe Lomurno & Gabriele Tedeschi (2024): *Effect Semantics for Quantum Process Calculi*. Available at `https://lceragioli.github.io/Preprint/EffectSemanticsforQuantumProcessCalculi.pdf`. Preprint.

[8] Lorenzo Ceragioli, Fabio Gadducci, Giuseppe Lomurno & Gabriele Tedeschi (2024): *Quantum Bisimilarity via Barbs and Contexts: Curbing the Power of Non-deterministic Observers*. *Proc. ACM Program. Lang.* 8(POPL), pp. 43:1269–43:1297, doi:10.1145/3632885.

[9] Bob Coecke & Aleks Kissinger (2017): *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*, 1 edition. Cambridge University Press, doi:10.1017/9781316219317.

[10] Timothy AS Davidson (2012): *Formal Verification Techniques Using Quantum Process Calculus*. Ph.D. thesis, University of Warwick.

[11] Yuxin Deng (2018): *Bisimulations for Probabilistic and Quantum Processes (Invited Paper)*. In Sven Schewe & Lijun Zhang, editors: *29th International Conference on Concurrency Theory (CONCUR 2018)*, *Leibniz International Proceedings in Informatics (LIPIcs)* 118, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, pp. 2:1–2:14, doi:10.4230/LIPIcs.CONCUR.2018.2.

[12] Yuxin Deng & Yuan Feng (2012): *Open Bisimulation for Quantum Processes*. In Jos C. M. Baeten, Tom Ball & Frank S. de Boer, editors: *Theoretical Computer Science*, Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, pp. 119–133, doi:10.1007/978-3-642-33475-7_9.

[13] Yuan Feng, Yuxin Deng & Mingsheng Ying (2014): *Symbolic Bisimulation for Quantum Processes*. *ACM Trans. Comput. Logic* 15(2), pp. 14:1–14:32, doi:10.1145/2579818.

[14] Fei Gao, SuJuan Qin, Wei Huang & QiaoYan Wen (2019): *Quantum Private Query: A New Kind of Practical Quantum Cryptographic Protocol*. *Sci. China Phys. Mech. Astron.* 62(7), p. 70301, doi:10.1007/s11433-018-9324-6.

[15] Stanley Gudder (2008): *Quantum Markov Chains*. *Journal of Mathematical Physics* 49(7), p. 072105, doi:10.1063/1.2953952.

[16] H. Peter Gumm & Tobias Schröder (2001): *Monoid-Labeled Transition Systems*. *Electronic Notes in Theoretical Computer Science* 44(1), pp. 185–204, doi:10.1016/S1571-0661(04)80908-3.

[17] Ichiro Hasuo (2011): *The Microcosm Principle and Compositionality of GSOS-Based Component Calculi*. In Andrea Corradini, Bartek Klin & Corina Cîrstea, editors: *Algebra and Coalgebra in Computer Science*, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 222–236.

[18] Ichiro Hasuo & Naohiko Hoshino (2011): *Semantics of Higher-Order Quantum Computation via Geometry of Interaction*. In: *2011 IEEE 26th Annual Symposium on Logic in Computer Science*, pp. 237–246, doi:10.1109/LICS.2011.26.

[19] Teiko Heinosaari & Mário Ziman (2011): *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement*. Cambridge University Press.

[20] Matthew Hennessy (2012): *Exploring Probabilistic Bisimulations, Part I*. *Form. Asp. Comput.* 24(4-6), pp. 749–768, doi:10.1007/s00165-012-0242-7.

[21] Bart Jacobs (2010): *Convexity, Duality and Effects*. In: *Theoretical Computer Science: 6th IFIP TC 1/WG 2.2 International Conference, TCS 2010, Held as Part of WCC 2010, Brisbane, Australia, September 20-23, 2010. Proceedings 6*, Springer, pp. 1–19.

[22] Bart Jacobs (2011): *Probabilities, Distribution Monads, and Convex Categories*. *Theoretical Computer Science* 412(28), pp. 3323–3336, doi:10.1016/j.tcs.2011.04.005.

[23] Bart Jacobs (2016): *Introduction to Coalgebra: Towards Mathematics of States and Observation*. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press.

[24] Jules Jacobs & Thorsten Wißmann (2023): *Fast Coalgebraic Bisimilarity Minimization*. *Proc. ACM Program. Lang.* 7(POPL), pp. 52:1514–52:1541, doi:10.1145/3571245.

[25] Shin-ya Katsumata (2014): *Parametric effect monads and semantics of effect systems*. In: *Proceedings of the 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '14, Association for Computing Machinery, New York, NY, USA, p. 633–645, doi:10.1145/2535838.2535846.

[26] Marie Lalire (2006): *Relations among Quantum Processes: Bisimilarity and Congruence*. arXiv:quant-ph/0603274.

[27] Ai Liu & Meng Sun (2019): *A Coalgebraic Semantics Framework for Quantum Systems*. In Yamine Ait-Ameur & Shengchao Qin, editors: *Formal Methods and Software Engineering*, Lecture Notes in Computer Science, Springer International Publishing, Cham, pp. 387–402, doi:10.1007/978-3-030-32409-4_24.

[28] Gui-lu Long, Fu-guo Deng, Chuan Wang, Xi-han Li, Kai Wen & Wan-ying Wang (2007): *Quantum Secure Direct Communication and Deterministic Secure Quantum Communication*. *Front. Phys. China* 2(3), pp. 251–272, doi:10.1007/s11467-007-0050-3.

[29] Lawrence S. Moss (1999): *Coalgebraic Logic*. *Annals of Pure and Applied Logic* 96(1-3), pp. 277–317, doi:10.1016/S0168-0072(98)00042-6.

[30] Michael A. Nielsen & Isaac L. Chuang (2010): *Quantum Computation and Quantum Information*, 10th anniversary ed edition. Cambridge University Press, Cambridge ; New York.

[31] Ali Ibnun Nurhadi & Nana Rachmana Syambas (2018): *Quantum Key Distribution (QKD) Protocols: A Survey*. In: *2018 4th International Conference on Wireless and Telematics (ICWT)*, pp. 1–5, doi:10.1109/ICWT.2018.8527822.

[32] Hiroshi Ogawa (2014): *Coalgebraic Approach to Equivalences of Quantum Systems*. Master's thesis, University of Tokyo.

[33] J.J.M.M. Rutten (2000): *Universal Coalgebra: A Theory of Systems*. *Theoretical Computer Science* 249(1), pp. 3–80, doi:10.1016/S0304-3975(00)00056-6.

[34] Ana Sokolova & Erik P. de Vink (2004): *Probabilistic Automata: System Types, Parallel Composition and Comparison*, pp. 1–43. Springer Berlin Heidelberg, Berlin, Heidelberg, doi:10.1007/978-3-540-24611-4_1. Available at `https://doi.org/10.1007/978-3-540-24611-4_1`.

[35] Peiying Zhang, Ning Chen, Shigen Shen, Shui Yu, Sheng Wu & Neeraj Kumar (2022): *Future Quantum Communications and Networking: A Review and Vision*. *IEEE Wireless Commun.*, pp. 1–8, doi:10.1109/MWC.012.2200295.

A Proof Sketches

Theorem 2. *Each effect morphism $m : \mathbb{E} \to \mathbb{F}$ yields a natural transformation $m \circ \cdot : D_{\mathbb{E}} \Rightarrow D_{\mathbb{F}}$. Moreover, if m is injective then the components of $m \circ \cdot$ are injective.*

Proof sketch. Naturality of $m \circ \cdot$ follows from the fact that m preserves the (partial) sum and that distributions have finite support. If m is injective, the components of $m \circ \cdot$ are also injective because if two different $\Delta, \Theta \in D_{\mathbb{E}}(X)$ differ on $x \in X$, then $m \circ \Delta$ and $m \circ \Theta$ will differ on the same x. □

The evolution of density operators is given as a *trace preserving superoperator* $\mathscr{E} : DM_{\mathscr{H}_A} \to DM_{\mathscr{H}_B}$, a function defined by its *Kraus operator sum decomposition* $\{E_i\}_i$ for a finite set of indexes $i = 1, \ldots, n \times m$, satisfying that $E_i \in \mathbb{C}^{m \times n}, \mathscr{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ and $\sum_i E_i^\dagger E_i = \mathbb{I}_n$, where n and m are the dimension of Hilbert space $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively. The tensor product of density operators $\rho \otimes \sigma$ is defined as their Kronecker product, and of superoperators $\mathscr{E} \otimes \mathscr{F}$ as the superoperator having Kraus decomposition $\{E_i \otimes F_j\}_{i,j}$ with $\{E_i\}_i$ and $\{F_j\}_j$ Kraus decompositions of $\mathscr{E}$ and $\mathscr{F}$.

A morphism between effects is the dual of a superoperator. Let $\mathscr{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ be a superoperator. Its dual is the superoperator $\mathscr{E}^\dagger(L) = \sum_i E_i^\dagger L E_i$.

Theorem 3 (Effect Monoid of Quantum Effect)**.** *Quantum effects carry a commutative $\mathscr{S}$-graded effect monoid structure, given by:*

- *The effect algebra $\mathbb{E}_C$ is $\mathbb{Q}_{\mathscr{H}_C}$ for any collection of systems $C \in \mathscr{P}(Sys)$*
- *The operator $\nabla_{C,D} : \mathbb{E}_C \otimes \mathbb{E}_D \to \mathbb{E}_{C \uplus D}$ is denoted as $\boxtimes$ and defined by $L_1 \boxtimes L_2 = Sort_{C,D}(L_1 \otimes_k L_2)$, where $\otimes_k$ is the Kronecker product between $L_1 \in \mathbb{Q}_{\mathscr{H}_C}$ and $L_2 \in \mathbb{Q}_{\mathscr{H}_D}$, and $Sort_{C,D}$ is the unitary transformation which "sorts" the Hilbert Space $\mathscr{H}_C \otimes \mathscr{H}_D$ into $\mathscr{H}_{C \uplus D}$.*
- *The effect morphism $\eta : 2 \to \mathbb{E}_\emptyset$ defined by $\{0 \mapsto 0, 1 \mapsto 1\}$.*

Proof sketch. Recall that $Sort_{C,D}$ is the transformation on quantum effects (i.e. a dual superoperator) which applies a unitary permutation matrix from $\mathscr{H}_C \otimes \mathscr{H}_D$ to $\mathscr{H}_{C \uplus D}$. Unitality follows from the definition of Kronecker product and from $Sort_{C,\emptyset}$ being the identity transformation. For commutativity, we check that $Swap_{C,D}$, the superoperator that permutes $\mathscr{H}_C \otimes \mathscr{H}_D$ in $\mathscr{H}_D \otimes \mathscr{H}_C$, respects sorting:

$$Sort_{D,C}(L_2 \otimes L_1) = Sort_{D,C}(Swap_{C,D}(L_1 \otimes L_2)) = (Sort_{D,C} \circ Swap_{C,D})(L_1 \otimes L_2) = Sort_{C,D}(L_1 \otimes L_2).$$

For associativity, we have that

$$Sort_{C,D \uplus E}(L_1 \otimes Sort_{D,E}(L_2 \otimes L_3)) = (Sort_{C,D \uplus E} \circ (Id \otimes Sort_{D,E}))(L_1 \otimes L_2 \otimes L_3) = Sort_{C,D,E}(L_1 \otimes L_2 \otimes L_3)$$

and similarly for $Sort_{C \uplus D,E}(Sort_{C,D}(L_1 \otimes L_2) \otimes L_3)$, where Id is the identity superoperator, and $Sort_{C,D,E}$ is the superoperator going from $\mathscr{H}_C \otimes \mathscr{H}_D \otimes \mathscr{H}_E$ to $\mathscr{H}_{C \uplus D \uplus E}$. □

Theorem 4 (Graded monads of graded effect monoids)**.** *If $\{\mathbb{E}_m\}$ is an M-graded effect monoid, there is a graded monad $\{D_{\mathbb{E}_m}\}$ with unit $\eta : Id \to D_{\mathbb{E}_0}$ and multiplication $\mu_{m,n} : D_{\mathbb{E}_m} D_{\mathbb{E}_n} \to D_{\mathbb{E}_{m+n}}$ given by*

$$\eta(x) = 1_{\mathbb{E}_0} \bullet x \qquad \mu_{m,n}(\sum_i e_i \bullet \Delta_i)x = \sum_i \nabla_{m,n}(e_i, \Delta_i(x))$$

Proof sketch. The proof proceeds as in the total case: the unit and multiplication of the monad are defined with the graded monoid structure of the weights, and the associativity and unitality conditions follow from the graded monoid ones. □

Theorem 5. *If $\{\mathbb{E}_m\}$ is a commutative graded effect monoid, then $\{D_{\mathbb{E}_m}\}$ is a commutative graded monad with its canonical strength.*

Proof sketch. As any **Set**-endofunctor, $D_{\mathbb{E}_m}$ has a canonical left-strength $\sigma_{m,X,Y} : X \times D_{\mathbb{E}_m} Y \to D_{\mathbb{E}_m}(X \times Y)$ given by

$$(x, (\sum_i e_i \bullet y_i)) \mapsto \sum_i e_i \bullet x, y_i$$

and a corresponding right strength given by the monoidal structure of **Set**. Both transformations yield indeed a strong graded monad. It's easy to see that the morphism $\mu_{m,n} \circ D_{\mathbb{E}_m} \sigma_n \circ \tau_m$ brings $(\sum_i e_i \bullet x_i, \sum_j e'_j \bullet y_j)$ into $\sum_{i,j} \nabla_{n,m}(e'_j, e_i) \bullet (x,y)$, while $\mu_{n,m} \circ D_{\mathbb{E}_n} \tau_m \circ \sigma_n$ brings it to $\sum_{i,j} \nabla_{m,n}(e_i, e'_j) \bullet (x,y)$. The two coincides whenever ∇ is commutative. □

Theorem 6. *Let $\mathbb{E}$ be an effect algebra. Let $X \xrightarrow{c} D_{\mathbb{E}} X$ and $Y \xrightarrow{d} D_{\mathbb{E}} Y$ be two $D_{\mathbb{E}}$-coalgebras. For any $x \in X$ and $y \in Y$: (i) $x \sim_{AM} y \Longrightarrow x \sim_k y$, and (ii) $x \sim_k y \Longrightarrow x \sim_{AM} y$ if and only if $\mathbb{E}$ is decomposable.*

Proof sketch. The first point has been demonstrated in [2, Corollary 8].

The *if* case of the second point follows from the composition of two previously demonstrated results. In [33, Theorem 5.13], the authors prove that if a commutative monoid is positive and decomposable then the functor $D_{\mathbb{E}}$ weakly preserves weak pullbacks. Such demonstration is also applicable to partial commutative monoid, of which effect algebras are a special case. Again, [2, Corollary 8] prove that if $D_{\mathbb{E}}$ preserves weak pullback then $\sim_k$ implies $\sim_{AM}$.

Finally, the *only if* case is demonstrated through contrapositivity. Assume we have effects $a,b,c,d \in \mathbb{E}$ that are not decomposable, i.e. $a+b = c+d$ and there are no $e_{11}, e_{12}, e_{21}, e_{22} \in \mathbb{E}$ such that $a = e_{11} + e_{12}$, $b = e_{21} + e_{22}$, $c = e_{11} + e_{21}$ and $d = e_{12} + e_{22}$. Let $s = a+b = c+d$.

Consider three coalgebras:

1. (X,c) with $X = \{x_1, x_2, x_3\}$ and $c(x_1) = \{x_2 \mapsto a, x_3 \mapsto b\}$.
2. (Y,d) with $Y = \{y_1, y_2, y_3\}$ and $d(y_1) = \{y_2 \mapsto c, y_3 \mapsto d\}$.
3. (Z,z) with $Z = \{z_1, z_2\}$ and $z(z_1) = \{z_2 \mapsto s\}$.

It is straightforward to show that $X \xrightarrow{m_1} Z \xleftarrow{m_2} X$ with $m_1(x_1) = m_2(y_1) = z_1$ and $m_1(x_2) = m_1(x_3) = m_2(y_1) = m_2(y_2) = z_2$, $m_1(x_4) = m_2(x_4) = z_3$ is a cocongruence, therefore $x_1 \sim_k y_1$. By [33, Lemma 5.5], a relation $R \subseteq X \times Y$ is an AM-bisimulation if and only if for every $(a,b) \in R$ there is a $|X| \times |Y|$ matrix, $(m_{x,y})$, with entries from $\mathbb{E}$ such that

- there are all but finitely many $m_{x,y} = 0$;
- if $m_{x,y} \neq 0$ then $(x,y) \in R$;
- $\forall x \in X.\, c(a)(x) = \sum_{y \in Y} m_{x,y}$;
- $\forall y \in Y.\, d(b)(y) = \sum_{x \in X} m_{x,y}$;

Let us consider the case for the pair (x_1, y_1). The requirements can be represented by the following table.

0	0	0	0
0	m_{x_2,y_2}	m_{x_2,y_3}	a
0	m_{x_3,y_2}	m_{x_3,y_3}	b
0	c	d	

However, by assumption of non-decomposability there are no m_{x_2,y_2}, m_{x_2,y_3}, m_{x_3,y_2}, m_{x_3,y_3} that can satisfy such requirements, hence $x_1 \not\sim_{AM} y_1$. □

Theorem 8. *Let $\alpha : F \Rightarrow G$ be a natural transformation between two functors F and $G : \mathbf{Set} \to \mathbf{Set}$. If all components α_X of α are injective then the induced functor $\alpha \circ \cdot$ reflects the kernel bisimilarity.*

Proof sketch. Let $X \xrightarrow{m_1} Z \xleftarrow{m_2} Y$ be a cocongruence between any two coalgebras $X \xrightarrow{\alpha_X \circ c} GX$ and $Y \xrightarrow{\alpha_Y \circ d} GY$. Let R be the pullback in **Set** of such cospan, witnessing the bisimilarity between the two G-coalgebras. [2, Theorem 5] prove that the same cocongruence is also witness for the kernel bisimulation between the coalgebras $X \xrightarrow{c} FX$ and $Y \xrightarrow{d} FY$. Therefore, R is also witness of the bisimilarity between the two F-coalgebras. □

Proposition 1. *If the dimension of $\mathscr{H}$ is grater or equal than 2, then $\mathbb{Q}_{\mathscr{H}}$ is not decomposable.*

Proof sketch. Take the following equality $|0\rangle\langle 0| + |1\rangle\langle 1| = |+\rangle\langle +| + |-\rangle\langle -|$. By [19, Proposition 1.63], if R is a positive rank-1 operator, T a positive operator and $T \sqsubseteq R$, then $T = pR$ for some $p \in [0,1]$. Recall that the partial order of quantum effects is the Löwner order, and that the considered effects are positive rank-1 operator. Let $|0\rangle\langle 0| = e_{11} + e_{12}$ for some $e_{11}, e_{12} \in \mathscr{H}_C$. It must be that $e_{11} = p_{11}\,|0\rangle\langle 0|$ for some $p_{11} \in [0,1]$, and similarly $e_{12} = p_{12}\,|0\rangle\langle 0|$, with $p_{11} + p_{12} = 1$. But in order to satisfy the decomposability requirement, $|+\rangle\langle +| = e_{11} + e_{21}$ for some $e_{21} \in \mathscr{H}_C$. Then, $e_{11} = p'_{11}\,|+\rangle\langle +|$ for some $p'_{11} \in [0,1]$. Thus, $p_{11}\,|0\rangle\langle 0| = p'_{11}\,|+\rangle\langle +|$, which holds if and only if $p_{11} = p'_{11} = 0$. Assume without loss of generality that $e_{11} = 0$. The same line of reasoning can show that $p_{12}\,|0\rangle\langle 0| = p'_{12}\,|-\rangle\langle -|$ holds if and only if $p_{12} = p'_{12} = 0$. Hence, both e_{11} and e_{12} must be 0, thus it is impossible to decompose $|0\rangle\langle 0| + |1\rangle\langle 1| = |+\rangle\langle +| + |-\rangle\langle -|$. This construction generalizes easily to dimensions grater than two. □

Theorem 9. *Let (X,c) and (Y,d) be two $D_{\mathbb{L}}$-coalgebras, and let $x \in X$ and $y \in Y$. If x and y are kernel bisimilar in $\downarrow_\rho(X,c)$ and $\downarrow_\rho(Y,d)$ for any ρ, then x and y are kernel bisimilar in (X,c) and (Y,d).*

Proof sketch. Assume x and y are kernel bisimilar in $\downarrow_\rho(X,c)$ and $\downarrow_\rho(Y,d)$ for any ρ. By Lemma 1 a $\widehat{\rho}$ such that the effect algebra homomorphism $L \mapsto tr(L\widehat{\rho})$ is injective. Then, the components of the natural transformation induced by this homomorphism are injective Theorem 2. By assumption, x and y are bisimilar also in $\downarrow_{\widehat{\rho}}(X,c)$ and $\downarrow_{\widehat{\rho}}(Y,d)$. The result then follows by Theorem 7, as $\downarrow_{\widehat{\rho}}$ reflects kernel bisimilarity. □

Theorem 10. *Let (X,c) and (Y,d) be Q_C-coalgebras, and let $x \in X$ and $y \in Y$: $x \sim_k y$ in (X,c) and (Y,d) holds if and only if $x \sim_k y$ holds in $(\alpha_{lpp} \circ \cdot)(X,c)$ and $(\alpha_{lpp} \circ \cdot)(Y,d)$.*

Proof sketch. Assume x and y are kernel bisimilar in (X,c) and (Y,d), then the injective effect algebra homomorphism $L \mapsto \lambda\rho.tr(L\rho)$ defines a natural transformation $\alpha_{lpp} : Q_c \to D_{fc}$ with injective components Lemma 1. By Theorem 7, $\alpha_{lpp} \circ \cdot$ is a functor from $\mathbf{Set}_{Q_C}$ to $\mathbf{Set}_{f_C}$: it preserves both bisimilarities as every functor, moreover, it also reflects kernel bisimilarity since $\alpha_{lpp} : Q_c \to D_{fc}$ has injective components. □

Theorem 11. *Suppose $\{\mathbb{E}_m\}$ is an M-graded effect monoid and M an effect algebra, with $-$ its induced difference operator. There is an extensible graded monad $\{D_{\mathbb{E}_m}\}$ with unit $\eta : Id \to D_{\mathbb{E}_0}$ and multiplication $\mu_{m,n} : D_{\mathbb{E}_m} D_{\mathbb{E}_n} \to D_{\mathbb{E}_{m+n}}$ defined as in Theorem 4 and extension $\xi_{m \preceq n}$*

$$\xi_{m \preceq n}\left(\sum_i e_i \bullet x_i\right) = \sum_i \nabla_{m,n-m}(e_i, 1_{\mathbb{E}_{n-m}}) \bullet x_i.$$

Proof sketch. The desired conditions follow from the unitality and associativity of ∇, and from the fact that $\nabla_{m,n}(1,1) = 1$, since it is an effect bihomomorphism. □

Lemma 2. *$\mathscr{S} = \langle \mathscr{P}(Sys), \emptyset, \uplus, Sys \setminus \cdot \rangle$ is an effect algebra, with $\setminus$ the usual set difference.*

Proof sketch. $Sys \setminus C_1 = C_2$ is the unique element in $\mathscr{P}(Sys)$ such that $C_1 \uplus C_2 = Sys = Sys \setminus \emptyset$ by definition. Finally, by construction, $C \uplus Sys$ is defined only if $C \cap Sys = \emptyset$, i.e. C must be $\emptyset$. □

Lemma 3. *Given two coalgebra homorphisms $f : c \to c', g : d \to d'$, the synchronization operator $\cdot|\cdot$ makes the following diagram commute*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad f \times g \quad} & X' \times Y' \\ \big\downarrow{\scriptstyle c|d} & & \big\downarrow{\scriptstyle c'|d'} \\ \mathscr{P}(T_m X \times T_n Y)^L & \xrightarrow[\mathscr{P}(T_m f \times T_n g)^L]{} & \mathscr{P}(T_m X' \times T_n Y')^L \end{array}$$

Proof sketch. The commutativity of the above diagram correspond to verifying that

- For all $s \parallel t \xrightarrow{\mu} \langle \Delta, \Theta \rangle$, there exists Δ', Θ' such that $f(s) \parallel g(t) \xrightarrow{\mu} \langle \Delta', \Theta' \rangle$ and $\Delta' = f(\Delta), \Theta' = g(\Theta)$
- For all $f(s) \parallel g(t) \xrightarrow{\mu} \langle \Delta', \Theta' \rangle$, there exists Δ, Θ such that $s \parallel t \xrightarrow{\mu} \langle \Delta, \Theta \rangle$ and $\Delta' = f(\Delta), \Theta' = g(\Theta)$

which we can prove by cases on the definition on $\cdot|\cdot$, thanks to the hypothesis on f, which correspond to

- For all $s \xrightarrow{\mu} \Delta$, there exists Δ' such that $f(s) \xrightarrow{\mu} \Delta'$ and $\Delta' = f(\Delta)$
- For all $f(s) \xrightarrow{\mu} \Delta'$, there exists Δ such that $s \xrightarrow{\mu} \Delta$ and $\Delta' = f(\Delta)$

and similarly for g. □

Theorem 12. *Given a transformation $\alpha : T_m X \times T_n Y \to T_{m+n}(X \times Y)$, we can construct a functor $\parallel$: $\mathbf{Set}_{\mathscr{P}(T_m)^L} \times \mathbf{Set}_{\mathscr{P}(T_n)^L} \to \mathbf{Set}_{\mathscr{P}(T_{m+n})^L}$ defined by*

$$(X, c) \parallel (Y, d) = (X \times Y, \mathscr{P}(\alpha)^L \circ (c|d)) \qquad f \parallel g = f \times g$$

Proof sketch. Checking that $\parallel$ is a functor amounts to checking the commutativity of the following diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\quad f \times g \quad} & X' \times Y' \\ \big\downarrow{\scriptstyle c|d} & & \big\downarrow{\scriptstyle c'|d'} \\ \mathscr{P}(T_m X \times T_n Y)^L & \xrightarrow[\mathscr{P}(T_m f \times T_n g)^L]{} & \mathscr{P}(T_m X' \times T_n Y')^L \\ \big\downarrow{\scriptstyle \mathscr{P}(\alpha)^L} & & \big\downarrow{\scriptstyle \mathscr{P}(\alpha)^L} \\ \mathscr{P}(T_{m+n}(X \times Y))^L & \xrightarrow[\mathscr{P}(T_{m+n}(f \times g))^L]{} & \mathscr{P}(T_{m+n}(X' \times Y'))^L \end{array}$$

which holds since the bottom square commutes by naturality of α, and the top squares commute by Lemma 3. □

Theorem 13. *If $s \sim_k t$ then, for any ρ, $s \downarrow_\rho \sim t \downarrow_\rho$. Moreover, if $s' \sim t'$ then $s \parallel s' \sim t \parallel t'$.*

Proof sketch. For the first, notice that $L \mapsto L\downharpoonright_\rho$ is an effect morphism thanks to the linearity of the partial trace. Thus, it yields a natural transformation and a functor by Theorem 2 and Theorem 7, and functors preserve bisimilarity. For the second, thanks to functoriality, we know that $\parallel$ maps cospans in cospans:

$$\left(\begin{array}{ccc} (X,c) & & (Y,d) \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \\ (Z,z) & & \end{array} \quad \Big\| \quad \begin{array}{ccc} (X',c') & & (Y',d') \\ \downarrow{\scriptstyle f'} & \swarrow{\scriptstyle g'} & \\ (Z',z') & & \end{array}\right) = \begin{array}{ccc} (X,c)\parallel(X',c') & & (Y,d)\parallel(Y',d') \\ \downarrow{\scriptstyle f\times g} & \swarrow{\scriptstyle f'\times g'} & \\ (Z,z)\parallel(Z',z') & & \end{array}$$

Then, it is easy to see that the **Set**-pullback of $X \times X' \to Z \times Z' \leftarrow Y \times Y'$ contains all and only the couples $s \parallel s', t \parallel t'$ such that $f(s) = g(t)$ and $f(s') = g(t')$. In other words, the cartesian product of two bisimulations is a bisimulation. $\square$