Continuous Domains for Function Spaces Using Spectral Compactification

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Abstract

We introduce a continuous domain for function spaces over topological spaces which are not core-compact. Notable examples of such topological spaces include the real line with the upper limit topology, which is used in solution of initial value problems with temporal discretization, and various infinite dimensional Banach spaces which are ubiquitous in functional analysis and solution of partial differential equations. If a topological space $\mathbb{X}$ is not core-compact and $\mathbb{D}$ is a non-singleton bounded-complete domain, the function space $[\mathbb{X} \to \mathbb{D}]$ is not a continuous domain. To construct a continuous domain, we consider a spectral compactification $\mathbb{Y}$ of $\mathbb{X}$ and relate $[\mathbb{X} \to \mathbb{D}]$ with the continuous domain $[\mathbb{Y} \to \mathbb{D}]$ via a Galois connection. This allows us to perform computations in the native structure $[\mathbb{X} \to \mathbb{D}]$ while computable analysis is performed in the continuous domain $[\mathbb{Y} \to \mathbb{D}]$, with the left and right adjoints used for moving between the two function spaces.

Keywords: domain theory, compactification, Stone duality

1 Introduction

The tight link between topology and the theory of computation is well-known and has been investigated extensively in the literature. This link is clearly manifested in the theory of domains [17], which have, in particular, provided a natural computational framework for mathematical analysis. This line of research was initiated by Edalat's work on dynamical systems [4]. Ever since, domains have been used for the study of several other concepts and operators of mathematical analysis, e. g., exact real number computation [12,5], differential equation solving [10], stochastic processes [2], reachability analysis of hybrid systems [9,18], and robustness analysis of neural networks [22].

In such applications, when the topological spaces involved have some desirable properties (e. g., metrizability, local compactness, etc.) the construction of the domain model can be relatively straightforward. In the absence of favourable properties, however, domain models do not arise naturally and one may look for *substitute* constructions, an example of which can be found in [15] for robustness analysis of systems with state spaces that are not (locally) compact.

Another example is encountered in the solution of initial value problems (IVPs). For the Picard method of IVP solving, continuous domains of functions arise naturally [8,10,14]. The situation is slightly different for the methods

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that are based on temporal discretization (e. g., Euler and Runge-Kutta methods). While it is still possible to use classical domain models when an imperative style of computation is adopted [7], a functional implementation via the fixpoint operator requires a substitute domain construction [6]. Let us discuss this in more detail. Assume that the following IVP is given:

$$\begin{cases} y'(t) = f(y(t)), \\ y(t_0) = y_0, \end{cases} \tag{1}$$

in which $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field, for some natural number $n \geq 1$. Assume that a solution exists over a lifetime of $[t_0, T]$, for some $T > t_0$. In a domain-theoretic framework, one would search for a solution of (1) in the space of functions from $[t_0, T]$ to the interval domain:

$$\mathbb{IR}^n_\perp := \{\mathbb{R}^n\} \cup \left\{ \prod_{i=1}^{n} [a_i, b_i] \,\middle|\, \forall i \in \{1, \ldots, n\} : a_i, b_i \in \mathbb{R} \text{ and } a_i \leq b_i \right\}, \tag{2}$$

ordered by superset relation, i. e., $\forall X, Y \in \mathbb{IR}^n_\perp : X \sqsubseteq Y \overset{\Delta}{\iff} X \supseteq Y$. Hence, the set $\mathbb{R}^n$ is the least element of the interval domain $\mathbb{IR}^n_\perp$.

For applications such as differential equation solving, the interval domain $\mathbb{IR}^n_\perp$ is considered with the Scott topology. What is equally important is the topology on the interval $[t_0, T]$. For the Picard method, the Euclidean topology on $[t_0, T]$ is the suitable topology. As the Euclidean topology over $[t_0, T]$ is compact, the space of functions from $[t_0, T]$ to $\mathbb{IR}^n_\perp$—which are continuous with respect to the Euclidean topology on $[t_0, T]$ and the Scott topology on $\mathbb{IR}^n_\perp$—ordered by pointwise ordering is a continuous domain.

The Euclidean topology, however, is not suitable in a functional framework in the presence of temporal discretization. To see this, note that, by integrating both sides of (1), we obtain $y(t + h) = y(t) + \int_t^{t+h} f(y(\tau))\, \mathrm{d}\tau$, for all $t \in [t_0, T]$ and $h \in [0, T - t]$. This can be written as:

$$y(t + h) = y(t) + i(t, h), \tag{3}$$

in which the integral $i(t, h)$ represents the dynamics of the solution from t to $t + h$. Thus, a general schema for validated solution of the IVP (1) with temporal discretization may be envisaged as follows:

(i) For some $k \geq 1$, consider the partition $Q = (q_0, \ldots, q_k)$ of the interval $[t_0, T]$.

(ii) Let $Y(t_0) := y_0$.

(iii) For each $j \in \{0, \ldots, k - 1\}$ and $h \in (0, q_{j+1} - q_j]$:

$$Y(q_j + h) := Y(q_j) + I(q_j, h), \tag{4}$$

where $I(q_j, h)$ is an interval enclosure of the integral factor $i(q_j, h)$ from equation (3). The operator I, in general, depends on several parameters, including (enclosures of) the vector field and its derivatives, the enclosure $Y(q_j)$, the index j, etc.

In (4), the operator '+' denotes interval addition, and for the method to be validated, the term $I(q_j, h)$ must account for all the inaccuracies, e. g., floating-point error, truncation error, etc.

In step (iii) of the schema, the solver moves forward in time, from q_j to q_{j+1}. This requires keeping the state, i. e., the solution up to the partition point q_j, and referring to this state in iteration j. As such, the schema has an imperative style, and indeed encompasses various validated approaches to IVP solving with temporal discretization in the literature, including [7]. This is in contrast with the functional style adopted in the definition of the Picard operator in [8,10], and in language design for real number computation. For instance, the languages designed in [12,13,3] for computation over real numbers and real functions are functional languages based on lambda calculus, with their denotational semantics provided by domain models.

In a functional framework, the solution of the IVP (1) is obtained as the fixpoint of a higher-order operator. Domain models are particularly suitable for fixpoint computations of this type. A straightfoward (but, flawed) way of obtaining a fixpoint formulation for the above general schema is to define a functional Φ over interval functions as follows:

$$\Phi(Y)(x) := \begin{cases} y_0, & \text{if } x = t_0, \\ Y(q_j) + I(q_j, x - q_j), & \text{if } q_j < x \le q_{j+1}. \end{cases}$$

The fixpoint of this operator (if it exists) will be the right choice. The problem is that, the enclosures obtained by applying Φ do not have upper (respectively, lower) semi-continuous upper (respectively, lower) bounds. Hence, by [6, Proposition 2.10], this results in approximations of the solution of the IVP which are not continuous with respect to the Euclidean topology on $[t_0, T]$, and can only be continuous with respect to the so-called upper limit topology. Recall that the upper limit topology has as its base the collection $\{(a, b] \mid a, b \in \mathbb{R}\}$ of half-open intervals. This topology is known not to be locally compact (see, e. g., [6, Proposition 4.5]). This shortcoming motivated the substitute construction presented in [6], where a more detailed justification of why the upper limit topology is needed can also be found, together with fixpoint formulations of Euler and Runge-Kutta operators.

In a more general setting, assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a T_0 topological space, and $\mathbb{D}$ is a non-singleton bounded-complete domain (bc-domain). From [11], we know that the function space $[\mathbb{X} \to \mathbb{D}]$ is a bc-domain if and only if $\mathbb{X}$ is core-compact. Thus, the main challenge is to develop a computational framework for the function space $[\mathbb{X} \to \mathbb{D}]$ when $\mathbb{X}$ is not core-compact, e. g., when $\mathbb{X}$ is the interval $[t_0, T]$ endowed with the upper limit topology, and $\mathbb{D} = \mathbb{IR}^n_\perp$ with the Scott topology. In [6], a domain is constructed as a substitute for this function space via abstract bases. In contrast with [6], the aim of the current article is to show that we can work directly on the space $\mathbb{X}$ and obtain the same substitute via Stone duality. To be more specific, using the well-known results in Stone duality, we construct a topological space $\hat{\mathbb{X}}$ with the following properties:

- $\hat{\mathbb{X}}$ is a core-compact (in fact, spectral) space and $\mathbb{X}$ can be embedded into $\hat{\mathbb{X}}$ as a dense subspace.
- The function spaces $[\mathbb{X} \to \mathbb{D}]$ and $[\hat{\mathbb{X}} \to \mathbb{D}]$ are related via a Galois connection.

Such a construction is useful for computable analysis within a domain framework. When $\mathbb{X}$ is not core-compact, the non-continuous directed-complete partial order (dcpo) $[\mathbb{X} \to \mathbb{D}]$ is used for implementation of algorithms, whereas analysis of computability is carried out over the continuous domain $[\hat{\mathbb{X}} \to \mathbb{D}]$, subject to the existence of a suitable effective structure over $[\hat{\mathbb{X}} \to \mathbb{D}]$.

1.1 Related Work

Compactification is a fundamental concept in topology. Classical examples such as Stone-Čech and one-point compactification [19] have been introduced primarily for Hausdorff topological spaces. In the non-Hausdorff setting, Smyth's stable compactification [20] is the closest to ours. In fact, our construction can be obtained as a special case of Smyth's stable compactification by considering the so-called fine quasi-proximities, resulting in compactifications that are spectral. In [20], this special case is refered to as spectralization, whereas we use the term *spectral compactification* to emphasize the compactification aspect of the construction. Spectral compactification is indeed an important special case of stable compactification which is suitable for computational purposes. Here, we keep the presentation simple. Specifically, we do not use quasi-proximities which form the foundation of Smyth's construction. This is sufficient for us and we obtain all the basic properties that we need in this simpler framework. We point out that spectral compactification is fundamentally different from Stone-Čech and one-point compactifications in that, even when a space $\mathbb{X}$ is compact and Hausdorff, its spectral compactification may not be T_1 (for an example, see [20, page 338]).

Another aspect of our work here is the idea of a substitute construction. Such constructions can be useful when the topological spaces do not have favourable properties. In [6], we used the idea of a substitute construction in the context of IVP solving. Another example is presented in [15], in the context of robustness analysis. In [15], we studied robustness analysis of systems with state spaces $\mathbb{S}$ which are not (locally) compact. In such cases, the lattice $\hat{\mathbb{C}}(\mathbb{S})$ of closed subsets of $\mathbb{S}$ (under superset relation) may not be continuous, let alone ω-continuous. The lattice of closed subsets is central to robustness analysis. Hence, we construct an ω-continuous lattice $\mathbb{L}$ which is related to $\hat{\mathbb{C}}(\mathbb{S})$ via a suitable adjunction.

1.2 Structure of the Paper

The preliminaries, including a brief reminder of basic concepts from domain theory and Stone duality, are presented in Section 2. In Section 3, we establish a Galois connection between the function spaces $[\mathbb{X} \to \mathbb{D}]$ and $[\mathbb{Y} \to \mathbb{D}]$, where $\mathbb{D}$ is a bc-domain, $\mathbb{X}$ and $\mathbb{Y}$ are topological spaces, and $\mathbb{X}$ is densely embedded in $\mathbb{Y}$. A detailed account of the spectral compactification of topological spaces is presented in Section 4. A continuous domain for the space of functions from an arbitrary T_0 space $\mathbb{X}$ to a bc-domain $\mathbb{D}$ is constructed in Section 5 based on spectral compactification of $\mathbb{X}$, and we prove that the result is equivalent to the construction based on abstract bases developed in [6]. We conclude the article with some remarks in Section 6.

2 Preliminaries

Basic familiarity with domain theory and Stone duality [1,16] will be helpful in understanding the main results of the paper. In this section, we present a brief reminder of the preliminary concepts, and establish some notations and definitions.

For arbitrary sets X and Y, by $X \subseteq_f Y$ we mean X is a finite subset of Y. Assume that $(D, \sqsubseteq)$ is a partially ordered set (poset) and $A \subseteq D$. We define $\downarrow A := \{x \in D \mid \exists a \in A : x \sqsubseteq a\}$, and when A is a singleton $\{a\}$, we may simply write $\downarrow a$ instead of $\downarrow \{a\}$. We denote the join (also known as the least upper bound) of A by $\bigvee A$, and the meet (also known as the greatest lower bound) of A by $\bigwedge A$, whenever they exist. A subset $A \subseteq D$ is said to be directed if it is non-empty and $\forall x, y \in A : \exists z \in A : x \sqsubseteq z$ and $y \sqsubseteq z$, in which case, we write $A \subseteq_{dir} D$. The poset $(D, \sqsubseteq)$ is said to be a directed-complete partial order (dcpo) if $\forall A \subseteq_{dir} D : \bigvee A$ exists in D. The poset $(D, \sqsubseteq)$ is said to be pointed if it has a bottom element $\bot$.

Assume that $(D, \sqsubseteq)$ is a dcpo and let $x, y \in D$. The element x is said to be *way-below* y—written as $x \ll y$—if for every directed subset A of D, if $y \sqsubseteq \bigvee A$, then there exists an element $d \in A$ such that $x \sqsubseteq d$. An element $x \in D$ is said to be *finite* if $x \ll x$.

For every element x of a dcpo $\mathbb{D} \equiv (D, \sqsubseteq)$, let $\twoheaddownarrow x := \{a \in D \mid a \ll x\}$. A subset $B \subseteq D$ is said to be a *basis* for $\mathbb{D}$ if for every element $x \in D$, the set $B_x := \twoheaddownarrow x \cap B$ is a directed subset and $x = \bigvee B_x$. A dcpo is said to be (ω-)continuous if it has a (countable) basis, and it is said to be (ω-)algebraic if it has a (countable) basis consisting entirely of finite elements.

Definition 2.1 [Domain] We call $\mathbb{D} \equiv (D, \sqsubseteq)$ a domain if it is a continuous pointed dcpo.

Apart from order-theoretic structure, domains also have a topological structure. Assume that $\mathbb{D} \equiv (D, \sqsubseteq)$ is a poset. A subset $O \subseteq D$ is said to be *Scott open* if it has the following properties:

(1) It is an upper set, i. e., $\forall x \in O, \forall y \in D : x \sqsubseteq y \implies y \in O$.

(2) For every directed set $A \subseteq D$ for which $\bigvee A$ exists, if $\bigvee A \in O$ then $A \cap O \neq \emptyset$.

The collection of all Scott open subsets of a poset $\mathbb{D}$ forms a T_0 topology $\sigma_{\mathbb{D}}$, refered to as the Scott topology. A function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is said to be Scott continuous if it is continuous with respect to the Scott topologies on $\mathbb{D}_1$ and $\mathbb{D}_2$. Scott continuity can be stated purely in order-theoretic terms, i. e., a map $f : (D_1, \sqsubseteq_1) \to (D_2, \sqsubseteq_2)$ between two posets is Scott continuous if and only if it is monotonic and preserves the suprema of directed sets, i. e., for every directed set $X \subseteq D_1$ for which $\bigvee X$ exists, we have $f(\bigvee X) = \bigvee f(X)$ [16, Proposition 4.3.5].

A poset $(D, \sqsubseteq)$ is said to be a lattice if it is closed under binary join and binary meet. A lattice $(D, \sqsubseteq)$ is called:

- bounded if it has both a bottom element $\bot$ and a top element $\top$.
- complete if $\forall A \subseteq D : \bigvee A$ exists in D. Note that every complete lattice must be bounded, with $\bot = \bigvee \emptyset$ and $\top = \bigvee D$.
- distributive if $\forall x, y, z \in D : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

By a continuous lattice we mean a complete lattice with a basis. Other variants (i. e., ω-continuous, algebraic, and ω-algebraic) are defined accordingly. Of particular interest to our discussion are the arithmetic lattices, i. e., continuous distributive lattices $(D, \sqsubseteq)$ with the following property:

$$\forall x, y, z \in D : \quad (x \ll y \text{ and } x \ll z) \implies x \ll y \wedge z.$$

For every topological space $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$, the poset $(\tau_{\mathbb{X}}, \subseteq)$ of open subsets of X ordered by subset relation is a complete distributive lattice, in which $\bot = \emptyset$ and $\top = X$. Furthermore, we have:

$$\forall A \subseteq \tau_{\mathbb{X}} : \quad \bigvee A = \bigcup A \text{ and } \bigwedge A = (\bigcap A)^{\circ},$$

where $(\cdot)^\circ$ denotes the interior operator. When the lattice $(\tau_{\mathbb{X}}, \subseteq)$ is continuous, the topological space $\mathbb{X}$ is said to be *core-compact*. Core-compactness is a desirable property which guarantees that we obtain 'well-behaved' function spaces.

Recall that a dcpo $(D, \sqsubseteq)$ is *bounded-complete* if each bounded subset $A \subseteq D$ has a join $\bigvee A \in D$. Let $(D, \sqsubseteq_0)$ be a bounded-complete domain (bc-domain). We let $\mathbb{D} \equiv (D, \sigma_{\mathbb{D}})$ denote the topological space with the carrier set D endowed with the Scott topology $\sigma_{\mathbb{D}}$. The space $[\mathbb{X} \to \mathbb{D}]$ of functions $f : X \to D$ which are $(\tau_{\mathbb{X}}, \sigma_{\mathbb{D}})$ continuous can be ordered pointwise by defining:

$$\forall f, g \in [\mathbb{X} \to \mathbb{D}] : \quad f \sqsubseteq g \iff \forall x \in X : f(x) \sqsubseteq_0 g(x).$$

It is straightfoward to verify that the poset $([\mathbb{X} \to \mathbb{D}], \sqsubseteq)$ is directed-complete and $\forall x \in X : (\bigvee_{i \in I} f_i)(x) = \bigvee \{f_i(x) \mid i \in I\}$, for any $\{f_i \mid i \in I\} \subseteq_{dir} [\mathbb{X} \to \mathbb{D}]$. By 'well-behaved' function spaces we mean those for which the dcpo $([\mathbb{X} \to \mathbb{D}], \sqsubseteq)$ is continuous:

Theorem 2.2 *For any topological space $\mathbb{X}$ and non-singleton bc-domain $\mathbb{D}$, the function space $([\mathbb{X} \to \mathbb{D}], \sqsubseteq)$ is a bc-domain $\iff$ $\mathbb{X}$ is core-compact.*

Proof. For the $(\Leftarrow)$ direction, see [11, Proposition 2]. A proof of the $(\Rightarrow)$ direction can also be found on [11, pages 62 and 63]. □

The connection between topology and order theory runs much deeper than stated so far in our discussion. We briefly mention some more results on this as they will be needed later on, but the interested reader may refer to [1,16] for a more comprehensive account of the connection.

A complete lattice $\mathbb{L}$ is called a frame if it satisfies the infinite distributivity law $x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$. It is not difficult to verify that, for every topological space $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$, the complete lattice $(\tau_{\mathbb{X}}, \subseteq)$ is indeed a frame. A map $f : \mathbb{K} \to \mathbb{L}$ between two complete lattices is called a frame homomorphism if it preserves finite meets and arbitrary joins. We let $\mathcal{Frm}$ denote the category of frames and frame homomorphisms, and we let $\mathcal{Top}$ denote the category of topological spaces and continuous functions. The map that assigns $(\tau_{\mathbb{X}}, \subseteq)$ to $\mathbb{X}$ can be extended to a functor $\Omega : \mathcal{Top} \to \mathcal{Frm}^{op}$ by mapping every continuous function $f : \mathbb{X} \to \mathbb{Y}$ to $\Omega(f) : \Omega(\mathbb{Y}) \to \Omega(\mathbb{X})$ defined by $\forall O \in \tau_{\mathbb{Y}} : \Omega(f)(O) = f^{-1}(O)$.

Going in the opposite direction, i. e., recovering a topological space from the lattice $\mathbb{L}$ of its open sets, is the core of Stone duality. A subset F of a complete lattice $\mathbb{L} \equiv (L, \sqsubseteq)$ is called a filter if it is non-empty and satisfies the following two conditions:

(i) F is an upper set, i. e., $\forall x \in F, y \in L : x \sqsubseteq y \implies y \in F$.

(ii) F is downward directed, i. e., $\forall x, y \in F : x \wedge y \in F$.

A filter $F \subseteq L$ is said to be completely prime if $\forall A \subseteq L : \bigvee A \in F \implies A \cap F \neq \emptyset$. Notice the similarity with the definition of Scott open sets, except that here we allow $A \subseteq L$ to be arbitrary, not just directed. In particular, every completely prime filter is Scott open. When $\mathbb{L} = (\tau_{\mathbb{X}}, \subseteq)$, the filter of all the open neighborhoods of any given $x \in X$ is completely prime. Taking that as a guide, for any complete lattice $\mathbb{L} \equiv (L, \sqsubseteq)$, by a point of $\mathbb{L}$ we mean a completely prime filter $F \subseteq L$. We let $\mathrm{pt}(\mathbb{L})$ denote the set of points of $\mathbb{L}$ with the so-called hull-kernel topology, with open sets $O_u := \{x \in \mathrm{pt}(\mathbb{L}) \mid u \in x\}$, where u ranges over all the elements of L [16, Proposition 8.1.13]. The map pt can be extended to a functor $\mathrm{pt} : \mathcal{Frm}^{op} \to \mathcal{Top}$ as follows: for any morphism $g : \mathbb{L} \to \mathbb{K}$ in $\mathcal{Frm}^{op}$ (i. e., a frame homomorphism $g : \mathbb{K} \to \mathbb{L}$) the function $\mathrm{pt}(g) : \mathrm{pt}(\mathbb{L}) \to \mathrm{pt}(\mathbb{K})$ maps every completely prime filter x of $\mathbb{L}$ to $g^{-1}(x)$. It is well-known that pt is right adjoint to Ω:

$$\mathcal{Frm}^{op} \underset{\Omega}{\overset{\mathrm{pt}}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \mathcal{Top}$$

Of particular interest are the cases where this adjunction restricts to an equivalence between sub-categories of $\mathcal{Top}$ and $\mathcal{Frm}^{op}$. For a detailed account, see [1, Section 7].

Let us call an arithmetic lattice in which the top element is finite (i. e., $\top \ll \top$) a fully arithmetic lattice. The above adjunction restricts to an equivalence

$$\mathcal{A}fal^{op} \underset{\Omega}{\overset{\text{pt}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{S}pec \tag{5}$$

between the opposite of the category $\mathcal{A}fal$ of algebraic fully arithmetic lattices and frame homomorphisms on the one hand, and the category $\mathcal{S}pec$ of the so-called *spectral* spaces on the other [1, Theorem 7.2.22.]. A spectral space is a compact, sober, coherent, and *strongly locally compact* space. This last property is of relevance to our discussion later. Let $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$ be a topological space. We call a subset $Q \subseteq Y$ *compact-open* if it is both compact and open.

Definition 2.3 We say that $\mathbb{Y}$ is a *strongly locally compact* space if its topology has a base of compact-open subsets.

Proposition 2.4 *Let $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$ be a topological space. A set $Q \subseteq Y$ is compact-open if and only if Q is a finite element of the complete lattice $(\tau_{\mathbb{Y}}, \subseteq)$, i. e., $Q \ll Q$.*

Proof. Straightforward. □

Lemma 2.5 *A topological space $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$ is strongly locally compact if and only if $(\tau_{\mathbb{Y}}, \subseteq)$ is an algebraic lattice.*

Proof. To prove the ($\Rightarrow$) direction, we first note that for any topological space $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$, the lattice $(\tau_{\mathbb{Y}}, \subseteq)$ is complete. So, it remains to show that $(\tau_{\mathbb{Y}}, \subseteq)$ is algebraic. Take any open set $O \in \tau_{\mathbb{Y}}$. As $\mathbb{Y}$ is assumed to be strongly locally compact, for each $x \in O$, there exists a compact-open $Q_x \in \tau_{\mathbb{Y}}$ such that $x \in Q_x \subseteq O$, which implies that $O = \bigcup_{x \in O} Q_x$. Hence, by Proposition 2.4, the set of all compact-open subsets of Y forms a basis of finite elements for $(\tau_{\mathbb{Y}}, \subseteq)$.

To prove the ($\Leftarrow$) direction, take any open set $O \in \tau_{\mathbb{Y}}$. As $(\tau_{\mathbb{Y}}, \subseteq)$ is assumed to be algebraic, then $O = \bigvee\{Q \ll O \mid Q \text{ is finite}\}$, which implies that $\forall x \in O : \exists Q_x \ll O : x \in Q_x \subseteq O$ and Q_x is a finite element of $(\tau_{\mathbb{Y}}, \subseteq)$. By Proposition 2.4, this Q_x must be compact-open. □

In this article, we start our construction with bounded distributive lattices of open sets, which are closely related to algebraic fully arithmetic lattices. Assuming that $\mathbb{L}_1$ and $\mathbb{L}_2$ are two bounded distributive lattices, a map $f : \mathbb{L}_1 \to \mathbb{L}_2$ is said to be a *bounded lattice homomorphism* if it is a monotone map that preserves all finite joins and all finite meets. Let $\mathcal{BDL}at$ denote the category of bounded distributive lattices and bounded lattice homomorphisms. Then, there is an equivalence of categories:

$$\mathcal{A}fal \underset{\text{Idl}}{\overset{\mathcal{K}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{BDL}at \tag{6}$$

in which, for any algebraic fully arithmetic lattice $\mathbb{L}$, $\mathcal{K}(\mathbb{L})$ is the bounded distributive lattice of finite elements of $\mathbb{L}$. The functor $\text{Idl} : \mathcal{BDL}at \to \mathcal{A}fal$ is ideal completion.

By composing diagrams (5) and (6), we obtain:

$$\mathcal{BDL}at^{op} \underset{\mathcal{K}^{op}}{\overset{\text{Idl}^{op}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{A}fal^{op} \underset{\Omega}{\overset{\text{pt}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{S}pec \tag{7}$$

A detailed account of this equivalence, in the framework of the current article, will be presented in Section 4.

3 Basic Galois Connection

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ and $\mathbb{Y} \equiv (Y, \tau_{\mathbb{Y}})$ are two topological spaces.

Definition 3.1 [Quasi-embedding, Embedding] A continuous map $\iota : \mathbb{X} \to \mathbb{Y}$ is said to be:

(i) a quasi-embedding of $\mathbb{X}$ into $\mathbb{Y}$ if it is *relatively open*, i. e., $\forall U \in \tau_{\mathbb{X}} : \exists V \in \tau_{\mathbb{Y}} : U = \iota^{-1}(V)$.

(ii) an embedding of $\mathbb{X}$ into $\mathbb{Y}$ if it is an injective quasi-embedding.

Also, recall that the map ι is said to be dense if $\iota(X)$ is a dense subset of Y.

Let $\mathcal{T}op_0$ denote the category of T_0 topological spaces and continuous maps. Over T_0 spaces, the two notions of Definition 3.1 coincide:

Proposition 3.2 *Assume that $\mathbb{X} : \mathcal{T}op_0$, $\mathbb{Y} : \mathcal{T}op$, and $\iota : \mathbb{X} \to \mathbb{Y}$ is relatively open. Then ι must be injective. As a consequence, when $\mathbb{X} : \mathcal{T}op_0$, every quasi-embedding $\iota : \mathbb{X} \to \mathbb{Y}$ is an embedding.*

Proof. Let $x_1 \neq x_2 \in X$. Since $\mathbb{X}$ is assumed to be T_0, without loss of generality, we assume that there exists an open $U \in \tau_{\mathbb{X}}$ such that $x_1 \in U$ and $x_2 \notin U$. As ι is relatively open, there exists an open $V \in \tau_{\mathbb{Y}}$ such that $U = \iota^{-1}(V)$. Hence, $\iota(x_1) \in V$ while $\iota(x_2) \notin V$. Therefore, $\iota(x_1) \neq \iota(x_2)$. □

Let us assume that $\mathbb{D}$ is a bc-domain with Scott topology. If $\iota : \mathbb{X} \to \mathbb{Y}$ is a dense quasi-embedding, then, as we will see, the two function spaces $[\mathbb{X} \to \mathbb{D}]$ and $[\mathbb{Y} \to \mathbb{D}]$ are related via a *Galois connection*:

Definition 3.3 [Category $\mathcal{P}o$, Galois connection $F \dashv G$] We let $\mathcal{P}o$ denote the category of posets and monotonic maps. A Galois connection in the category $\mathcal{P}o$ between two posets $\mathbb{C} \equiv (C, \sqsubseteq_{\mathbb{C}})$ and $\mathbb{D} \equiv (D, \sqsubseteq_{\mathbb{D}})$ is a pair of monotonic maps:

$$\mathbb{D} \underset{F}{\overset{G}{\rightleftarrows}} \mathbb{C} \quad (\top)$$

such that:

$$\forall x \in C : \forall y \in D : x \sqsubseteq_{\mathbb{C}} G(y) \iff F(x) \sqsubseteq_{\mathbb{D}} y.$$

In this case, we call $F : \mathbb{C} \to \mathbb{D}$ the left adjoint and $G : \mathbb{D} \to \mathbb{C}$ the right adjoint, and write $F \dashv G$.

Our aim is to show that the two function spaces are related via the following Galois connection:

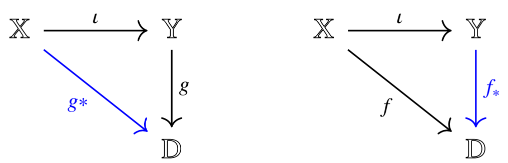
$$[\mathbb{X} \to \mathbb{D}] \underset{(\cdot)^*}{\overset{(\cdot)_*}{\rightleftarrows}} [\mathbb{Y} \to \mathbb{D}], \quad (\top)$$

in which:

$$\forall g \in [\mathbb{Y} \to \mathbb{D}] : \quad g^* \coloneqq g \circ \iota, \tag{8}$$

and:

$$\forall f \in [\mathbb{X} \to \mathbb{D}] : \forall y \in Y : \quad f_*(y) \coloneqq \bigvee \left\{ \bigwedge f(\iota^{-1}(U)) \,\middle|\, y \in U \in \tau_{\mathbb{Y}} \right\}. \tag{9}$$



Proposition 3.4 *The map $(\cdot)^* : [\mathbb{Y} \to \mathbb{D}] \to [\mathbb{X} \to \mathbb{D}]$ is well-defined, i. e.:*

$$\forall g \in [\mathbb{Y} \to \mathbb{D}] : g^* \in [\mathbb{X} \to \mathbb{D}].$$

Proof. Follows from the fact that ι is continuous. □

Proposition 3.5 *For all $f \in [\mathbb{X} \to \mathbb{D}]$ and $y \in Y$, $f_*(y)$ is well-defined.*

Proof. As the quasi-embedding ι is dense, for all non-empty $U \in \tau_{\mathbb{Y}} : \iota^{-1}(U) \neq \emptyset$, which implies that $f(\iota^{-1}(U)) \neq \emptyset$. Since D is assumed to be a bc-domain, it must have the infima of all non-empty subsets [1, Proposition 4.1.2]. Hence, $\bigwedge f(\iota^{-1}(U))$ exists.

Next, we must show that the set $A \coloneqq \left\{ \bigwedge f(\iota^{-1}(U)) \,\middle|\, y \in U \in \tau_{\mathbb{Y}} \right\}$ is directed. Take $U_1, U_2 \in \tau_{\mathbb{Y}}$ such that $y \in U_1$ and $y \in U_2$. Then, $y \in U_1 \cap U_2$, hence $U_1 \cap U_2 \neq \emptyset$, and $\bigwedge f(\iota^{-1}(U_1 \cap U_2)) \in A$ is an upper bound of both $\bigwedge f(\iota^{-1}(U_1))$ and $\bigwedge f(\iota^{-1}(U_2))$. Therefore, A is directed and $\bigvee A$ exists. □

Proposition 3.6 *The map $(\cdot)_* : [\mathbb{X} \to \mathbb{D}] \to [\mathbb{Y} \to \mathbb{D}]$ is well-defined, i. e., $\forall f \in [\mathbb{X} \to \mathbb{D}] : f_* \in [\mathbb{Y} \to \mathbb{D}]$.*

Proof. We must prove that for all $f \in [\mathbb{X} \to \mathbb{D}]$, f_* is continuous with respect to $\tau_{\mathbb{Y}}$ and the Scott topology $\sigma_{\mathbb{D}}$. It suffices to show that, for any $e \in D$, the set $f_*^{-1}(\Uparrow e)$ is open. Take any $y \in Y$. Then:

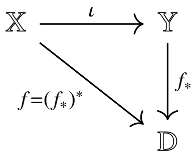
$$\begin{aligned} y \in f_*^{-1}(\Uparrow e) &\implies e \ll f_*(y) \\ \text{(by (9))} &\implies \exists U_0 \in \tau_{\mathbb{Y}} : e \ll \bigwedge f(\iota^{-1}(U_0)) \text{ and } y \in U_0. \end{aligned}$$

Note that U_0 is an open neighborhood of y. We claim that $U_0 \subseteq f_*^{-1}(\Uparrow e)$. Take any arbitrary $\hat{y} \in U_0$. Then:

$$e \ll \bigwedge f(\iota^{-1}(U_0)) \sqsubseteq \bigvee \left\{ \bigwedge f(\iota^{-1}(U)) \,\middle|\, \hat{y} \in U \in \tau_{\mathbb{Y}} \right\} = f_*(\hat{y}).$$

Hence, $\hat{y} \in f_*^{-1}(\Uparrow e)$. □

Proposition 3.7 *For every $f \in [\mathbb{X} \to \mathbb{D}]$, we have $f = f_* \circ \iota$. In particular, $f = (f_*)^*$.*



Proof. Assume that $x \in X$. For any $U \in \tau_{\mathbb{Y}}$ satisfying $\iota(x) \in U$ we have $x \in \iota^{-1}(U)$. As a result, $\bigwedge f(\iota^{-1}(U)) \sqsubseteq f(x)$, which implies that $f_*(\iota(x)) = \bigvee \left\{ \bigwedge f(\iota^{-1}(U)) \,\middle|\, \iota(x) \in U \in \tau_{\mathbb{Y}} \right\} \sqsubseteq f(x)$.

Next, we show that $f(x) \sqsubseteq f_*(\iota(x))$. Take an arbitrary $e \ll f(x)$. Hence, $f^{-1}(\Uparrow e)$ is an open neighborhood of x. As $\iota : \mathbb{X} \to \mathbb{Y}$ is relatively open, for some $U \in \tau_{\mathbb{Y}}$ we have $f^{-1}(\Uparrow e) = \iota^{-1}(U)$. Also, note that $\iota(x) \in U$. Thus, we have:

$$\begin{aligned} \left(\forall y \in \iota^{-1}(U) : e \ll f(y)\right) &\implies e \sqsubseteq \bigwedge f(\iota^{-1}(U)) \\ \text{(by (9) and } \iota(x) \in U) &\implies e \sqsubseteq f_*(\iota(x)). \end{aligned}$$

In conclusion, $f(x) \sqsubseteq f_*(\iota(x))$. □

Let us briefly recall the concept of a monotone section-retraction pair. Assume that $\mathbb{D}$ and $\mathbb{E}$ are two posets. A pair of maps $s : \mathbb{D} \to \mathbb{E}$ and $r : \mathbb{E} \to \mathbb{D}$ is called a monotone section-retraction pair if s and r are monotone and $r \circ s = \mathrm{id}_{\mathbb{D}}$. In this case, $\mathbb{D}$ is said to be a monotone retract of $\mathbb{E}$. It is straightforward to verify that if s and r form a section-retraction pair, then s must be injective and r must be surjective. For a more detailed account of section-retraction pairs the reader may refer to [1, Section 3.1.1].

Lemma 3.8 *The maps $(\cdot)_*$ and $(\cdot)^*$ form a monotone section retraction pair between $[\mathbb{X} \to \mathbb{D}]$ and $[\mathbb{Y} \to \mathbb{D}]$.*

Proof. Monotonicity of $(\cdot)^*$ follows from that of composition, i. e.:

$$\forall g, g' \in [\mathbb{Y} \to \mathbb{D}] : \quad g \sqsubseteq g' \implies g \circ \iota \sqsubseteq g' \circ \iota.$$

Monotonicity of $(\cdot)_*$ follows from monotonicity of meet and join. To be more precise, take $f, f' \in [\mathbb{X} \to \mathbb{D}]$ satisfying $f \sqsubseteq f'$, and assume that $y \in Y$. Then, $\forall U \in \tau_{\mathbb{Y}} : \bigwedge f(\iota^{-1}(U)) \sqsubseteq \bigwedge f'(\iota^{-1}(U))$. Thus, $\bigvee \left\{ \bigwedge f(\iota^{-1}(U)) \,\middle|\, y \in U \in \tau_{\mathbb{Y}} \right\} \sqsubseteq \bigvee \left\{ \bigwedge f'(\iota^{-1}(U)) \,\middle|\, y \in U \in \tau_{\mathbb{Y}} \right\}$, which, by (9), implies that $f_*(y) \sqsubseteq f'_*(y)$.

By Proposition 3.7, $\forall f \in [\mathbb{X} \to \mathbb{D}] : f = (f_*)^*$. Therefore, $(\cdot)_*$ and $(\cdot)^*$ form a section retraction pair. □

Theorem 3.9 (Galois connection) *Assume that $\mathbb{X}$ and $\mathbb{Y}$ are two topological spaces, $\iota : \mathbb{X} \to \mathbb{Y}$ is a dense quasi-embedding, and $\mathbb{D}$ is a bc-domain. Then, the maps $(\cdot)^*$ and $(\cdot)_*$ (defined in (8) and (9), respectively) form a Galois connection:*

$$[\mathbb{X} \to \mathbb{D}] \underset{(\cdot)^*}{\overset{(\cdot)_*}{\rightleftarrows}} [\mathbb{Y} \to \mathbb{D}], \tag{10}$$

in the category $\mathcal{P}o$, in which, $(\cdot)_*$ is the right adjoint, and $(\cdot)^*$ is the left adjoint. Furthermore:

(i) The map $(\cdot)^*$ is surjective, and $(\cdot)_*$ is injective.

(ii) $(\cdot)^* \circ (\cdot)_* = id_{[\mathbb{X}\to\mathbb{D}]}$, i. e., $\forall f \in [\mathbb{X} \to \mathbb{D}] : (f_*)^* = f$.

(iii) The left adjoint $(\cdot)^*$ is Scott continuous.

Proof. Claims (i) and (ii) follow from Lemma 3.8. For any adjunction between two dcpos, the left adjoint is Scott continuous [1, Proposition 3.1.14]. So, given that both $[\mathbb{X} \to \mathbb{D}]$ and $[\mathbb{Y} \to \mathbb{D}]$ are dcpos, claim (iii) will also be established once we prove that the maps $(\cdot)^*$ and $(\cdot)_*$ form a Galois connection.

To that end, we must prove that, for any $f \in [\mathbb{X} \to \mathbb{D}]$ and $g \in [\mathbb{Y} \to \mathbb{D}] : g^* \sqsubseteq f \iff g \sqsubseteq f_*$. Equivalently:

$$\forall x \in X : g(\iota(x)) \sqsubseteq f(x) \iff \forall y \in Y : g(y) \sqsubseteq f_*(y).$$

To prove the $(\Leftarrow)$ implication, for any given $x \in X$, by assumption, $g(\iota(x)) \sqsubseteq f_*(\iota(x))$. This, together with Proposition 3.7 imply $\forall x \in X : g(\iota(x)) \sqsubseteq f(x)$.

To prove the $(\Rightarrow)$ implication, assume that $y \in Y$ is given, and consider an arbitrary $e \ll g(y)$. From the assumption, we obtain $\forall x \in \iota^{-1}(g^{-1}(\Uparrow e)) : g(\iota(x)) \sqsubseteq f(x)$, which implies that:

$$\bigwedge_{x\in\iota^{-1}(g^{-1}(\Uparrow e))} g(\iota(x)) \sqsubseteq \bigwedge f\left(\iota^{-1}(g^{-1}(\Uparrow e))\right). \tag{11}$$

On the other hand, for any $x \in \iota^{-1}(g^{-1}(\Uparrow e))$, we have $e \ll g(\iota(x))$, which, together with (11), implies $e \sqsubseteq \bigwedge f\left(\iota^{-1}(g^{-1}(\Uparrow e))\right)$. As $y \in g^{-1}(\Uparrow e)$, we obtain $e \sqsubseteq \bigvee \left\{\bigwedge f(\iota^{-1}(U)) \,\middle|\, y \in U \in \tau_{\mathbb{Y}}\right\} = f_*(y)$, where, in the last equality, we have used (9). As $e \ll g(y)$ was arbitrary, we conclude that $g(y) \sqsubseteq f_*(y)$. □

4 Core-Compactification via Spectral Spaces

In applications, the Galois connection of Theorem 3.9 is useful when one of the spaces has certain desirable properties that the other does not have, for instance, when $\mathbb{X}$ is not core-compact, but $\mathbb{Y}$ is. Recall from Theorem 2.2 that, whenever $\mathbb{D}$ is a bc-domain and $\mathbb{Y}$ is core-compact, then $[\mathbb{Y} \to \mathbb{D}]$ is also a bc-domain. In fact, there is an explicit description of a basis for $[\mathbb{Y} \to \mathbb{D}]$ consisting of *step functions*, as we will explain below.

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a topological space, and $\mathbb{D} \equiv (D, \sqsubseteq)$ is a pointed directed-complete partial order (pointed dcpo), with bottom element $\perp$. Then, for every open set $O \in \tau_{\mathbb{X}}$, and every element $b \in D$, we define the single-step function $b\chi_O : X \to D$ as follows:

$$b\chi_O(x) := \begin{cases} b, & \text{if } x \in O, \\ \perp, & \text{if } x \in X \setminus O. \end{cases}$$

For any set $\{b_i\chi_{O_i} \mid i \in I\}$ of single-step functions, the supremum $\bigvee_{i\in I} b_i\chi_{O_i}$ exists if and only if $\{b_i\chi_{O_i} \mid i \in I\}$ satisfies the following consistency condition:

$$\forall J \subseteq I : \quad \bigcap_{j\in J} O_j \neq \emptyset \implies \exists b_J \in D : \forall j \in J : b_j \sqsubseteq b_J.$$

By a step-function we mean the join of a consistent finite set of single-step functions.

Lemma 4.1 *Assume that:*

(1) $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$ is a core-compact space and $B_{\mathbb{Y}}$ is a basis for the continuous lattice $(\tau_{\mathbb{Y}}, \subseteq)$.

(2) $\mathbb{D}$ is a bc-domain and $D_0 \subseteq D$ is a basis for $\mathbb{D}$.

Then:

$$\forall f \in [\mathbb{Y} \to \mathbb{D}] : \quad f = \bigvee \left\{b\chi_U \,\middle|\, U \ll f^{-1}(\Uparrow b), U \in B_{\mathbb{Y}}, b \in D_0\right\}. \tag{12}$$

In particular, $[\mathbb{Y} \to \mathbb{D}]$ *is a bc-domain with a basis* $\mathbb{B}$ *of step-functions of the form:*

$$\mathbb{B} = \left\{ \bigvee_{i \in I} b_i \chi_{U_i} \,\middle|\, I \text{ is finite, } \{b_i \chi_{U_i} \mid i \in I\} \text{ is consistent}, \forall i \in I : U_i \in B_{\mathbb{Y}}, b_i \in D_0 \right\}.$$

Proof. The proof is a straightforward modification of the proof of [11, Lemma 1(c)]. Note that, as $\mathbb{Y}$ is assumed to be core-compact, the lattice $(\tau_{\mathbb{Y}}, \subseteq)$ is continuous. Furthermore, any basis (e. g., $B_{\mathbb{Y}}$) of the continuous lattice $(\tau_{\mathbb{Y}}, \subseteq)$ is a topological base of $\mathbb{Y}$. With that in mind, we have:

$$\forall y \in Y, \forall b \in D : \quad b \ll f(y) \iff y \in f^{-1}(\Uparrow b) \iff \exists U \in B_{\mathbb{Y}} : y \in U \ll f^{-1}(\Uparrow b), \tag{13}$$

Thus:

$$\begin{aligned} \forall y \in Y : \quad \bigvee \left\{ b\chi_U \,\middle|\, U \ll f^{-1}(\Uparrow b), U \in B_{\mathbb{Y}}, b \in D_0 \right\}(y) &= \bigvee \left\{ b \in D_0 \,\middle|\, \exists U \in B_{\mathbb{Y}} : y \in U \ll f^{-1}(\Uparrow b) \right\} \\ (\text{by } (13)) &= \bigvee \{ b \in D_0 \mid b \ll f(y) \} \\ (\text{since } D_0 \text{ is a basis of } D) &= f(y). \end{aligned}$$

□

Going back to the Galois connection of Theorem 3.9, we know that the left adjoint is always Scott-continuous. Nonetheless, when $\mathbb{Y}$ is core-compact and $\mathbb{X}$ is not, the right adjoint cannot be Scott-continuous:

Proposition 4.2 *Assume that* $\mathbb{Y}$ *is core-compact and* $\mathbb{D}$ *is a bc-domain. If the right adjoint* $(\cdot)_*$ *in the Galois connection* (10) *is Scott continuous and* D *is not a singleton, then* $\mathbb{X}$ *must be core-compact.*

Proof. By Lemma 3.8 and Theorem 3.9, the pair $((\cdot)_*, (\cdot)^*)$ forms a monotone section-retraction, with a Scott continuous retraction map $(\cdot)^*$. When the section $(\cdot)_*$ is also Scott continuous, the dcpo $[\mathbb{X} \to \mathbb{D}]$ becomes a continuous retract of the continuous domain $[\mathbb{Y} \to D]$. By [1, Theorem 3.1.4], any continuous retract of a continuous domain is also a continuous domain, hence $[\mathbb{X} \to \mathbb{D}]$ must be a continuous domain. By Theorem 2.2, this means that $\mathbb{X}$ must be core-compact. □

We now establish a working definition of what constitutes a core-compactification:

Definition 4.3 [Core-compactification] We say that a core-compact space $\mathbb{X}'$ is a *core-compactification* of the topological space $\mathbb{X}$ if $\mathbb{X}$ can be embedded as a dense sub-space of $\mathbb{X}'$.

Some classical compactification methods yield core-compactifications of topological spaces. For instance:

(i) Let $\mathbb{X}$ be the set $\mathbb{R}^n$ with the Euclidean topology. Then, the one-point (Alexandroff) compactification $\mathbb{R}^n \cup \{\infty\}$ is a core-compactification of $\mathbb{R}^n$.

(ii) Let $\mathbb{X}$ be any Tychonoff space. Then, the Stone-Čech compactification $\beta\mathbb{X}$ is a core-compactification of $\mathbb{X}$ [19, Chapter 5].

The classical compactification methods, however, are not suitable for our objectives. For instance, the one-point compactification provides the right result only when applied to locally compact (hence, core-compact) spaces, whereas our focus here is mainly on non-core-compact spaces, even though the method that we present is applicable to all T_0 spaces. The Stone-Čech compactification leads to a dense embedding when applied to any Tychonoff space, but an explicit description of the resulting space $\beta\mathbb{X}$ is lacking even for simple topological spaces $\mathbb{X}$, and as such, it is not suitable for computational purposes. The compactification obtained by our method, on the other hand, is designed for computational purposes.

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a topological space. We say that $\Omega_0 \subseteq \tau_{\mathbb{X}}$ is a *ring of open subsets of* $\mathbb{X}$ if it is closed under finite unions and finite intersections. In particular, Ω_0 must contain $\emptyset = \cup\emptyset$ and $X = \cap\emptyset$. We say that Ω_0 *separates points* if $\forall x, y \in X : \exists O \in \Omega_0 : (x \in O \wedge y \notin O) \vee (x \notin O \wedge y \in O)$.

Definition 4.4 [Viable base] Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a topological space. We say that $\Omega_0 \subseteq \tau_{\mathbb{X}}$ is a *viable base* for $\mathbb{X}$ if it is a ring of open sets that forms a base for the topology $\tau_{\mathbb{X}}$.

Remark 4.5 If $\mathbb{X}$ is assumed to be T_0, then there is always at least one viable base, i. e., $\Omega_0 = \tau_{\mathbb{X}}$.

Proposition 4.6 *If $\mathbb{X}$ is a T_0 topological space and Ω_0 is a viable base, then Ω_0 separates points.*

Proof. Assume that $x \neq y \in X$. As $\tau_{\mathbb{X}}$ is T_0, there exists an open set $U \in \tau_{\mathbb{X}}$ that separates x and y. Without loss of generality, let us assume that $x \in U$ and $y \notin U$. Since Ω_0 is a base of the topology, there exists a $U' \in \Omega_0$ such that $x \in U' \subseteq U$. Hence, $y \notin U'$. □

Any ring Ω_0 of open sets is a bounded distributive lattice with $\bigwedge := \bigcap$ and $\bigvee := \bigcup$. Assume that Ω_0 is a ring of open sets and let $\mathcal{L} := \mathrm{Idl}(\Omega_0)$ be the ideal completion of $(\Omega_0, \subseteq)$. From diagram (7), we know that $\mathcal{L}$ must be an algebraic fully arithmetic lattice. The principal ideals $\downarrow \emptyset = \{\emptyset\}$ and $\downarrow X = \Omega_0$ are the bottom and the top elements of $\mathcal{L}$, respectively. We mention some basic properties of $\mathcal{L}$.

Proposition 4.7 *Every ideal $I \in \mathcal{L}$ is closed under finite unions.*

Proof. Assume that $x, y \in I$. As I is an ideal, then $\exists z \in I : x \subseteq z$ and $y \subseteq z$. Hence, $x \cup y \subseteq z$. On the other hand, as Ω_0 is closed under finite unions and I is a lower set, we must have $x \cup y \in I$. □

Proposition 4.8 *Assume that $I_1, I_2 \in \mathcal{L}$. Then, $\forall O_1 \in I_1, O_2 \in I_2 : O_1 \cap O_2 \in I_1 \cap I_2$.*

Proof. Follows from the fact that I_1 and I_2 are lower sets and Ω_0 is closed under finite intersections. □

Proposition 4.9 *$\mathcal{L}$ is a complete lattice, in which, for every subset $A \subseteq \mathcal{L}$, we have $\bigwedge A = \bigcap A$.*

Proof. If $A = \emptyset$, then both sides are equal to the top element of the lattice, i. e., $\bigwedge A = \bigcap A = \downarrow X = \Omega_0$. Next, assume that $A = \{I_j \mid j \in J\}$ for some non-empty index set J. We must show that $\hat{I} := \bigcap_{j \in J} I_j$ is indeed an ideal:

- $\hat{I}$ is non-empty: This is because $\forall j \in J : \emptyset \in I_j$.
- $\hat{I}$ is a lower set: This follows immediately from the fact that $\forall j \in J : I_j$ is an ideal.
- $\hat{I}$ is directed: Let $x, y \in \bigcap_{j \in J} I_j$. By Proposition 4.7, $\forall j \in J : x \cup y \in I_j$, which implies that $x \cup y \in \bigcap_{j \in J} I_j$.

The ideal $\hat{I}$ is a lower bound of $\{I_j \mid j \in J\}$ because $\forall j \in J : \hat{I} \subseteq I_j$. Furthermore, if $\tilde{I} \in \mathcal{L}$ is any other lower bound of $\{I_j \mid j \in J\}$, then $\forall j \in J : \tilde{I} \subseteq I_j$, which implies that $\tilde{I} \subseteq \bigcap_{j \in J} I_j = \hat{I}$. Therefore, $\hat{I}$ is the infimum of $\{I_j \mid j \in J\}$. □

In a complete lattice, suprema can be obtained using infima. To be more precise, for any subset $A \subseteq \mathcal{L}$, let $\breve{A} := \{z \in \mathcal{L} \mid \forall x \in A : x \subseteq z\}$ be the set of all upper bounds of A. Then, we have $\bigvee A = \bigwedge \breve{A}$. In the following proposition, however, we present an explicit description of the suprema of subsets of $\mathcal{L}$: [3]

Proposition 4.10 *In the complete lattice $\mathcal{L}$, for every non-empty subset $A = \{I_j \mid j \in J\}$, we have:*

$$\bigvee A = \bigcup \left\{ \downarrow \cup_{j \in J_0} O_j \,\middle|\, J_0 \subseteq_f J, \forall j \in J_0 : O_j \in I_j \right\}. \tag{14}$$

In other words, for any finite combination of open sets $\{O_j \mid j \in J_0\}$ taken from the ideals in A, we form the principal ideal $\downarrow \cup_{j \in J_0} O_j$, and finally take the union of all these principal ideals.

Proof. Let $\hat{I} := \bigcup \left\{ \downarrow \cup_{j \in J_0} O_j \,\middle|\, J_0 \subseteq_f J, \forall j \in J_0 : O_j \in I_j \right\}$. First, we show that the set $\hat{I}$ is an ideal:

- $\hat{I}$ is a lower set: Take any $x \in \hat{I}$. Then, for some finite set $J_0 \subseteq_f J$ and a collection $\{O_j \mid j \in J_0\}$ satisfying $\forall j \in J_0 : O_j \in I_j$, we have $x \in \downarrow \cup_{j \in J_0} O_j$, which implies that $x \subseteq \cup_{j \in J_0} O_j$. Hence, any $y \in \hat{I}$ satisfying $y \subseteq x$ must also satisfy $y \subseteq \cup_{j \in J_0} O_j$, i. e., $y \in \downarrow \cup_{j \in J_0} O_j$.
- $\hat{I}$ is directed: Let $x, y \in \hat{I}$. Then, for some finite sets $J_x, J_y \subseteq_f J$ and for some collections $\{O_j \mid j \in J_x\}$ and $\{O'_j \mid j \in J_y\}$ satisfying $\forall j \in J_x : O_j \in I_j$ and $\forall j \in J_y : O'_j \in I_j$, we have $x \subseteq \cup_{j \in J_x} O_j$ and $y \subseteq \cup_{j \in J_y} O'_j$. Let $J_0 := J_x \cup J_y$ and define:

$$\begin{cases} \forall j \in J_0 \setminus J_x : O_j = \emptyset, \\ \forall j \in J_0 \setminus J_y : O'_j = \emptyset. \end{cases}$$

[3] Also, see [16, Exercise 9.5.11].

As such, $x \cup y \subseteq (\cup_{j \in J_x} O_j) \cup (\cup_{j \in J_y} O'_j) = \cup_{j \in J_0}(O_j \cup O'_j)$, which implies that $x \cup y \in \downarrow \cup_{j \in J_0}(O_j \cup O'_j)$. By Proposition 4.7, $\forall j \in J_0 : O_j \cup O'_j \in I_j$. Hence, $x \cup y \in \hat{I}$.

The ideal $\hat{I}$ is an upper bound of $\{I_j \mid j \in J\}$. This is because, for every $j \in J$ and $O \in I_j$, we have $O \in \downarrow O \subseteq \hat{I}$.

Assume that $\tilde{I} \in \mathcal{L}$ is another upper bound of $\{I_j \mid j \in J\}$. As $\tilde{I}$ is a directed lower set, then for any finite set $J_0 \subseteq_f J$ and collection $\{O_j \mid j \in J_0\}$ satisfying $\forall j \in J_0 : O_j \in I_j$, we must have $\bigcup_{j \in J_0} O_j \in \tilde{I}$, which implies that $\downarrow \bigcup_{j \in J_0} O_j \subseteq \tilde{I}$. Therefore, $\hat{I} \subseteq \tilde{I}$. □

Recall that a lattice is said to be spatial if it is order isomorphic to $(\tau_{\mathbb{M}}, \subseteq)$ for some topological space $\mathbb{M} \equiv (M, \tau_{\mathbb{M}})$.

Proposition 4.11 *The lattice* $\mathcal{L} = \mathrm{Idl}(\Omega_0)$ *is spatial.*

Proof. Follows from the fact that every continuous distributive lattice is spatial [1, Lemma 7.2.15]. □

As such, $\mathrm{Idl}(\Omega_0)$ is (order isomorphic to) the lattice of open subsets of a topological space $\hat{\mathbb{X}}_{\Omega_0}$ which we regard as a spectral compactification of $\mathbb{X}$:

Definition 4.12 [Spectral compactification: $\hat{\mathbb{X}}_{\Omega_0}$] Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a T_0 topological space and $\Omega_0 \subseteq \tau_{\mathbb{X}}$ is a viable base of $\mathbb{X}$. By the spectral compactification of $\mathbb{X}$ generated by Ω_0 we mean the topological space:

$$\hat{\mathbb{X}}_{\Omega_0} \equiv (\hat{X}_{\Omega_0}, \hat{\tau}) := \mathrm{pt}(\mathcal{L}) = \mathrm{pt}(\mathrm{Idl}(\Omega_0)),$$

in which $\hat{\tau}$ is the hull-kernel topology. When Ω_0 is clear from the context, we use the simpler notation $\hat{\mathbb{X}}$.

Recall that, in the hull-kernel topology $\hat{\tau}$ on $\hat{\mathbb{X}}_{\Omega_0}$, every open set is of the form:

$$O_I := \{y \in \hat{X}_{\Omega_0} \mid I \in y\}, \tag{15}$$

in which I ranges over $\mathrm{Idl}(\Omega_0)$.

Proposition 4.13 $\forall I_1, I_2 \in \mathrm{Idl}(\Omega_0) : I_1 \subseteq I_2 \implies O_{I_1} \subseteq O_{I_2}$.

Proof. Take any $y \in O_{I_1}$. Then, by (15), $I_1 \in y$. As y is a filter and $I_1 \subseteq I_2$, then $I_2 \in y$, which implies that $y \in O_{I_2}$. Hence, $O_{I_1} \subseteq O_{I_2}$. □

Proposition 4.14 *Assume that* $I \in \mathrm{Idl}(\Omega_0)$ *and for a family* $\{I_j \mid j \in J\} \subseteq \mathrm{Idl}(\Omega_0)$*, we have* $I = \bigvee_{j \in J} I_j$*. Then,* $O_I = \bigcup_{j \in J} O_{I_j}$.

Proof. The fact that $O_I \supseteq \bigcup_{j \in J} O_{I_j}$ follows from Proposition 4.13. To prove the $\subseteq$ direction, for any $y \in O_I$:

$$\begin{array}{rcl} y \in O_I & \implies & I \in y \\ (\text{since } I = \bigvee_{j \in J} I_j) & \implies & \bigvee_{j \in J} I_j \in y \\ (y \text{ is completely prime}) & \implies & \exists j \in J : I_j \in y \\ (\text{by (15)}) & \implies & y \in O_{I_j}. \end{array}$$

□

Corollary 4.15 *The set* $\{O_{\downarrow W} \mid W \in \Omega_0\}$ *forms a base for the hull-kernel topology* $\hat{\tau}$ *on* $\hat{\mathbb{X}}_{\Omega_0}$.

Proof. Follows from Proposition 4.14, and the fact that $\{\downarrow W \mid W \in \Omega_0\}$ forms a (domain-theoretic) basis for the lattice $\mathrm{Idl}(\Omega_0)$. □

We know that $\hat{\mathbb{X}}_{\Omega_0}$ is a spectral space, hence, it is core-compact. In what follows, we show that, whenever Ω_0 separates points, $\hat{\mathbb{X}}_{\Omega_0}$ is indeed a core-compactification of $\mathbb{X}$. We consider the map $\iota : \mathbb{X} \to \hat{\mathbb{X}}_{\Omega_0}$ defined by:

$$\forall x \in X : \quad \iota(x) := \{I \in \mathrm{Idl}(\Omega_0) \mid x \in \cup I\}\,. \tag{16}$$

Proposition 4.16 *The map ι defined in (16) is well-defined, i. e., $\forall x \in X : \iota(x)$ is a completely prime filter.*

Proof. We first prove that, $\forall x \in X : \iota(x)$ is a filter:

- $\iota(x) \neq \emptyset$, because $\Omega_0 \in \iota(x)$.
- $\iota(x)$ is an upper set as a consequence of the monotonicity of union. To be more precise, assume that $I \in \iota(x)$ and $I' \in \mathrm{Idl}(\Omega_0)$ satisfy $I \subseteq I'$. Then, $x \in \cup I \subseteq \cup I'$, which implies that $x \in \cup I'$. Hence, $I' \in \iota(x)$.
- $\iota(x)$ is downward directed: Let $I_1, I_2 \in \iota(x)$. Then, for some $O_1 \in I_1$ and $O_2 \in I_2$, we must have $x \in O_1$ and $x \in O_2$. Thus, $x \in O_1 \cap O_2$. By Proposition 4.8, $O_1 \cap O_2 \in I_1 \cap I_2$. By Proposition 4.9, we must have $I_1 \wedge I_2 \in \iota(x)$.

Next, we prove that $\iota(x)$ is completely prime. Assume that $A := \{I_j \mid j \in J\} \subseteq \mathrm{Idl}(\Omega_0)$ and $\bigvee A \in \iota(x)$. Note that $A \neq \emptyset$, otherwise we would have:

$$\bigvee A = \bigvee \emptyset = \bot_{\mathrm{Idl}(\Omega_0)} = \downarrow \emptyset \notin \iota(x).$$

Hence, A must be non-empty. By (14), for some finite set $J_0 \subseteq_f J$ and a collection $\{O_j \mid j \in J_0 \text{ and } O_j \in I_j\}$, we must have $x \in \cup_{j \in J_0} O_j$. Thus, $\exists j_0 \in J_0 : x \in O_{j_0} \subseteq \cup I_{j_0}$, which implies that $I_{j_0} \in \iota(x)$. □

Proposition 4.17 *The map $\iota : \mathbb{X} \to \hat{\mathbb{X}}_{\Omega_0}$ is continuous and $\forall I \in \mathrm{Idl}(\Omega_0) : \iota^{-1}(O_I) = \cup I$.*

Proof. For all $I \in \mathrm{Idl}(\Omega_0)$ and $x \in X$, we have: $x \in \cup I \iff I \in \iota(x) \iff \iota(x) \in O_I$. □

Using Proposition 4.17, for the case of $\mathbb{Y} = \hat{\mathbb{X}}_{\Omega_0}$, we obtain the following alternative formulation of (9):

Proposition 4.18 *For any $f \in [\mathbb{X} \to \mathbb{D}]$, we have:*

$$\forall \hat{x} \in \hat{X}_{\Omega_0} : \quad f_*(\hat{x}) = \bigvee \left\{ \bigwedge f(\cup I) \,\middle|\, I \in \hat{x} \right\}.$$

Proof. Every open set $\hat{O}$ in the hull-kernel topology $\hat{\tau}$ is of the form O_I (as in (15)) for some ideal $I \in \mathrm{Idl}(\Omega_0)$ and we know that $\hat{x} \in O_I \iff I \in \hat{x}$. Furthermore, by Proposition 4.17, $\forall I \in \mathrm{Idl}(\Omega_0) : \iota^{-1}(O_I) = \cup I$. □

To show that the map $\iota : \mathbb{X} \to \hat{\mathbb{X}}_{\Omega_0}$ is a quasi-embedding, we must use the assumption that Ω_0 is a base. It is straightforward to verify that, for any open set $W \in \tau_{\mathbb{X}}$ (i. e., not necessarily in Ω_0) we have $\downarrow W := \{U \in \Omega_0 \mid U \subseteq W\} \in \mathrm{Idl}(\Omega_0)$. With that in mind:

Proposition 4.19 *For every open set $W \in \tau_{\mathbb{X}}$ and $x \in X$, we have $x \in W \iff \downarrow W \in \iota(x)$.*

Proof. The ($\Leftarrow$) direction is straightforward. For the ($\Rightarrow$) implication, since Ω_0 is a base, we have:

$$x \in W \implies \exists U \in \Omega_0 : x \in U \subseteq W \implies x \in \bigcup \downarrow W \implies \downarrow W \in \iota(x).$$

□

Proposition 4.20 *For every open set $W \in \tau_{\mathbb{X}}$, we have $\iota(W) = O_{\downarrow W} \cap \iota(X)$.*

Proof. By using Proposition 4.19, we obtain $\forall x \in X : x \in W \iff \downarrow W \in \iota(x) \iff \iota(x) \in O_{\downarrow W}$. □

Corollary 4.21 *The map $\iota : \mathbb{X} \to \hat{\mathbb{X}}_{\Omega_0}$ is a topological quasi-embedding.*

Proof. Follows from Propositions 4.17, and 4.20. □

Lemma 4.22 *The quasi-embedding $\iota : \mathbb{X} \to \hat{\mathbb{X}}_{\Omega_0}$ is dense.*

Proof. From Propositions 4.17, and 4.20, we obtain: $\forall I \in \mathrm{Idl}(\Omega_0) : O_I \cap \iota(X) = \iota(\cup I)$. □

So far, we have not used separation of points:

Proposition 4.23 *Whenever Ω_0 separates points, the map $\iota : \mathbb{X} \to \hat{\mathbb{X}}_{\Omega_0}$ is injective.*

Proof. Let $x, y \in X$, and assume that $x \neq y$. Since Ω_0 separates points, there exists an open set $W \in \Omega_0$ which includes one point but not the other. Without loss of generality, we assume that $x \in W$ and $y \notin W$. Then $\downarrow W \in \iota(x)$ but $\downarrow W \notin \iota(y)$. □

To summarize, we have proven that:

Theorem 4.24 (Core-compactification) *Assume that $\mathbb{X}$ is a T_0 topological space. If Ω_0 is a viable base of $\mathbb{X}$, then the spectral space $\hat{\mathbb{X}}_{\Omega_0}$ is a core-compactification of $\mathbb{X}$.*

Example 4.25 [rational upper limit topology] Consider the set $B_{(\mathbb{Q}]} := \{(a, b] \mid a, b \in \mathbb{Q}\}$ of left half-open intervals with rational end-points. The collection $B_{(\mathbb{Q}]}$ forms a base for what we refer to as the *rational upper limit topology*. Let $\mathbb{R}_{(\mathbb{Q}]} \equiv (\mathbb{R}, \tau_{(\mathbb{Q}]})$ denote the topological space with $\mathbb{R}$ as the carrier set endowed with the rational upper limit topology $\tau_{(\mathbb{Q}]}$. As for a viable Ω_0, an immediate option is $\tau_{(\mathbb{Q}]}$. Yet, considering Corollary 4.15, it may not lead to a second-countable $\hat{\mathbb{X}}_{\Omega_0}$.

Instead, we can take Ω_0 to consist of all the finite unions of elements of $B_{(\mathbb{Q}]}$. This is a countable set which can be effectively enumerated. By Corollary 4.15, the spectral space $\hat{\mathbb{X}}_{\Omega_0}$ must also be second-countable.

The rational upper limit topology is used in [6] for solution of IVPs with temporal discretization. In [6], the domain $[\mathbb{Y} \to \mathbb{D}]$ is constructed by rounded ideal completion of a suitable abstract basis of step functions. In this article, we work directly on $\mathbb{X}$. As we will see (Theorem 5.6) the two approaches lead to equivalent outcomes.

Remark 4.26 We have presented the construction of the spectral compactification by first considering the ideal completion $\mathrm{Idl}(\Omega_0)$ of Ω_0, and then applying the pt functor. The same result can be obtained by directly considering the prime filters of Ω_0 [1, Proposition 7.2.23]. We opted for the two-step construction because, in Section 5, we will need to refer to some properties of $\mathrm{Idl}(\Omega_0)$.

5 Continuous Domain of Functions

The aim here is to present a framework for computation over function spaces. Starting from a topological space $\mathbb{X}$ and a bc-domain $\mathbb{D}$, if $\mathbb{X}$ is core-compact, then $[\mathbb{X} \to \mathbb{D}]$ is a bc-domain, and all that remains is to determine whether $[\mathbb{X} \to \mathbb{D}]$ admits a suitable effective structure. If it does not, or if $\mathbb{X}$ is not core-compact to begin with, then our aim is to consider the substitute bc-domain $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$, for a viable base Ω_0 of $\mathbb{X}$.

Theorem 5.1 *Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a topological space and $\Omega_0 \subseteq \tau_{\mathbb{X}}$ is a viable base of $\mathbb{X}$. Let $\mathbb{D} \equiv (D, \sqsubseteq)$ be a bc-domain and assume that $D_0 \subseteq D$ is a basis for $\mathbb{D}$. Then, $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$ is a bc-domain with a basis $\hat{\mathbb{B}}$ of step-functions of the form:*

$$\hat{\mathbb{B}} = \left\{ \bigvee_{i \in I} b_i \chi_{O_{\downarrow W_i}} \;\middle|\; I \text{ is finite}, \left\{ b_i \chi_{O_{\downarrow W_i}} \;\middle|\; i \in I \right\} \text{ is consistent}, \forall i \in I : W_i \in \Omega_0, b_i \in D_0 \right\}. \tag{17}$$

Proof. The fact that $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$ is a bc-domain follows from Theorem 2.2. By Corollary 4.15, the set $\{O_{\downarrow W} \mid W \in \Omega_0\}$ forms a base for the hull-kernel topology $\hat{\tau}$ on $\hat{\mathbb{X}}_{\Omega_0}$. Hence, from Lemma 4.1, we deduce:

$$\forall f \in [\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}] : \quad f = \bigvee \left\{ b\chi_{O_{\downarrow W}} \;\middle|\; \downarrow W \ll f^{-1}(\Uparrow b), W \in \Omega_0, b \in D_0 \right\},$$

which implies that $\hat{\mathbb{B}}$ is indeed a basis for $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$. □

Corollary 5.2 *If Ω_0 is countable and $\mathbb{D}$ is ω-continuous, then $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$ is also ω-continuous.*

Proof. If $\mathbb{D}$ is ω-continuous, then D_0 can be chosen to be countable. By (17), $\hat{\mathbb{B}}$ is a countable basis for $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$. □

Regarding the way-below relation over $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$, we can derive a useful formulation for the basis elements, i. e., step functions of the form (17). Recall that a core-compact space $\mathbb{Y} \equiv (Y, \tau_{\mathbb{Y}})$ is called *stable* if $U \ll V$ and $U \ll V'$ imply $U \ll V \cap V'$, for all $U, V, V' \in \tau_{\mathbb{Y}}$. When $\mathbb{Y}$ is stable, based on [11, Lemma 1 and Proposition 5], we know that, for any step function $\bigvee_{i\in I} b_i\chi_{O_i} \in [\mathbb{Y} \to \mathbb{D}]$ and any arbitrary function $g \in [\mathbb{Y} \to \mathbb{D}]$:

$$\bigvee_{i\in I} b_i\chi_{O_i} \ll g \iff \forall i \in I : O_i \ll g^{-1}(\Uparrow b_i). \tag{18}$$

We point out that stability is used for the ($\Rightarrow$) direction, i. e., for the ($\Leftarrow$) implication to hold, it suffices for $\mathbb{Y}$ to be core-compact [11, Lemma 1].

Let us now take $\mathbb{Y} = \hat{\mathbb{X}}_{\Omega_0}$ and recall that every spectral space is stable. Hence, we can use (18). Assume that, for some finite index set J, we have $g := \bigvee_{j\in J} b'\chi_{O_{\downarrow W'_j}}$. It is easy to show that $\forall b \in D \setminus \{\bot\} : \exists J_b \subseteq J : g^{-1}(\Uparrow b) = \bigcup_{j\in J_b} O_{\downarrow W'_j}$. By Proposition 4.14, we have:

$$\bigcup_{j\in J_b} O_{\downarrow W'_j} = O_{\bigvee_{j\in J_b}\downarrow W'_j} = O_{\downarrow \bigcup_{j\in J_b} W'_j}. \tag{19}$$

According to Proposition 4.11, the complete lattice $\mathcal{L} = \mathrm{Idl}(\Omega_0)$ is spatial and we have:

$$\Omega(\hat{\mathbb{X}}_{\Omega_0}) \cong \Omega(\mathrm{pt}(\mathrm{Idl}(\Omega_0))) \cong \mathrm{Idl}(\Omega_0). \tag{20}$$

For instance, we have:

$$\begin{aligned} & O_{\downarrow W} \ll O_{\downarrow W'} && (\text{in } (\hat{\tau}, \subseteq)) \\ (\text{by } 20) \iff\ & \downarrow W \ll \downarrow W' && (\text{in } (\mathrm{Idl}(\Omega_0), \subseteq)) \\ (\text{by [1, Proposition 2.2.22]}) \iff\ & W \subseteq W' && (\text{in } \Omega_0). \end{aligned}$$

In fact, we have:

Lemma 5.3 *The way-below relation on step functions of (17) can be expressed as:*

$$\bigvee_{i\in I} b_i\chi_{O_{\downarrow W_i}} \ll \bigvee_{j\in J} b'_j\chi_{O_{\downarrow W'_j}} \iff \forall i \in I : W_i \subseteq U_i,$$

in which $U_i \in \Omega_0$ *satisfies* $O_{\downarrow U_i} = \left(\bigvee_{j\in J} b'_j\chi_{O_{\downarrow W'_j}}\right)^{-1}(\Uparrow b_i)$.

Proof. This follows from (18), (19), and the fact that Ω_0 is closed under finite unions. □

5.1 Construction Using Abstract Bases

Lemma 5.3 suggests an alternative approach to obtaining a domain of functions based on *abstract bases* without referring to Stone duality:

Definition 5.4 [Abstract basis] A pair $(B, \lhd)$ consisting of a set B and a binary relation $\lhd\ \subseteq B \times B$ is said to be an abstract basis if the relation $\lhd$ is transitive and satisfies the following interpolation property:

- For every finite subset $A \subseteq_f B$ and element $x \in B : A \lhd x \implies \exists y \in B : A \lhd y \lhd x$.

Here, by $A \lhd x$ we mean $\forall a \in A : a \lhd x$.

In this approach, we work directly with step functions in $[\mathbb{X} \to \mathbb{D}]$. Specifically, we consider:

$$\mathbb{B}_{\mathrm{abs}} := \left\{ f : X \to D \,\middle|\, f = \bigvee_{i\in I} b_i\chi_{O_i}, I \text{ is finite}, \forall i \in I : O_i \in \Omega_0 \text{ and } b_i \in D_0 \right\}. \tag{21}$$

As for the binary relation $\lhd$, considering Lemma 5.3, we define:

$$\bigvee_{i\in I} b_i\chi_{O_i} \lhd \bigvee_{j\in J} b'_j\chi_{O'_j} \iff \forall i \in I : O_i \subseteq (\bigvee_{j\in J} b'_j\chi_{O'_j})^{-1}(\Uparrow b_i). \tag{22}$$

This is indeed the approach taken in [6], where $\mathbb{X}$ is the real line endowed with the rational upper limit topology (Example 4.25) and $\mathbb{D}$ is the interval domain $\mathbb{IR}^n_\perp$ of (2). In [6], a domain $\mathcal{W}$ is constructed as the rounded ideal completion of $(\mathbb{B}_{\text{abs}}, \lhd)$ and the Galois connection of Theorem 3.9 is obtained with $[\mathbb{Y} \to \mathbb{D}]$ replaced by $\mathcal{W}$:

$$[\mathbb{X} \to \mathbb{D}] \underset{(\cdot)^*}{\overset{(\cdot)_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathcal{W}.$$

The aim in [6] has been solution of IVPs with temporal discretization. In that context, the computation is carried out over the (non-continuous) dcpo $[\mathbb{X} \to \mathbb{D}]$, while the theoretical analyses (including computable analysis) is carried out over the continuous domain $\mathcal{W}$. The Galois connection provides the bridge between the two.

We expect this to be the general rule. When $\mathbb{X}$ is not core-compact, the non-continuous dcpo $[\mathbb{X} \to \mathbb{D}]$ is still useful for implementation of algorithms. But it cannot be used for computable analysis. With careful choice of Ω_0, it is possible to endow $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$ with an effective structure which makes it suitable for computable analysis of the relevant problem.

To demonstrate that the two approaches are equivalent, we show that the (abstract) bases $(\hat{\mathbb{B}}, \ll)$ of (17) and $(\mathbb{B}_{\text{abs}}, \lhd)$ of (21) are isomorphic. We define a map $\gamma : \hat{\mathbb{B}} \to \mathbb{B}_{\text{abs}}$ by $\forall f \in \hat{\mathbb{B}} : \gamma(f) := f \circ \iota = f^*$. It is straightforward to verify that, for any $\bigvee_{i\in I} b_i \chi_{W_i} \in \mathbb{B}_{\text{abs}}$, we have $\bigvee_{i\in I} b_i \chi_{W_i} = \gamma\left(\bigvee_{i\in I} b_i \chi_{O_{\downarrow W_i}}\right)$. It is also easy to verify that $\gamma : \hat{\mathbb{B}} \to \mathbb{B}_{\text{abs}}$ and its inverse $\gamma^{-1} : \mathbb{B}_{\text{abs}} \to \hat{\mathbb{B}}$ are monotonic and bijective. We must show, however, that γ preserves the way-below relations.

Lemma 5.5 $\forall f, g \in \hat{\mathbb{B}} : \quad f \ll g \iff \gamma(f) \lhd \gamma(g).$

Proof. This is almost immediate from Lemma 5.3 and (22), except that we must show that the relations hold regardless of how the step-functions are represented. In other words, whenever $\bigvee_{k\in K} c_k \chi_{U_k} = \gamma\left(\bigvee_{i\in I} b_i \chi_{O_{\downarrow W_i}}\right)$ and $\bigvee_{\ell\in L} c'_\ell \chi_{U'_\ell} = \gamma\left(\bigvee_{j\in J} b'_j \chi_{O_{\downarrow W'_j}}\right)$:

$$\bigvee_{i\in I} b_i \chi_{O_{\downarrow W_i}} \ll \bigvee_{j\in J} b'_j \chi_{O_{\downarrow W'_j}} \iff \bigvee_{k\in K} c_k \chi_{U_k} \lhd \bigvee_{\ell\in L} c'_\ell \chi_{U'_\ell}. \tag{23}$$

This follows from the fact that γ is monotone and bijective. For instance, to prove the ($\Leftarrow$) implication in (23), assume that $\bigvee_{k\in K} c_k \chi_{U_k} \lhd \bigvee_{\ell\in L} c'_\ell \chi_{U'_\ell}$. Then, we have:

$$\begin{cases} \gamma\left(\bigvee_{k\in K} c_k \chi_{O_{\downarrow U_k}}\right) = \bigvee_{k\in K} c_k \chi_{U_k} = \gamma\left(\bigvee_{i\in I} b_i \chi_{O_{\downarrow W_i}}\right), \\ \gamma\left(\bigvee_{\ell\in L} c'_\ell \chi_{O_{\downarrow U'_\ell}}\right) = \bigvee_{\ell\in L} c'_\ell \chi_{U'_\ell} = \gamma\left(\bigvee_{j\in J} b'_j \chi_{O_{\downarrow W'_j}}\right). \end{cases}$$

As γ is bijective, we must have $\bigvee_{k\in K} c_k \chi_{O_{\downarrow U_k}} = \bigvee_{i\in I} b_i \chi_{O_{\downarrow W_i}}$ and $\bigvee_{\ell\in L} c'_\ell \chi_{O_{\downarrow U'_\ell}} = \bigvee_{j\in J} b'_j \chi_{O_{\downarrow W'_j}}$. The result now follows from Lemma 5.3 and (22). □

The approach based on abstract bases also provides us with the same continuous domain of functions:

Theorem 5.6 *Assume that the domain $\mathcal{W}$ is the rounded ideal completion of* $(\mathbb{B}_{\text{abs}}, \lhd)$. *Then* $\mathcal{W} \cong [\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$.

Proof. Follows from the fact that the (abstract) bases $(\hat{\mathbb{B}}, \ll)$ and $(\mathbb{B}_{\text{abs}}, \lhd)$ are isomorphic. □

6 Concluding Remarks

We have investigated the basic properties of a computational framework for function spaces over topological spaces that are not core-compact. To that end, we considered spectral compactification of a given space and presented the construction directly without referring to quasi-proximities. In our framework, for a space $\mathbb{Y}$ to be a compactification of a space $\mathbb{X}$, there must exist a *dense* embedding $\iota : \mathbb{X} \to \mathbb{Y}$ (Definition 4.3). By using quasi-proximities, one can require the existence of a so-called *basis* embedding $\iota : \mathbb{X} \to \mathbb{Y}$, which is a stronger condition [20]. It will be worth investigating how our results can be strengthened with the basis embedding requirement.

In our framework, computability is analyzed in the continuous domain $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$. In Type-II Theory of Effectivity (TTE) [21], computability is analyzed via admissible representations of the function space $\mathbb{D}^{\mathbb{X}}$. Part of our future work is the investigation of how the two approaches are related.

Regarding applications of the framework, it has provided a suitable semantic model for solution of IVPs. To be more precise, in [6], we constructed a domain using abstract bases for solution of IVPs with temporal discretization. In Theorem 5.6 of the current article, we showed that the same domain (up to isomorphism) can be obtained using the construction of the current article. We believe that spectral compactification will be useful in domain theoretic solution of partial differential equations (PDEs) as well.

Spectral compactification provides another angle on the construction obtained via abstract bases in [6]. Even though, in practice, the two approaches lead to isomorphic function spaces, we believe that the construction of the current article has some theoretical advantages. Compactification is a central topic in topology, and as we have pointed out earlier, our construction can be obtained as a special case of Smyth's stable compactification by considering fine quasi-proximities [20]. It will be interesting to see if any concrete application necessitates consideration of stable compactifications obtained via quasi-proximities other than the fine one, in the same way that solution of IVPs with temporal discretization led us to the investigation of spectral compactification.

References

[1] Abramsky, S. and A. Jung, *Domain theory*, in: S. Abramsky, D. M. Gabbay and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168, Clarendon Press, Oxford (1994).

[2] Bilokon, P. and A. Edalat, *A domain-theoretic approach to Brownian motion and general continuous stochastic processes*, Theoretical Computer Science **691**, pages 10–26 (2017).
https://doi.org/10.1016/j.tcs.2017.07.016

[3] Di Gianantonio, P., A. Edalat and R. Gutin, *A language for evaluating derivatives of functionals using automatic differentiation*, Electronic Notes in Theoretical Informatics and Computer Science **3** (2023). Proceedings of MFPS 2023.
https://doi.org/10.46298/entics.12303

[4] Edalat, A., *Dynamical systems, measures and fractals via domain theory*, Information and Computation **120**, pages 32–48 (1995).

[5] Edalat, A., *Domains for computation in mathematics, physics and exact real arithmetic*, Bull. Symbolic Logic **3**, pages 401–452 (1997).

[6] Edalat, A., A. Farjudian and Y. Li, *Recursive solution of initial value problems with temporal discretization*, Theoretical Computer Science page 114221 (2023).
https://doi.org/10.1016/j.tcs.2023.114221

[7] Edalat, A., A. Farjudian, M. Mohammadian and D. Pattinson, *Domain theoretic second-order Euler's method for solving initial value problems*, Electr. Notes in Theoret. Comp. Sci. **352**, pages 105–128 (2020). The 36th Mathematical Foundations of Programming Semantics Conference, 2020 (MFPS 2020), Paris, France.

[8] Edalat, A. and A. Lieutier, *Domain theory and differential calculus (functions of one variable)*, Mathematical Structures in Computer Science **14**, pages 771–802 (2004).

[9] Edalat, A. and D. Pattinson, *Denotational semantics of hybrid automata*, The Journal of Logic and Algebraic Programming **73**, pages 3–21 (2007).

[10] Edalat, A. and D. Pattinson, *A domain-theoretic account of Picard's theorem*, LMS Journal of Computation and Mathematics **10**, pages 83–118 (2007).

[11] Erker, T., M. H. Escardó and K. Keimel, *The way-below relation of function spaces over semantic domains*, Topology and its Applications **89**, pages 61–74 (1998).
https://doi.org/10.1016/S0166-8641(97)00226-5

[12] Escardó, M. H., *PCF extended with real numbers*, Theoretical Computer Science **162**, pages 79–115 (1996).

[13] Farjudian, A., *Shrad: A language for sequential real number computation*, Theory Comput. Syst. **41**, pages 49–105 (2007).

[14] Farjudian, A. and M. Konečný, *Time complexity and convergence analysis of domain theoretic Picard method*, in: W. Hodges and R. de Queiroz, editors, *Proceedings of WoLLIC '08*, volume 5110 of *Lecture Notes in Artificial Intelligence*, pages 149–163, Springer (2008).

[15] Farjudian, A. and E. Moggi, *Robustness, Scott continuity, and computability*, Mathematical Structures in Computer Science **33**, pages 536–572 (2023).
https://doi.org/10.1017/S0960129523000233

[16] Goubault-Larrecq, J., *Non-Hausdorff topology and domain theory*, Cambridge University Press (2013).

[17] Mislove, M. W., *Topology, domain theory and theoretical computer science*, Topology and its Applications **89**, pages 3–59 (1998).

[18] Moggi, E., A. Farjudian, A. Duracz and W. Taha, *Safe & robust reachability analysis of hybrid systems*, Theoretical Computer Science **747**, pages 75–99 (2018).

[19] Munkres, J. R., *Topology*, Prentice Hall, Upper Saddle River, 2nd edition (2000).

[20] Smyth, M. B., *Stable compactification I*, Journal of the London Mathematical Society **s2-45**, pages 321–340 (1992). https://doi.org/10.1112/jlms/s2-45.2.321

[21] Weihrauch, K., *Computable Analysis, An Introduction*, Springer (2000).

[22] Zhou, C., R. A. Shaikh, Y. Li and A. Farjudian, *A domain-theoretic framework for robustness analysis of neural networks*, Mathematical Structures in Computer Science **33**, pages 68–105 (2023). https://doi.org/10.1017/S0960129523000142