Extended Abstract: A Convenient Topological Setting for Higher-Order Probability Theory

Benedikt Peterseim, March 28, 2024

In this talk, we outline the construction of a probability monad on the category of compactly generated weakly Hausdorff (CGWH) spaces, thus providing a *convenient topological setting for higher-order probability theory.*

Background: probability monads. Probability monads are one of the centrepieces of category-theoretically informed approaches to probability theory. The fundamental idea of a probability monad is that forming the space of probability measures should yield a monad on a suitable base category of "sample spaces". This was first proposed by Giry in [Gir81], who constructed two probability monads: the Giry monad on the category of measurable spaces and a probability monad on the category of Polish spaces (i.e. completely metrisable, separable topological spaces), also referred to as the Giry monad. Since then, many variations of this idea have been studied [Kei08, FP19, FPR21, VB22]; see also [nLa24] for a quick overview.

Higher-order probability theory. The title of this talk is inspired by the paper *A Convenient Setting for Higher-Order Probability Theory* [HKSY17], which intoduces *quasi-Borel spaces* as a cartesian closed substitute for measurable spaces. We instead work with the category CGWH of CGWH spaces, a (if not *the*) standard cartesian closed substitue for topological spaces. The role of cartesian closedness is to provide a setting for probability theory that accommodates higher-order functions, hence *higher-order probability theory.* Such higher-order settings arise naturally in the context of probabilistic programming languages; see [HKSY17] for further details.

The Riesz probability monad. Our main result is the construction of a probability monad on CGWH, along with establishing some additional properties and structure, such as enrichment and commutativity. Being based on a novel version of the Riesz representation theorem, we call this monad the *Riesz probability monad.* We outline this construction below.

The Baire probability monad. An important cartesian closed subcategory of CGWH is the category QCB of weakly Hausdorff quotients of countably based (QCB) spaces [BSS07, Section 4]. The Riesz probability monad $\mathcal{P}$ restricts to a commutative enriched monad on QCB, and whenever X is a QCB space, $\mathcal{P}(X)$ consists exactly of the Baire measures on X. For this reason, we call the resulting monad on QCB the *Baire probability monad.*

The Baire probability monad is strongly affine. Weakly Hausdorff QCB spaces do not only form a well-behaved category, they are also well-behaved objects in that they share many countability properties with Polish spaces [BSS07, Propositions 4.6, 4.7, 4.8]. One consequence of this is that product measures on QCB spaces are particularly simple to understand in this situation, from which we deduce that the Baire probability monad on QCB is *strongly affine* [Jac16, Definition 1]. The strongly affine property ensures that independence and determinism interact as we would expect. Interestingly, it *fails* for the probability monad quasi-Borel spaces. This results in an example of a probability measure on a product space (in the sense of quasi-Borel spaces) which has a deterministic marginal, but is nevertheless *not* the product of its marginals [FGHL$^+$23, Proposition 2.14; Section 3]. One may interpret this as saying that, in the setting of quasi-Borel spaces, random variables need not be independent from deterministic ones. Working with the Baire probability monad on QCB allows us to avoid such counterintuitive phenomenona.

Relation to classical probability monads. The Baire probability monad on QCB further restricts to the classical Giry monad on the category Pol on Polish spaces. Likewise, the Riesz probability monad

$$
\begin{tikzcd}
& \mathsf{CGWH} \arrow[loop above, "\text{``Riesz monad''}"] & \\
\mathsf{QCB} \arrow[ur, hook] \arrow[loop above, "\text{``Baire monad''}"] & & \mathsf{CompHaus} \arrow[ul, hook'] \arrow[loop below, "\text{``Radon monad''}"] \\
\mathsf{Pol} \arrow[u, hook] \arrow[loop below, "\text{``Giry monad''}"] & &
\end{tikzcd}
$$

Figure 1: Relations between different probability monads. The hooked arrows are fully faithful functors. Each of these can be (trivially) extended to a morphism of monads.

restricts to the well-known Radon monad on the category $\mathsf{CompHaus}$ of compact Hausdorff spaces. The situation is summarised in Figure 1.

Outline of the construction. We now summarise the construction of the Riesz and Baire probability monads in several steps. We write $\mathbb{K}$ for the field of real or complex numbers.

***k*-regular Baire measures.** First, we define the set $\mathcal{M}(X)$ of $\mathbb{K}$-valued *k-regular Baire measures* on a CGWH space X. Here, k-regularity is a very mild regularity condition. When X is a QCB space, every $\mathbb{K}$-valued Baire measure is k-regular, so $\mathcal{M}(X)$ is simply the set of Baire measures on X.

A Riesz representation theorem. We then prove a Riesz representation theorem,

$$\mathcal{M}(X) = C_b(X)',$$

identifying $\mathcal{M}(X)$ with the continuous dual space $C_b(X)'$ of the space $C_b(X)$ of continuous bounded functions on X, *equipped with its natural CGWH topology.* This natural CGWH topology on $C_b(X)$ is defined as that of the sequential colimit (in CGWH),

$$C_b(X) := \operatorname{colim}_{n\in\mathbb{N}} (n \cdot D)^X,$$

where $D \subseteq \mathbb{K}$ is the unit disc. Note that this is *not* the Banach space topology on $C_b(X)$, unless X is compact.

The endofunctor $\mathcal{M}$. The space $C_b(X)$ is an example of a *linear CGWH space*, a notion analogous to, but generally distinct from, that of a topological vector space. Due to the cartesian closedness of CGWH, the continuous dual space V' of a linear CGWH space V inherits a natural CGWH topology from the exponential $\mathbb{K}^V \supseteq V'$, under which it becomes the *natural dual* $V^\wedge$ of V. Thus, under the identification $\mathcal{M}(X) = C_b(X)^\wedge$, the space of k-regular measures becomes a CGWH space. In this way, $\mathcal{M}$ becomes an endofunctor on CGWH, with the pushforward of measures as the action on morphisms.

The monad structure. The endofunctor $\mathcal{M} = C_b(-)^\wedge$ is now given by a double-dualisation-type operation, which allows us to endow it with a commutative CGWH-enriched monad structure in a seamless way. The resulting *Riesz monad* $\mathcal{M}$ restricts to a (commutative enriched) monad on the category QCB of weakly Hausdorff QCB spaces. Since $\mathcal{M}(X)$ consists exactly of the Baire measures on X when X is a QCB space, we call the resulting monad the *Baire monad*.

Passing to probability measures. Finally, passing to the subspace

$$\mathcal{P}(X) \subseteq \mathcal{M}(X)$$

of probability measures yields the Riesz and Baire *probability* monads on CGWH and QCB.

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