On Kleisli liftings and decorated trace semantics

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Abstract

It is well known that Kleisli categories provide a natural language to model side effects. For instance, in the theory of coalgebras, behavioural equivalence coincides with language equivalence (instead of bisimilarity) when nondeterministic automata are modelled as coalgebras living in the Kleisli category of the powerset monad. In this paper, our aim is to establish decorated trace semantics based on language and ready equivalences for conditional transition systems (CTSs) with/without upgrades. To this end, we model CTSs as coalgebras living in the Kleisli category of a relative monad. Our results are twofold. First, we reduce the problem of defining a Kleisli lifting for the machine endofunctor in the context of a relative monad to the classical notion of Kleisli lifting. Second, we provide a recipe based on indexed categories to construct a Kleisli lifting for general endofunctors.

Keywords: Kleisli liftings, Relative monads, Conditional Transition Systems, Indexed categories

1 Introduction

Coalgebras [27] are a categorical generalisation of labelled transition systems (LTSs) and state-based systems in general, where the branching type is parameterised by an endofunctor over a category. Coalgebra homomorphisms between any two coalgebras are behaviour preserving maps between the underlying sets of states; often they correspond to some form of functional bisimulations. Under certain restrictions—for instance, when the underlying endofunctor over the category **Set** of sets is bounded [27]—the final coalgebra exists which can be seen as a universe of all coalgebras of the same type.

As coalgebra homomorphims in the category **Set** of sets correspond to functional bisimulations, the behavioural equivalence induced by the unique coalegbra homomorphism into the final coalgebra coincides with some form of bisimilarity. Nevertheless, there are many interesting notions of behavioural equivalences other than bisimilarity; for instance, decorated trace equivalences (like trace/language/failure/ready equivalences) on the states of an LTS (see the linear time-branching time spectrum [30]).

The clue to get coarser notions than bisimilarity is to consider TB-coalgebras with side effects, where T is a monad modelling the implicit side effects (like the powerset monad) and $B = \mathbb{A} \times \_$ is the endofunctor modelling the explicity branching with the set $\mathbb{A}$ of actions. Moreover, if $B\colon \mathbf{Set} \to \mathbf{Set}$ has a Kleisli

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lifting $\overline{B}$, then a TB-coalgebras living in **Set** could also be viewed as a $\overline{B}$-coalgebra living in the Kleisli category $\mathbf{K\ell}(T)$ and the unique coalgebra homomorphism, in the context of LTSs, maps a state to the set of traces generated by the state [19]. This idea was developed further by Hasuo et al. [13] by showing the behavioural equivalence in the chosen Kleisli category coincides with (probabilistic) language equivalence; in other words, language equivalence for various types of automata can be captured coinductively.

In a similar spirit, one can recover various forms of decorated trace equivalences coinductively by moving from the category **Set** of sets to an Eilenberg-Moore category (cf. [7,17]) and graded algebras (cf. [23,10]) induced by graded monads. Nevertheless, these coinductive characterisations of decorated trace equivalences based on Eilenberg-Moore categories and graded monads [11] require a preprocessing step to determinise the given coalgebra (generalising the well-known determinisation procedure of an automaton); as a result, there is an exponential blowup on the underlying state space unlike in the Kleisli case.

In this paper, our aim is to use a coalgebraic machinery to synthesise decorated trace semantics (language equivalence, failure and ready equivalences) for conditional transition systems (CTS). CTSs [1,4,5,6] are generalisations of traditional LTSs where each transition is guarded by a condition. CTSs come in two flavours based on whether the set of conditions are ordered or unordered. In the unordered case, CTSs and featured transition systems [22], a well-known operational model for software product lines, are equally expressive and we are able to characterise the above three decorated trace equivalence coinductively. In the ordered case, CTSs can model adaptive software product lines where certain features (encoded as conditions) can get upgraded to better versions modelled by the order relation; in this case, we present coinductive characterisations of language and ready equivalences (but not for failure equivalences). Nonetheless, for both types of systems, decorated trace semantics coarser than conditional bisimilarity [1,4,5,6] are not yet developed.

Our first contribution is to model CTSs without upgrades as $T^G B$-coalgebras, where $B = \mathbb{A} \times \_ + O$ is the endofunctor modelling the explicit branching with the set $\mathbb{A}$ of actions/alphabet and the set O of observations attributed to various notions of decorated traces. The essential difference with the case of an LTS is to model the implicit branching by a relative monad T^G [2] induced by the powerset monad and the writer comonad $G = \mathbb{K} \times \_$. This is to handle the 'conditional' transition relation $\to \subseteq X \times \mathbb{A} \times \mathbb{K} \times X$ of a CTS with a set X of state space. Operationally, a CTS (without upgrade) executes by selecting a condition and, henceforth, behaviour evolves like in an LTS. This state-transition structure enriched with conditions from $\mathbb{K}$ can be modelled as a coalgebra of type $\mathbb{K} \times X \to \mathcal{P}(\mathbb{K} \times (\mathbb{A} \times X + O))$; or simply as an arrow $X \to \mathbb{A} \times X + O$ in the Kleisli category $\mathbf{K\ell}(T^G)$ induced by the relative monad T^G. The set O of observations attributed to various notions of decorated traces (cf. Section 5). Now the final coalgebra homomorphism $X \to \mathbb{A}^\star \times O$ in $\mathbf{K\ell}(T^G)$ is a function of type $\mathbb{K} \times X \to \mathcal{P}(\mathbb{K} \times \mathbb{A}^\star \times O)$ matching our intuition of mapping a state $x \in X$ and a condition to the set of decorated traces generated by x and the condition obtained after executing a decorated trace (cf. Theorem 28).

For CTSs with upgrades we move to the category **Pos** of partially ordered sets and order preserving functions as morphisms with T fixed to be the downset monad (cf. Section 5). However, the difference with [6] is that we consider relative monads in this paper and thus, coalgebras in the Kleisli category $\mathbf{K\ell}(T^G)$ induced by a relative monad T^G. Just like one needs to define a Kleisli lifting of an endofunctor in the classical case [19,13], we prove a similar result (cf. Lemma 8) in the context of relative monad T^G. The conditions, though, of this lemma on the existence of Kleisli lifting $\widehat{B}\colon \mathbf{K\ell}(T^G) \to \mathbf{K\ell}(T^G)$ are quite strong; for instance, G does not preserve $B = \mathbb{A} \times \_ + O$ because $GBX \ncong BGX$ even when $O = 1$.

Despite this hurdle we are able to reduce the problem of defining a Kleisli lifting $\widehat{B}$ for the endofunctor $B = \mathbb{A} \times \_ + O$ in the context of a relative monad to the classical notion of Kleisli lifting [13,19,24]. Furthermore, we were able to use a result by Freyd [12] to prove that the initial algebra and final coalgebra for the functor $\widehat{B}$ coincide under the conditions that $\mathbf{K\ell}(T)$ is **Cppo**-enriched and the ω-directed joins commute with coproducts (cf. Theorem 10). These form the second contribution of the paper—paving a way to characterise decorated trace equivalences in a coinductive manner.

Our final contribution is to provide a recipe to construct a Kleisli lifting of a functor $F\colon \mathbf{C} \to \mathbf{C}$ in the context of just a monad $T\colon \mathbf{C} \to \mathbf{C}$. The key idea here is the following correspondence

$$\frac{X \to TY}{\overline{X \times IY \to \Omega,}}$$

which says that a Kleisli arrow is in one-to-one correspondence with a predicate over $X \times IY$ (here, $I\colon \mathbf{C} \to$

$\mathbf{C}$ is an endofunctor and Ω is a truth value object in the category $\mathbf{C}$). Once we have this correspondence, we can employ techniques from coalgebraic modal logic (like predicate and relation liftings) to define a distributive law ϑ of type $FT \Rightarrow TF$, which is equivalent to a Kleisli lifting of F (cf. [24]). In particular, under certain technical assumptions (**A3.1**-**A3.4**) we are able to define ϑ (cf. Proposition 12) as the transpose (under the above correspondence) of a relation lifting applied to the relation $\in_X : TX \times IX \to \Omega$. This relation coincides with the membership relation when $T = \mathcal{P}$ is the powerset monad and $I = \mathrm{Id}$ is the identity functor on **Set**. Moreover, if the relation lifting preserves the diagonal relation Δ (defined as the transpose of the unit of T) and the relation composition (defined using the Kleisli composition), then the constructed natural transformation ϑ is well-behaved with the unit (cf. Lemma 18) and multiplication (cf. Lemma 19) of the monad T, respectively. As a result, in the context of $\mathbf{C} = \mathbf{Set}$ or $\mathbf{C} = \mathbf{Pos}$, we obtain that the constructed natural transformation ϑ is a distributive law of type $FT \Rightarrow TF$ whenever F preserves the weak pullback squares.

2 Preliminaries

The objective of this section is to set the notations for this paper and recall the preliminaries related to coalgebraic modelling in a Kleisli category from [13].

We assume familiarity with basic category theory and the theory of coalgebras. We use meta-predicates $X, Y \in \mathbf{C}$ and $f \in \mathbf{C}(X, Y)$ to denote objects X, Y and an arrow f of the category C, respectively. If $X, Y \in \mathbf{C}$ have a coproduct, we write $\iota_X : X \to X + Y$ for the inclusion map. Dually, we write the projection map $\mathrm{pr}_X : X \times Y \to X$ whenever the product $X \times Y$ exists. Moreover, when $f' : X' \to Y' \in \mathbf{C}$ and the coproducts $X + X'$ and $Y + Y'$ exist, we denote by $f + f' : X + X' \to Y + Y'$ the unique arrow from the universal property of coproduct $X + X'$ such that the equations $\iota_Y \circ f = (f + f') \circ \iota_X$ and $\iota_{Y'} \circ f' = (f + f') \circ \iota_{X'}$ hold. Dually, if $X \times X'$ and $Y \times Y'$ exist, we define $f \times f' : X \times X' \to Y \times Y'$ to be the unique map given by $f \circ \mathrm{pr}_X, f' \circ \mathrm{pr}_{X'}$ and the universal property of $Y \times Y'$.

2.1 Coalgebras in a Kleisli category

We fix a category $\mathbf{C}$ and a monad (T, η, μ) on $\mathbf{C}$. Recall the Kleisli category $\mathbf{K\ell}(T)$ induced by $T : \mathbf{C} \to \mathbf{C}$:

$$\frac{X \in \mathbf{C}}{X \in \mathbf{K\ell}(T)} \qquad \frac{f : X \to TY \in \mathbf{C}}{f : X \to Y \in \mathbf{K\ell}(T)}.$$

The Kleisli composition $g \bullet f$ of two arrows $f : X \to Y, g : Y \to Z$ is given by the composition $\mu_Z \circ Tg \circ f$.

Throughout this section, we fix a coalgebra $c : X \to TBX \in \mathbf{C}$, where B is an endofunctor on $\mathbf{C}$. Typical examples are nondeterministic automata (NDAs), when $B = \mathbb{A} \times \_ + 1$ and $T = \mathcal{P}$, or their probabilistic/weighted variant, when T is sub-distribution monad [13] or semiring monad [17].

It is well known—for instance in the context of NDAs—that the coalgebra homomorphisms correspond to functional bisimulations which are too strong to capture language equivalence (either by taking the span or cospan of coalgebra homomorphisms). This mismatch is avoided, as first noted in [19], by moving to the Kleisli category $\mathbf{K\ell}(\mathcal{P})$. In particular, NDAs can also be seen as the coalgebra $X \to \overline{B}X \in \mathbf{K\ell}(\mathcal{P})$, where $\overline{B}$ is the Kleisli lifting of the endofunctor $B = \mathbb{A} \times \_ + 1$.

In general, a functor $\overline{B} : \mathbf{K\ell}(T) \to \mathbf{K\ell}(T)$ is a *Kleisli lifting* of $B : \mathbf{C} \to \mathbf{C}$ whenever the following square (drawn on the left) commutes, i.e. $\overline{B} \circ L' = L' \circ B$. Here L' is the free functor that maps an object X to the free algebra (TX, μ_X) and is left adjoint to the forgetful functor $R' : \mathbf{K\ell}(T) \to \mathbf{C}$ with $R'(X) = TX$ (for $X \in \mathbf{K\ell}(T)$) and $R'(f) = \mu'_X \circ Tf$ (for $f : X \to Y \in \mathbf{K\ell}(T)$).

$$\begin{array}{ccc} \mathbf{K\ell}(T) & \xrightarrow{\overline{B}} & \mathbf{K\ell}(T) \\ {\scriptstyle L'}\uparrow & & \uparrow{\scriptstyle L'} \\ \mathbf{C} & \xrightarrow{B} & \mathbf{C} \end{array} \qquad \begin{array}{ccc} TB & \xrightarrow{\vartheta} & BT \\ {\scriptstyle \eta_B}\uparrow & \nearrow_{B\eta} & \\ B & & \end{array} \quad (1) \qquad \begin{array}{ccccc} BTT & \xrightarrow{\vartheta_T} & TBT & \xrightarrow{T\vartheta} & TTB \\ {\scriptstyle B\mu}\downarrow & & & & \downarrow{\scriptstyle \mu_B} \\ BT & & \xrightarrow{\vartheta} & & TB \end{array} \quad (2)$$

Moreover, $\overline{B}$ is a Kleisli lifting of B [13,24] iff there is a natural transformation $\vartheta : BT \Rightarrow TB$ satisfying the laws indicated above in the middle and on the right. Such natural transformations were coined $\mathbf{K\ell}$-laws in [17]. The upshot of having a Kleisli lifting is that (probabilistic) language equivalence can be captured in a coinductive manner [13], i.e. whenever the final $\overline{B}$-coalgebra exists.

Hasuo et al. [13] presented two conditions (of increasing strengths) on a Kleisli lifting $\overline{B}$ that ensured when the initial B-algebra in $\mathbf{C}$ (if it exist) coincides with the final $\overline{B}$-coalgebra in $\mathbf{K\ell}(T)$. In this paper, we will use the following result due to Freyd [12] and generalise it to the level of relative monads. The other condition in [13, Theorem 3.3] requires that $\overline{B}$ is locally monotone instead of locally continuous; though we work with stronger assumption since B in our case studies will be locally continuous.

Definition 1. A category $\mathbf{C}$ is a **Cppo**-enriched category whenever its hom-set forms a ω-cpo with a bottom and the composition of arrows is a continuous function. In particular,

- for each $X, Y \in \mathbf{C}$, the set $\mathbf{C}(X,Y)$ is partially ordered $\preceq_{X,Y}$ with a bottom $\bot_{X,Y} \in \mathbf{C}(X,Y)$ (we drop the subscripts whenever it is clear from the context);
- for every increasing ω-chains $(f_i \in \mathbf{C}(X,Y))_{i\in\mathbb{N}}$ (i.e. $f_i \preceq f_{i+1}$), the join $\bigvee_{i\in\mathbb{N}} f_i \in \mathbf{C}(X,Y)$ exists.
- for every increasing ω-chains $(f_i \in \mathbf{C}(X,Y))_{i\in\mathbb{N}}$ and every $g \in \mathbf{C}(Y,Y'), h \in (X',X)$ we have

$$g \circ (\bigvee_{i\in\mathbb{N}} f_i) = \bigvee_{i\in\mathbb{N}} g \circ f_i \quad \text{and} \quad (\bigvee_{i\in\mathbb{N}} f_i) \circ h = \bigvee_{i\in\mathbb{N}} f_i \circ h.$$

Theorem 2 ([12,13]). *Let $\mathbf{K\ell}(T)$ be a **Cppo**-enriched category whose composition is left strict (i.e. $\bot_{Y,Y} \bullet f = \bot_{Y,Y}$ for every $f \in \mathbf{K\ell}(T)(X,Y)$) and $\overline{B}\colon \mathbf{K\ell}(T) \to \mathbf{K\ell}(T)$ be a locally continuous endofunctor. Then an initial algebra $\alpha\colon B(\mu_B) \xrightarrow{\cong} \mu_B \in \mathbf{C}$ (if it exists) induces a final coalgebra $B(\alpha)\colon \mu_B \to \overline{B}(\mu_B) \in \mathbf{K\ell}(T)$.*

2.2 Decorated trace equivalences coinductively

In this subsection, we apply Theorem 2 to characterise failure and ready equivalences using coinduction, i.e. we will characterise these equivalences as the equivalence induced by a unique coalgebra homomorphism from the underlying coalgebra to the final coalgebra. Though the presentation is new and the results follow directly from the above theorem, but we do *not* claim novelty (perhaps this is folklore). To the best of our knowledge, these decorated trace equivalences were only characterised by considering coalgebras in Eilenberg-Moore categories [7,17] or in the setting of graded algebras [10], but not in a Kleisli setting.

Throughout this subsection, we fix the endofunctor $B = \mathbb{A} \times \_ + O$ where $\mathbb{A}$ and O are some fixed sets with O indicating the observations that make these decorated trace equivalences distinct among themselves. So a labelled transition system (LTS) enriched with observations from O is a coalgebra $c\colon X \to \mathcal{P}BX$.

Proposition 3. *For the endofunctor $B = \mathbb{A} \times \_ + O$ on **Set**, the initial algebra exists and is given by $\mu_B = \mathbb{A}^\star \times O$ (the product of the sets of finite words induced by $\mathbb{A}$ and observations). Moreover, the algebra $h + h'\colon B\mu_B \xrightarrow{\cong} \mu_B$ is given by $h'(o) = (\epsilon, o)$ and $h(a,w,o) = (aw, o)$ (for $a \in \mathbb{A}, w \in \mathbb{A}^\star, o \in O$).*

Moreover, $\mathbf{K\ell}(\mathcal{P})$ is a **Cppo**-enriched category (cf. [13]) where the order is given by the subset inclusion and $\bot$ is given by the empty relation (recall that $\mathbf{K\ell}(P)$ is isomorphic to the category **Rel** of sets as objects and relations as morphisms). The functor B has a Kleisli lifting $\overline{B}$, which acts on a relation $f\colon \mathbf{Rel}(X,Y)$ as follows (which can be derived using the machinery developed in Section 4; see Example 26):

$$\overline{B}f = \{(o,o) \mid o \in O\} \cup \{((a,x),(a,y)) \mid a \in \mathbb{A} \wedge x\ f\ y\}. \tag{3}$$

Now Theorem 2 becomes applicable and we have

Proposition 4. *The final coalgebra for $\overline{B}$ exists in $\mathbf{K\ell}(\mathcal{P})$ and is given by $\mathbb{A}^\star \times O$.*

Moreover, it is instructive to verify that the unique coalgebra homomorphism $f\colon X \to \mathbb{A}^\star \times O \in \mathbf{K\ell}(\mathcal{P})$ from the coalgebra $c\colon X \to \overline{B}X \in \mathbf{K\ell}(\mathcal{P})$ maps a state x to a set of tuples $(w,o) \in \mathbb{A}^\star \times O$ such that o is an observation after performing the trace w from the state x. In addition, by modelling *refusal sets* [30] in the coalgebra map c when $O = \mathcal{P}\mathbb{A}$, i.e. for any $x \in X$ we impose on $c(x) \subseteq \mathbb{A} \times X + \mathcal{P}\mathbb{A}$ the condition

$$o \in c(x) \iff o \text{ is the set of actions disallowed from the state } x$$

for any $o \in O$, we obtain that the unique coalgebra homomorphism f maps a state to its failure pairs [30]. A *failure pair* of a state x is a tuple (w, o) where w is a trace starting from x to some x' and, moreover, o is a 'refusal' set of actions disallowed from x'. As a result, we get a coinductive characterisation of failure equivalence. Similarly, modifying the above coalgebra map to a map with the condition

$$o \in c(x) \iff o \text{ is the set of actions enabled from the state } x$$

for any $o \in O$, obtain that the unique coalgebra homomorphism f maps a state to its ready pairs [30]. A *ready pair* of a state x is a tuple (w, o) where w is a trace starting from x to x' and o is a 'ready' set of actions enabled from the state x'. Thus, obtaining a coinductive characterisation of ready equivalence.

3 Relative monads, Kleisli categories, and Kleisli liftings

In Section 5 it will become apparent that CTSs are modelled as coalgebras living in a Kleisli category induced by a relative monad. Relative monads in Computer Science were introduced in [2]; in particular, they worked out the so-called Kleisli and Eilenberg-Moore constructions of a relative monad. Nevertheless, the question of Kleisli lifting of an endofunctor was not considered in *op. cit.* and is particularly relevant for coalgebraic modelling of CTSs with and without upgrades. Therefore, in this section, we are going to recall the Kleisli construction of a relative monad from [2] and give sufficient conditions that ensure that the resulting functor is a Kleisli lifting of a given endofunctor.

Definition 5. Given a functor [4] $G\colon \mathbf{C} \to \mathbf{C}$, then a *$G$-relative monad* [2] is given by the following data:

(i) an object mapping $T\colon \mathbf{C} \to \mathbf{C}$;

(ii) for every object $X \in \mathbf{C}$, there is a *unit* map $\eta_X \in \mathbf{C}(GX, TX)$;

(iii) for every arrow $f \in \mathbf{C}(GX, TY)$ there is a map $f^\sharp \in \mathbf{C}(TX, TY)$ called the *Kleisli lifting* of f satisfying the unit and associative laws, i.e. for any $g \in \mathbf{C}(GY, TZ)$ the following diagrams commute:

$$\begin{array}{ccc} GX & \xrightarrow{\eta_X} & TX \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f^\sharp} & \\ TY & & \end{array} \tag{4a}$$

$$\begin{array}{c} TX \\ {\scriptstyle \mathrm{id}_{TX}}\downarrow\;\downarrow{\scriptstyle \eta_X^\sharp} \\ TX \end{array} \tag{4b}$$

$$\begin{array}{ccc} TX & \xrightarrow{(g^\sharp \circ f)^\sharp} & TZ \\ \downarrow{\scriptstyle f^\sharp} & \nearrow{\scriptstyle g^\sharp} & \\ TY & & \end{array} \tag{4c}$$

Just like how a traditional monad gives rise to a Kleisli category, so does the relative monad in the manner explained next. Given a G-relative monad T, its *Kleisli category*, denoted $\mathbf{K}\boldsymbol{\ell}(T^G)$, is given by objects from $\mathbf{C}$ and maps between any X and Y by maps between GX and TY in $\mathbf{C}$, i.e.

$$\frac{X \in \mathbf{C}}{X \in \mathbf{K}\boldsymbol{\ell}(T^G)} \qquad \frac{f\colon GX \to TY \in \mathbf{C}}{f\colon X \to Y \in \mathbf{K}\boldsymbol{\ell}(T^G).} \tag{5}$$

The identity morphism on X is provided by η_X, which is a left and right unit according to Eqs. (4a) and (4b), respectively. Composition of two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{K}\boldsymbol{\ell}(T^G)$ is given by $g^\sharp \circ f$ in $\mathbf{C}$. Associativity of this composition is ensured by Eq. (4c).

The usual free-forgetful adjunction between the Kleisli category and its underlying base category gets a bit subtle in the presence of relative monads. The notion of adjunction generalises to that of a **(left) G-relative adjunction** $L \dashv R\colon G \to \mathbf{D}$ between an endofunctor G on $\mathbf{C}$ and another category $\mathbf{D}$. It consists of two functors $L\colon \mathbf{C} \to \mathbf{D}$ and $R\colon \mathbf{D} \to \mathbf{C}$ such that we have the natural bijection

$$\frac{GX \to RY \in \mathbf{C}}{LX \to Y \in \mathbf{D}}\,.$$

[4] Our presentation of relative monad is an instance of a more general formulation in [2] where G is not necessarily an endofunctor.

Proposition 6. *For any endofunctor G and monad T, both on $\mathbf{C}$, there is a G-relative adjunction $L \dashv R\colon G \to \mathbf{K}\boldsymbol{\ell}(T^G)$, called the* ***relative Kleisli adjunction****, given by the functors L, R:*

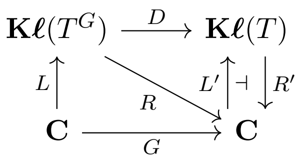
$$LX = X, \qquad Lf = \eta_{GY} \circ Gf, \tag{6a}$$
$$RX = TX, \qquad Rg = g^\sharp, \tag{6b}$$

where $f\colon X \to Y$ and $g\colon GX \to TY$.

Every monad induces a relative monad

Now fix an endofunctor G on a category $\mathbf{C}$, then every monad (T, η, μ) on $\mathbf{C}$ gives rise to a relative monad T^G with $T^G X = TGX$ (for $X \in \mathbf{C}$), the unit given by η_G, and $f^\sharp = \mu_{GY} \circ Tf$ (for $f\colon GX \to TGY$).

Proposition 7. *[2, § 2.4] The three categories are formally related as follows:*



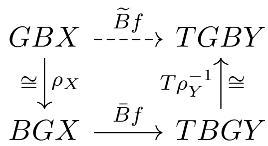
Where $L' \dashv R'$ is the classical Kleisli adjunction, $DX = GX$, and $Df = f$ (for $X, f \in \mathbf{K}\boldsymbol{\ell}(T^G)$).

Note that the construction T^G resembles the "Kleisli-like" construction $K^T_{G,T}$ from Hirsch's thesis [14, p. 44], where it was assumed that G is a comonad. He explores different ways to combine monads and comonads in programming language semantics and applies his results to security policies.

Kleisli lifting of an endofunctor

In the sequel, all our relative monads are induced by monads; so, in this section, we explore how to extend a given endofunctor $B\colon \mathbf{C} \to \mathbf{C}$ to an endofunctor $\widetilde{B}\colon \mathbf{K}\boldsymbol{\ell}(T^G) \to \mathbf{K}\boldsymbol{\ell}(T^G)$. Just like in the traditional case, we say an endofunctor $\widetilde{B}\colon \mathbf{K}\boldsymbol{\ell}(T^G) \to \mathbf{K}\boldsymbol{\ell}(T^G)$ is a *Kleisli lifting* of B iff $\widetilde{B} \circ L = L \circ B$.

Lemma 8. *If G preserves B, i.e. there is a natural isomorphism $\rho\colon GB \cong BG$, the existence of a Kleisli lifting $\mathbf{K}\boldsymbol{\ell}(T) \xrightarrow{\overline{B}} \mathbf{K}\boldsymbol{\ell}(T)$ of B implies the existence of a Kleisli lifting $\mathbf{K}\boldsymbol{\ell}(T^G) \xrightarrow{\widetilde{B}} \mathbf{K}\boldsymbol{\ell}(T^G)$ of B. In particular, $\widetilde{B}X = BX$ (for $X \in \mathbf{C}$) and $\widetilde{B}f$ (for an arrow $f\colon X \to Y \in \mathbf{K}\boldsymbol{\ell}(T^G)$) is defined as in (7) Moreover, if $\overline{B}$ is locally continuous (when $\mathbf{K}\boldsymbol{\ell}(T)$ is* **Cppo**-*enriched), then so is $\widetilde{B}$ as defined above.*



(7)

Unfortunately, unlike in the traditional case, we have to enforce some restriction on G (cf. Lemma 8) to characterise a Kleisli lifting in terms of certain distributive laws. Note that this condition is, perhaps, not that surprising when compared to other existing results that lift traditional results known on monads/adjunctions to relative monads/adjunctions. For instance, the well known result that colimits are preserved by left adjoints does *not* hold in general in the context of relative adjunctions; it is only those colimits that are preserved by G are preserved by the left adjoint in this new setting [29].

Kleisli lifting of Machine endofunctor $B = \mathbb{A} \times \_ + O$

The condition in Lemma 8 that G preserves the endofunctor B is too strong because $GBX \ncong BGX$ (for $X \in \mathbf{Set}$, any nonempty set O, and $G = \mathbb{K} \times \_$ with $\mathbb{K} \neq \emptyset$). In this section, we further impose the restriction on G to preserve coproducts, which allows us to define a Kleisli lifting for the machine endofunctor. Assume that the working category $\mathbf{C}$ has binary products and coproducts, so that we can define the machine endofunctor $B = \mathbb{A} \times \_ + O$, where $\mathbb{A}, O \in \mathbf{C}$ are two fixed objects in the category $\mathbf{C}$. For brevity, the functor $\mathbb{A} \times \_$ is denoted by A.

Throughout this section, we further assume that

A1 the functor G preserves coproducts;

A2 the functor G preserves A (cf. Lemma 8);

A3 $\overline{A}\colon \mathbf{K}\boldsymbol{\ell}(T) \to \mathbf{K}\boldsymbol{\ell}(T)$ is a Kleisli lifting of A (thus $\widetilde{A}\colon \mathbf{K}\boldsymbol{\ell}(T^G) \to \mathbf{K}\boldsymbol{\ell}(T^G)$ exists).

These assumptions allow us to define the mapping $\widehat{B}$, which will become our lifting: It maps an object X to BX and for a given arrow $f\colon X \to Y \in \mathbf{K}\boldsymbol{\ell}(T^G)$ we define (note **A2** ensures that Lemma 8 is applicable for the endofunctor A):

$$\begin{array}{ccc} G(AX+O) & \xrightarrow{\quad \widehat{B}f \quad} & TG(AY+O) \\ {\scriptstyle G\iota_{AX} \triangledown G\iota_O} \big\uparrow \cong & & \big\uparrow {\scriptstyle TG\iota_{AY} \triangledown TG\iota_O} \\ GAX + GO & \xrightarrow[\widetilde{A}f + \eta_{GO}]{} & TGAY + TGO \end{array} \tag{8}$$

where $f \triangledown f'\colon X + X' \to Y$, "codiagonal", is defined as the universal arrow of two morphisms $f\colon X \to Y$ and $f'\colon X' \to Y$ with joined codomain. It is actually the map given from the pair (f, f') as the adjunct of the adjunction between the coproduct and the diagonal functor $X \mapsto (X, X)$ from $\mathbf{C}$ to $\mathbf{Cat}(\{1,2\}, \mathbf{C})$. Later in Section 4 we will also use the dual version $\vartriangle$ of $\triangledown$.

Theorem 9. *The above mapping $\widehat{B}\colon \mathbf{K}\boldsymbol{\ell}(T^G) \to \mathbf{K}\boldsymbol{\ell}(T^G)$ is a functor. Moreover, $\widehat{B}$ is also a Kleisli lifting of B, i.e. $\widehat{B} \circ L = L \circ B$.*

Theorem 10. *Let T be a monad on $\mathbf{C}$ and G an endofunctor on $\mathbf{C}$.*

(i) *If G preserves colimits and the initial algebra $h\colon B(\mu_B) \xrightarrow{\cong} \mu_B$ of B exists in $\mathbf{C}$, then $Lh\colon \widehat{B}(\mu_B) \xrightarrow{\cong} \mu_B$ is the initial algebra of $\widehat{B}$ in $\mathbf{K}\boldsymbol{\ell}(T^G)$.*

(ii) *If $\mathbf{K}\boldsymbol{\ell}(T)$ is* **Cppo***-enriched, $\overline{A}$ is locally continuous, and the operation $\_ + g$ commutes with the ω-directed joins, i.e. for any increasing families of arrows $(f_i \in \mathbf{K}\boldsymbol{\ell}(T)(X, Z))_{i\in\mathbb{N}}$ we have*

$$\bigvee_{i\in\mathbb{N}} (f_i + g) = (\bigvee_{i\in\mathbb{N}} f_i) + g,$$

then $\widehat{B}$ is locally continuous.

As a result, μ_B is the final coalgebra of $(Lh)^{-1}\colon \mu_B \to \widehat{B}(\mu_B)$ of $\widehat{B}$ in $\mathbf{K}\boldsymbol{\ell}(T^G)$.

4 On constructing a Kleisli lifting

In the previous section and in the context of endofunctor $B = \mathbb{A} \times \_ + O$, we reduced the problem of defining a Kleisli lifting of $A = \mathbb{A} \times \_$ w.r.t. a relative monad T^G by simply defining the Kleisli lifting of A w.r.t. a monad T (cf. **A3**). The remaining assumptions **A1** and **A2** are straightforward to satisfy in our case studies (cf. Section 5). The objective of this section is to give a general recipe to construct a Kleisli lifting $\overline{F}\colon \mathbf{K}\boldsymbol{\ell}(T) \to \mathbf{K}\boldsymbol{\ell}(T)$ of an endofunctor $F\colon \mathbf{C} \to \mathbf{C}$. And recall from [17] that a Kleisli lifting $\overline{F}$ is in one-to-one correspondence with a $\mathbf{K}\boldsymbol{\ell}$-law $\vartheta\colon FT \Rightarrow TF$. The rest of this section is devoted to define such a $\mathbf{K}\boldsymbol{\ell}$-law ϑ internally using the framework of indexed categories/fibrations.

To motivate our assumptions that follow, consider the Kleisli category $\mathbf{K}\boldsymbol{\ell}(\mathcal{P})$ induced by the powerset monad. It is well known that the set of Kleisli arrows $\mathbf{Set}(X, \mathcal{P}Y)$ are in one-to-one correspondence with the set $\mathcal{P}(X \times Y)$ of binary relations, i.e.

$$\mathbf{Set}(X, \mathcal{P}Y) \;\cong\; \mathcal{P}(X \times Y) \;\cong\; \mathbf{Set}(X \times Y, 2),$$

where 2 is a two element set {0,1}. However, in the Kleisli category $\mathbf{K}\boldsymbol{\ell}(\mathcal{P}_\downarrow)$ for the downset monad $\mathcal{P}_\downarrow\colon \mathbf{Pos} \to \mathbf{Pos}$ over the category of posets, this idea of representing a Kleisli arrow as homming into 2 (ordered by the smallest poset generated by the relation $\{(0,1)\}$) is subtly different (see Proposition 30):

$$\mathbf{Pos}(X, \mathcal{P}_\downarrow Y) \cong \{R \subseteq X \times Y \mid R \text{ is up (down) closed in the first (second) argument}\} \cong \mathbf{Pos}(X \times Y^o, 2),$$

where Y^o denotes the dual poset of Y. Thus, in general, we require an endofunctor $I\colon \mathbf{C} \to \mathbf{C}$ such that the arrows $X \to TY$ can be internally represented as fibres $\Phi(X \times IY)$ in an indexed category.

A3.1 There is an indexed category $\Phi\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}$ with a bifibration structure, i.e. for each $f\colon X \to Y \in \mathbf{C}$ there is an adjoint situation $\exists_f \dashv f^*\colon \Phi X \to \Phi Y$. Note it is customary to write Φf as f^* (cf. [18]).

A3.2 There is an endofunctor $I\colon \mathbf{C} \to \mathbf{C}$ such that $F \circ I = I \circ F$.

A3.3 There is a monad (T, η, μ) on $\mathbf{C}$ with the following correspondence

$$\theta_{X,Y}\colon \mathbf{C}(X, TY) \cong \Phi(X \times IY) \qquad \text{(for each } X, Y \in \mathbf{C}\text{)}$$

such that the following diagrams commute for each $f\colon X \to X', g\colon Y \to Y' \in \mathbf{C}$.

A3.3a
$$\begin{array}{ccc} \mathbf{C}(X, TY) & \xrightarrow{\theta_{X,Y}} & \Phi(X \times IY) \\ {\scriptstyle \_\circ f}\uparrow & & \uparrow{\scriptstyle (f\times IY)^*} \\ \mathbf{C}(X', TY) & \xrightarrow{\theta_{X',Y}} & \Phi(X' \times IY) \end{array}$$

A3.3b
$$\begin{array}{ccc} \mathbf{C}(X, TY) & \xrightarrow{\theta_{X,Y}} & \Phi(X \times IY) \\ \downarrow{\scriptstyle Tg\circ\_} & & \downarrow{\scriptstyle \exists_{(X\times Ig)}} \\ \mathbf{C}(X, TY') & \xrightarrow{\theta_{X,Y'}} & \Phi(X \times IY') \end{array}$$

A3.4 There is an indexed morphism (aka predicate liftings) $\sigma\colon \Phi \Rightarrow \Phi F$.

Note that our technical objective of this section is to show that Assumptions **A3.1-A3.4** imply Assumption **A3**; hence the use of nesting in the above naming convention.

Some remarks are in order on the indexed category Φ. Intuitively, in our case-studies, **A3.1** will be the fibres $\Phi(X)$ containing predicates of type $X \to \Omega \in \mathbf{C}$ with $\Omega \in \mathbf{C}$ modelling the truth value object. For our purposes $\Omega = 2$ the two-pointed set with the order $0 < 1$ (when $\mathbf{C} = \mathbf{Pos}$). The left adjoint $\exists_f$ of f^* is used in categorical logic [18] to model the existential quantifier, which is originally due to Lawvere [21]. Below we will use such left adjoints to construct a relation lifting from the predicate lifting σ, which are used to define semantics of a modality in coalgebraic modal logics.

The general idea is to define a $\mathbf{K\ell}$-law $\vartheta\colon FT \Rightarrow TF$ internally using the language of fibred categories. For instance, thanks to the isomorphism in **A3.3**, $\vartheta_X\colon FTX \to TFX$ (for some $X \in \mathbf{C}$) can be defined internally by an element in the fibre $\Phi(FTX \times IFX)$. To this end, we start by the identity arrow id_{TX} and consider the element called 'membership relation'

$$\in_X \stackrel{\text{def}}{=} \theta_{TX,X}(\mathrm{id}_{TX}) \qquad \in \Phi(TX \times IX).$$

This notation is motivated by one of the case studies: when $\mathbf{C} = \mathbf{Set}$, $I = \mathrm{Id}$, and $T = \mathcal{P}$, this element will correspond to the element relation $\in_X \subseteq TX \times X$.

Note that technically $\in_X$ is an element in the fibre $\Phi(TX \times IX)$. Now applying the predicate lifting σ on $\in_X$ gives an element $\sigma_{TX\times IX}(\in_X) \in \Phi F(TX \times IX)$. We are now a step away from defining our distributive law. First recall the following arrows defined thanks to the universal property of product and the equation $I \circ F = F \circ I$ (cf. **A3.2**):

$$\lambda_{X,Y} = F(\mathrm{pr}_X) \vartriangle F(\mathrm{pr}_{IY})\colon F(X \times IY) \to FX \times FIY = FX \times IFY$$

where $f \vartriangle f'$, the "diagonal operation", is, in general, defined for any two arrows $f\colon X \to Y$ and $f\colon X \to Y'$ in $\mathbf{C}$ with common domain X. It is the dual to $\triangledown$ introduced in connection with Eq. (8). Our distributive law ϑ_X is simply the mapping of $\sigma_{TX\times IX}(\in_X)$ by the map

$$\theta^{-1}_{FTX,FX} \circ \exists_{\lambda_{TX,X}}\colon \Phi F(TX \times IX) \to \Phi(FTX \times IFX) \to \mathbf{C}(FTX, TFX);$$

thus, we define by composition of maps

$$\vartheta_X \stackrel{\text{def}}{=} \theta^{-1}_{FTX,FX} \circ \exists_{\lambda_{TX,X}} \circ \sigma_{TX\times IX}(\in_X) = \theta^{-1}_{FTX,FX} \circ \exists_{\lambda_{TX,X}} \circ \sigma_{TX\times IX} \circ \theta_{TX,X}(\mathrm{id}_{TX}). \tag{9}$$

Lemma 11. *Let $f\colon X \to Y \in \mathbf{C}$. Then we have $\exists_{TX\times If}(\in_X) = (Tf \times IY)^*(\in_Y)$.*

As a result, we get the following result which is one of the basic requirements for ϑ to be a $\mathbf{K\ell}$-law.

Proposition 12. *The map ϑ defined in (9) is a natural transformation of type $FT \Rightarrow TF$.*

On relation lifting

Before we establish the compatibility of ϑ with unit η and multiplication μ of the monad T, respectively, in the subsequent subsections, we need that the mapping

$$\Phi(X \times IY) \xrightarrow{\sigma_{X\times IY}} \Phi F(X \times IY) \xrightarrow{\exists_{\lambda_{X,IY}}} \Phi(FX \times IFY)$$

is a relation lifting in the following sense. Notice that, in the sequel, the fibres $\Phi(X \times IY)$ over the object $X \times IY \in \mathbf{C}$ are maps of type $X \times IY \to \Omega \in \mathbf{C}$, which we simply call as relations. Thus, the above map abbreviated $\tilde{\sigma}_{X,Y} = \exists_{\lambda_{X,IY}} \circ \sigma_{X\times IY}$ is a candidate to what is known as F-*relators* in the literature on coalgebras. Actually, there is no common consensus on the 'categorical' definition of an F-relator (cf. [16, Chapter 4]), however, for our purpose we require a relator to be an indexed morphism, i.e. a natural transformation of type $\Phi(\_ \times IX) \Rightarrow \Phi(\_ \times IFX)$ (for every $X \in \mathbf{C}$). Thus, in the parlance of relators, our natural transformation ϑ on a component is nothing but a relation lifting of the membership relation $\in$.

Definition 13. Given a commuting diagram below on the left, we say the Beck-Chevalley condition for (10a) holds iff the square (10b) on the right commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow k \\ Z & \xrightarrow{h} & Z' \end{array} \qquad (10a) \qquad\qquad \begin{array}{ccc} \Phi(X) & \xrightarrow{\exists_f} & \Phi(Y) \\ g^* \uparrow & & \uparrow k^* \\ \Phi(Z) & \xrightarrow{\exists_h} & \Phi(Z') \end{array} \qquad (10b)$$

Lemma 14. *If in (10b) $k^* \circ \exists_h \leq \exists_f \circ g^*$, then the Beck-Chevalley condition holds for (10a).*

Corollary 15. *Let $\mathbf{C}$ be* **Set** *or* **Pos** *and $\Omega = 2$ with the order generated by $0 < 1$ when $\mathbf{C} = \mathbf{Pos}$. If Φ is the indexed category of predicates, i.e. $\Phi = \mathbf{C}(\_, \Omega)$, then the Beck-Chevalley condition holds for the square in (10a) whenever it is a weak pullback square.*

Theorem 16. *The square Eq. (11a) below commutes for any $f : X \to X'$ and $g : Y \to Y'$.*

$$\begin{array}{ccc} F(X \times IY) & \xrightarrow{\lambda_{X,Y}} & FX \times FIY \\ \downarrow F(f\times Ig) & & \downarrow Ff\times FIg \\ F(X' \times IY') & \xrightarrow[\lambda_{X',Y'}]{} & FX' \times FIY' \end{array} \quad (11a) \qquad \begin{array}{ccc} F(X \times IY) & \xrightarrow{\lambda_{X,Y}} & FX \times IFY \\ F(f\times IY) \downarrow & & \downarrow Ff\times IFY \\ F(X' \times IY) & \xrightarrow[\lambda_{X',Y}]{} & FX' \times IFY \end{array} \quad (11b)$$

Moreover:

(i) *If the Beck-Chevalley condition holds in the special cases of Eq. (11b) (for any f) then the map $\tilde{\sigma}_{\_,Y} : \Phi(\_ \times IY) \Rightarrow \Phi(F\_ \times IFY)$ (for each $Y \in \mathbf{C}$) is a natural transformation (i.e. a relation lifting).*

(ii) *If F preserves weak-pullbacks, then Eq. (11a) is a weak pullback-square.*

(iii) *In the context of $\mathbf{C}$ is* **Set** *or* **Pos** *and $\Omega = 2$, the Beck-Chevalley condition holds for the above square whenever F preserves weak pullback squares.*

On compatibility of ϑ with η and μ

It turns out that these compatibility properties, i.e. Equations (1) and (2), are intrinsically related with Kleisli compositions. Note that because of **A3.3** we can define the composition $\odot$ of relations of type:

$$\odot : \Phi(Y \times IZ) \times \Phi(X \times IY) \to \Phi(X \times IZ) \qquad S \odot R = \theta_{X,Z}\left(\theta^{-1}_{Y,Z}(S) \bullet \theta^{-1}_{X,Y}(R)\right) \tag{12}$$

Proposition 17. *The identity relation $\Delta_X \stackrel{def}{=} \theta_{X,X}(\eta_X)$ is the unit to the relational composition $\odot$.*

Lemma 18. *If the Beck-Chevalley conditions holds for the squares in 11b (for every $f \in \mathbf{C}$) then the relation lifting $\tilde{\sigma}$ preserves the identity relation, i.e. $\tilde{\sigma}_{X,X}(\Delta_X) = \Delta_{FX}$ (for each $X \in \mathbf{C}$) if, and only if, ϑ is compatible with the unit η (i.e. Eq. (1) holds).*

Proof. First observe that

$$\dfrac{\dfrac{\tilde{\sigma}_{X,X}(\Delta_X) = \Delta_{FX}}{\tilde{\sigma}_{X,X}\theta_{X,X}(\eta_X) = \theta_{FX,FX}(\eta_{FX})}\text{ Def. of }\Delta}{\theta^{-1}_{FX,FX}\tilde{\sigma}_{X,X}\theta_{X,X}(\eta_X) = \eta_{FX}}\quad .$$

Next observe that the diagram

$$\begin{array}{ccccccc}
\mathbf{C}(TX,TX) & \xrightarrow{\theta_{TX,X}} & \Phi(TX \times IX) & \xrightarrow{\tilde{\sigma}_{TX,X}} & \Phi(FTX \times FIX) & \xrightarrow{\theta^{-1}_{FTX,FX}} & \mathbf{C}(FTX,FTX) \\
\downarrow{\scriptstyle \eta^*_X} & & \downarrow{\scriptstyle (\eta_X \times IX)^*} & & \downarrow{\scriptstyle (F\eta_X \times FIX)^*} & & \downarrow{\scriptstyle (F\eta_X)^*} \\
\mathbf{C}(X,TX) & \xrightarrow{\theta_{X,X}} & \Phi(X \times IX) & \xrightarrow{\tilde{\sigma}_{X,X}} & \Phi(FX \times FIX) & \xrightarrow{\theta^{-1}_{FX,FX}} & \mathbf{C}(FX,TX)
\end{array}$$

commutes by Theorem 16 and Assumption **A3.3**a. Chasing id_{TX} through the diagram gives

$$\dfrac{\left(\theta^{-1}_{FTX,FX} \circ \tilde{\sigma}_{TX,X} \circ \theta_{TX,X}(\mathrm{id}_{TX})\right) \circ F\eta_X = \theta^{-1}_{FX,FX}\tilde{\sigma}_{X,X}\theta_{X,X}(\mathrm{id}_{TX} \circ \eta_X)}{\vartheta_X \circ F\eta_X = \theta^{-1}_{FX,FX}\tilde{\sigma}_{X,X}\theta_{X,X}(\eta_X)}\ \textit{Eq. (9)}\quad .$$

Combining the observations of this an the preceding paragraph gives the lemma. □

Lemma 19. *Assume that the Beck-Chevalley conditions holds for the squares in 11b (for every $f \in \mathbf{C}$). If the relation lifting $\tilde{\sigma}$ preserves the relational composition, i.e. for each $R \in \Phi(X \times IY), S \in \Phi(Y \times IZ)$*

$$\tilde{\sigma}_{X,Z}(S \odot R) = \tilde{\sigma}_{FY,FZ}(S) \odot \tilde{\sigma}_{FX,FY}(R), \tag{13}$$

then ϑ is compatible with the multiplication μ (i.e. Eq. (2) holds). Conversely, if the compatibility with multiplication occurs, then Eq. (13) holds at least in the instance $R = \in_X$ and $S = \in_{TX}$.

Proof. First note that using Kleisli composition condition Eq. (2) can be expressed by

$$\theta_X \circ F\mu_X = \theta_X \bullet \theta_{TX} \tag{14}$$

for X ranging over all objects in $\mathbf{C}$. Second, we consider the same diagram as in the proof of Lemma 18 but with the multiplication μ in place of the unit η:

$$\begin{array}{ccccccc}
\mathbf{C}(TX,TX) & \xrightarrow{\theta_{TX,X}} & \Phi(TX \times IX) & \xrightarrow{\tilde{\sigma}_{TX,X}} & \Phi(FTX \times FIX) & \xrightarrow{\theta^{-1}_{FTX,FX}} & \mathbf{C}(FTX,FTX) \\
\downarrow{\scriptstyle \mu^*_X} & & \downarrow{\scriptstyle (\mu_X \times IX)^*} & & \downarrow{\scriptstyle (F\mu_X \times FIX)^*} & & \downarrow{\scriptstyle (F\mu_X)^*} \\
\mathbf{C}(TTX,TX) & \xrightarrow{\theta_{TTX,X}} & \Phi(TTX \times IX) & \xrightarrow{\tilde{\sigma}_{TTX,X}} & \Phi(FTTX \times FIX) & \xrightarrow{\theta^{-1}_{FTTX,FX}} & \mathbf{C}(FTTX,TX)
\end{array}$$

Again the diagram commutes due to Theorem 16 and Assumption **A3.3**a. Again we chase id_{TX} through the diagram. We obtain

$$\left(\theta^{-1}_{FTX,FX} \circ \tilde{\sigma}_{TX,X} \circ \theta_{TX,X}(\mathrm{id}_{TX})\right) \circ F\mu_X = \theta^{-1}_{FTTX,FX}\tilde{\sigma}_{TTX,X}\theta_{TTX,X} \circ (\mu_X \circ \mathrm{id}_{TTX})$$

which becomes using Eq. (9)

$$\begin{aligned}
\vartheta_X \circ F\mu_X &= \theta^{-1}_{FTTX,FX}\tilde{\sigma}_{TTX,X}\theta_{TTX,X}(\mu_X \circ \mathrm{id}_{TTX}) \\
&= \theta^{-1}_{FTTX,FX}\tilde{\sigma}_{TTX,X}\theta_{TTX,X}(\mathrm{id}_{TX} \bullet \mathrm{id}_{TTX})
\end{aligned}$$

$$= \theta^{-1}_{FTTX,FX}\tilde{\sigma}_{TTX,X}\theta_{TTX,X}(\theta^{-1}_{TX,X}(\in_X) \bullet \theta^{-1}_{TTX,TX}(\in_{TX}))$$
$$\stackrel{\text{Eq. (12)}}{=} \theta^{-1}_{FTTX,FX}\tilde{\sigma}_{TTX,X}(\in_X \odot \in_{TX})$$
$$\stackrel{\text{Eq. (13)}}{=} \theta^{-1}_{FTTX,FX}\big(\tilde{\sigma}_{TX,X}(\in_X) \odot \tilde{\sigma}_{TTX,TX}(\in_{TX})\big)$$
$$\stackrel{\text{Eq. (12)}}{=} \theta^{-1}_{FTX,FX}(\tilde{\sigma}_{TX,X}\in_X) \bullet \theta^{-1}_{FTTX,FTX}(\tilde{\sigma}_{TTX,TX}\in_{TX})$$
$$\stackrel{\text{Eq. (9)}}{=} \theta_X \bullet \theta_{TX}.$$

This concludes the forward direction by Eq. (14). For the converse direction note that in the single instance where Eq. (13) was used when $R = \in_X$ and $S = \in_{TX}$ is an equivalence since $\theta_{FTTX,FX}$ is bijective. All other steps in the last calculation were equivalences as well. □

Now we can state the main result of this section.

Theorem 20. *Let* **C** *be either* **Set** *or* **Pos** *and* $\Omega = 2$*. If F preserves weak pullbacks, then the natural transformation ϑ defined in Eq. (9) is a* $\mathbf{K}\boldsymbol{\ell}$*-law.*

We end this section by giving an example of $\mathbf{K}\boldsymbol{\ell}$-law which is a direct consequence of the above theorem. Moreover, the general results of neither [13, Lemma 2.4] nor [19, Section 4] are applicable in Example 21 since our functor F is not shapely [13] and T is a generalisation of powerset monad.

Example 21. We work with the Lawvere quantale [5] Ω and let $T = \mathcal{P}_\Omega$ be the Ω-valued powerset monad [15, Remark 1.2.3] on **Set** defined as $\mathcal{P}_\Omega = \Omega^X$ on objects and as $Tf(g)(y) = \inf_{f(x)=y} g(x)$ (for $f\colon X \to Y$) on arrows. Its unit $\eta_X\colon X \to \mathcal{P}_\Omega X$ is given by $\eta_X(x)(x') = 0$ if $x = x'$ and 1 (the empty meet) otherwise. Multiplication $\mu_X\colon \mathcal{P}_\Omega\mathcal{P}_\Omega X \to \mathcal{P}_\Omega X$ is defined as $\mu_X(G)(x) = \inf_{g\in\mathcal{P}_\Omega X} G(g) \oplus g(x)$. It is not hard to see that a Kleisli arrow $X \to Y \in \mathbf{K}\boldsymbol{\ell}(\mathcal{P}_\Omega)$ corresponds to a Ω-valued matrix of dimension $X \times Y$ (i.e. the functor I is set to be the identity functor); the latter are known as Ω-valued relations in [8]. The indexed category $\Phi(X \times Y)$ of Ω-valued relations forms a bifibration; the left adjoint $\exists_f$ (for a function $f\colon X \to Y$) is given by $\exists_f(p)(y) = \inf_{fx=y} p(x)$. Moreover, every weak pullback square in **Set** satisfies the Beck-Chevalley condition in this quantalic context.

Now consider the functor $F = \mathcal{D}$ the distribution functor and the predicate lifting $\sigma_X\colon \Omega^X \to \Omega^{\mathcal{D}X}$ given by the expectation $\mathbb{E}_\mu(p)$:

$$\sigma_X(p)(\mu) = \mathbb{E}_\mu(p) = \sum_{x\in X} p(x)\cdot\mu(x) \qquad \text{(for each } \mu \in \mathcal{D}X\text{)}$$

(note that the sum is automatically defined as it ranges over non-negative values and is in Ω due to the assumptions on p and μ). Furthermore, the relation $\in_X$ is given by evaluation $(p, x) \mapsto p(x)$ and the natural transformation $\lambda\colon \mathcal{D}(X \times Y) \to \mathcal{D}(X) \times \mathcal{D}(Y)$ maps a joint distribution $\omega \in \mathcal{D}(X \times Y) \mapsto (\sum_{y\in Y}\omega(\_, y), \sum_{x\in X}\omega(x, \_))$ to its corresponding marginal distributions. Thus, using the terminology of optimal transport, the left adjoint $\exists_{\lambda_X}(M)(\mu, \nu) = \inf_{\lambda_X(\omega)=(\mu,\nu)} M(\omega)$ computes the best possible coupling of a given pair of distributions μ, ν in M. This gives rise to a $\mathbf{K}\boldsymbol{\ell}$-law $\vartheta\colon \mathcal{D}\mathcal{P}_\Omega \to \mathcal{P}_\Omega\mathcal{D}$ as follows:

$$\vartheta_X(M)(\mu) = \inf_{\lambda_{TX,X}(\omega)=(M,\mu)} \mathbb{E}_\omega(\in_X) = \inf_{\lambda_{\mathcal{D}X,X}(\omega)=(M,\mu)} \sum_{(p,x)\in\mathcal{P}_\Omega X\times X} p(x)\cdot\omega(p,x),$$

where $M \in \mathcal{D}(\mathcal{P}_\Omega X)$ and $\mu \in \mathcal{D}(X)$.

5 A case study on conditional transition system (CTS)

In this section, we will apply Theorem 10 to synthesise language, failure and ready equivalences for CTSs. CTSs are a generalisation of labelled transitions systems (LTSs) aimed at modelling a family of LTSs in a

[5] The Lawvere quantale is given by the poset $([0,1], \geq)$ with the monoidal operation given by truncated addition $\oplus$, i.e. $r \oplus r' = \min(r + r', 1)$.

compact manner; thus, they are suited to formally model a software product line [4].

Definition 22. A *conditional transition system* (CTS) over an alphabet $\mathbb{A}$ and a finite poset $\mathbb{K}$ of conditions is a quadruple $(X, \mathbb{A}, \mathbb{K}, \rightarrow)$, where X is a set of states and $\rightarrow \subseteq X \times \mathbb{A} \times \mathbb{K} \times X$ is the transition relation satisfying the following condition (below we write $x \xrightarrow{a,k} y$ to denote the predicate $(x, a, k, y) \in \rightarrow$):

$$\forall_{x,y \in X, a \in \mathbb{A}, k, k' \in \mathbb{K}} \ (x \xrightarrow{a,k} y \wedge k' \leq k) \implies x \xrightarrow{a,k'} y.$$

The operational intuition behind a CTS with upgrades is as follows. A CTS starts executing its behaviour from a state x and by arbitrarily choosing a condition $k \in \mathbb{K}$. Note that all the transitions that are enabled at x and are guarded by a condition greater than or equal to k are activated, while the remaining transitions remain inactive. Henceforth, the system behaves like a traditional LTS, though at any point in its evolution the system may upgrade to a condition $k' \leq k$. If the set $\mathbb{K}$ is trivially ordered, then we call the system as a CTS without upgrades (originally introduced in [1]).

In the sequel we will fix a set of actions $\mathbb{A}$ that, whenever order is taken into account, is trivially ordered (i.e. by equality). For the systems we model, we additionally assume that the state space, usually denoted by X, is always trivially ordered. We define the behavioural notions for CTS as follows:

Definition 23 (Behavioural Equivalences)**.** Let $(X, \mathbb{A}, \mathbb{K}, \rightarrow)$ be a CTS over $\mathbb{K}$ then we define:

- Assume $\downarrow \subseteq X$ modelling the set of accepting/terminating states, then the k-language of a state $x \in X$:

$$L(x, k) = \{w \in \mathbb{A}^\star \mid \exists_{x'} \ x' \in X \wedge x \overset{w,k}{\twoheadrightarrow} x' \wedge x' \in \downarrow\},$$

  where $\overset{k}{\twoheadrightarrow} \subseteq X \times \mathbb{A}^\star \times X$ is the usual reachability relation on the state space.
- Taking into consideration that upgrades may allow additional steps, we call x and y equivalent for a condition $k \in \mathbb{K}$ iff $\forall k' \leq k : L(k', x) = L(k', y)$. Two states are *conditionally language equivalent*, if they are language equivalent for all $k \in \mathbb{K}$.
- We define the failure pairs, resp. ready pairs, of a state $x \in X$ for a condition $k \in \mathbb{K}$ as:

$$F(x, k) = \{(w, U) \mid \exists_{x' \in X} \ x \overset{w,k}{\twoheadrightarrow} x' \ \wedge \ \forall_{a \in U} \nexists_{x''} \ x' \xrightarrow{a,k} x''\}$$
$$R(x, k) = \{(w, U) \mid \exists_{x' \in X} \ x \overset{w,k}{\twoheadrightarrow} x' \ \wedge \ \forall_{a \in U} \exists_{x'' \in X} \ x' \xrightarrow{a,k} x''\}.$$

- Taking into consideration that upgrades may allow additional steps we call two states $x, y \in X$ failure (resp. ready) equivalent for a condition $k \in \mathbb{K}$ iff $F(x, k') = F(y, k')$ (resp. $R(x, k') = R(y, k')$) for all $k' \leq k$. We call them *conditionally failure equivalent* (resp. *conditionally ready equivalent*), if $F(x, k) = F(y, k)$ for all conditions $k \in \mathbb{K}$.

The three notions of behaviour we want to model coalgebraically can all be modelled by variations of the functor $\widehat{B}$ from Theorem 10 either for the Kleisli category induced by the powerset monad $\mathcal{P}$ on **Set** or the downset monad $\mathcal{P}_\downarrow$ on **Pos**. We can choose the behavioural notion by our choice of the set of observations O and we can vary between systems with and without upgrades by giving $\mathbb{K}$ an order or no order. The following table shows the modelling choices succinctly:

	$\mathbb{K}$ unordered	$\mathbb{K}$ ordered
$O = \{1\}$	$\hat{o} = (k, \bullet)$ if $x \in \downarrow$	$\hat{o} = (k, \bullet)$ if $x \in \downarrow$
$O = \mathcal{P}(\mathbb{A})$ (refusal sets)	$\hat{o} = \{(k, a) \mid \nexists_{x'} \ x \xrightarrow{k,a} x'\}$	undefined
$O = \mathcal{P}(\mathbb{A})$ (ready sets)	$\hat{o} = \{(k, a) \mid \exists_{x'} \ x \xrightarrow{k,a} x'\}$	$\hat{o} = \{(k', a) \mid k' \leq k \wedge \exists_{x'} \ x \xrightarrow{k',a} x'\}$

In particular, for a coalgebraic modelling of a CTS $(X, \mathbb{A}, \mathbb{K}, \rightarrow)$ (with or without upgrades), we consider the following function $\alpha \colon \mathbb{K} \times X \rightarrow T(\mathbb{K} \times \mathbb{A} \times X + O)$ where

$$\alpha(k, x) = \{(k', a, x') \mid x \xrightarrow{k',a} x' \wedge k' \leq k\} \cup \{\hat{o} \in \mathbb{K} \times O \mid \hat{o} \text{ as defined in the above table}\},$$

$T = \mathcal{P}, \mathbf{C} = \mathbf{Set}$ for CTSs without upgrades and $T = \mathcal{P}_{\downarrow}, \mathbf{C} = \mathbf{Pos}$ for CTSs with upgrades. In other words, the coalgebra map models local conditional behaviour, i.e. it models immediate transitions and immediate ready sets from a state at a particular condition. Note that $\alpha(k,x)$ need *not* be a downward closed set if we incorporate refusal sets in the ordered case like the ready sets, i.e. by including the clause $o = \{(k', a) \mid k' \leq k \wedge \nexists_{x'}\ x \xrightarrow{k',a} x'\}$ in the definition of α. This is because refusal sets are order reversing, unlike ready sets which are order preserving. As a result, we are *unable* to capture conditional failure equivalence coinductively for CTSs with upgrades; but we can without upgrades (cf. Theorem 28) since $\alpha(k,x)$ (for arbitrary $k \in \mathbb{K}, x \in X$) is downward closed when $\mathbb{K}$ is trivially ordered.

So the functor G throughout this section is the writer comonad $\mathbb{K} \times \_$. Moreover, $\mathbb{K} \times \_ \dashv \_^{\mathbb{K}}$ both in **Set** and **Pos**, we immediately have that Assumption **A1** is valid since left adjoints preserve colimits. In addition, Assumption **A2** holds because $GAX = \mathbb{K} \times \mathbb{A} \times X \cong \mathbb{A} \times \mathbb{K} \times X = AGX$. The next two subsections are on Assumption **A3**, i.e. a Kleisli lifting $\overline{A}$ exists for the functor $A \colon \mathbf{C} \to \mathbf{C}$ (given by $AX = \mathbb{A} \times X$ for $X \in \mathbf{C}$ and $\mathbf{C} \in \{\mathbf{Set}, \mathbf{Pos}\}$).

CTS without upgrades when $C = \mathbf{Set}$ and $T = \mathcal{P}$

Recall that a Kleisli category $X \to \mathcal{P}Y \in \mathbf{Set}$ is isomorphic to a binary relation $X \times Y \to 2$, where $2 = \{0, 1\}$. Thus, we let $I = \mathrm{Id}$ and $\Phi \colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Pos}$ be the indexed category of (Boolean) predicate, i.e. $\Phi X = \mathbf{Set}(X, 2) \cong \mathcal{P}X$ and f^* is given by the inverse image f^{-1} (for each $f \in \mathbf{Set}$). Moreover, Φ has the bifibration structure since the left adjoint $\exists_f$ (for a function $f \colon X \to Y \in \mathbf{Set}$) is given by the direct image $\exists_f(U) = \{fx \mid x \in U\}$ (for each $U \in \Phi X$). Thus, Assumptions **A3.1** and **A3.2** hold.

Proposition 24. *With the above definitions of I and Φ, Assumption **A3.3** is valid.*

To satisfy Assumption **A3.4**, consider the predicate lifting $\sigma_X \colon \Phi X \to \Phi(AX)$ given by the mapping

$$U \subseteq X \ \mapsto\ \sigma_X(U) = \{(a, x) \mid x \in U \wedge a \in \mathbb{A}\} = \mathbb{A} \times U.$$

Clearly, σ is an indexed morphism because f^* is a functor. Now we can compute the relation lifting $\tilde{\sigma}_{X,Y} \colon \Phi(X \times Y) \to \Phi(AX \times AY)$ to be:

$$\tilde{\sigma}_{X,Y} R = \exists_{\lambda_{X,Y}} \sigma_{X \times Y}(R) = \{\lambda_{X,Y}(a, x, y) \mid x\ R\ y \wedge a \in \mathbb{A}\} = \big\{\big((a, x), (a, y)\big) \mid x\ R\ y \wedge a \in \mathbb{A}\big\}.$$

Proposition 25. *The above relation lifting $\tilde{\sigma}$ preserves identity relations and relational composition.*

And thanks to Theorem 20, Eq. (9) gives a $\mathbf{K}\boldsymbol{\ell}$-law $\vartheta \colon A\mathcal{P} \Rightarrow \mathcal{P}A$. More concretely, on a component $X \in \mathbf{Set}$, it is given as $\vartheta_X = \theta^{-1}_{A\mathcal{P}X, AX} \circ \tilde{\sigma}_{\mathcal{P}X \times X}(\in_X) = \theta^{-1}_{A\mathcal{P}X, AX}\{((a, U), (a, x)) \mid x \in U\}$. Thus,

$$\vartheta_X(a, U) = \{(a, x) \mid x \in U\}.$$

So we obtain Kleisli liftings $\overline{A} \colon \mathbf{K}\boldsymbol{\ell}(\mathcal{P}) \to \mathbf{K}\boldsymbol{\ell}(\mathcal{P})$ of A (below $X \in \mathbf{Set}$ and $f \colon X \to \mathcal{P}Y \in \mathbf{Set}$)

$$\overline{A}X = AX = \mathbb{A} \times X \text{ and } \overline{A}f(a, x) = \{(a, y) \mid y \in fx\}.$$

Example 26. In the above paragraph, let $F \colon \mathbf{Set} \to \mathbf{Set}$ be $B = \mathbb{A} \times \_ + O$ (instead of just $\mathbb{A} \times \_$). Now the following predicate lifting $\sigma \colon \Phi X \to \Phi FX$ induces a relation lifting $\tilde{\sigma} \colon \Phi(X \times Y) \to \Phi(FX \times FY)$:

$$U \subseteq X \mapsto \sigma_X U = O \cup \{(a, x) \mid x \in U \wedge a \in \mathbb{A}\}$$
$$R \subseteq X \times Y \mapsto \tilde{\sigma}_X U = \Delta_O \cup \{((a, x), (a, y)) \mid x\ R\ y \wedge a \in \mathbb{A}\}$$

So from Theorem 20 we get a $\mathbf{K}\boldsymbol{\ell}$-law $\vartheta \colon F\mathcal{P} \Rightarrow \mathcal{P}F$, which on a component $X \in \mathbf{Set}$ is $\vartheta_X(a, U) = \{a\} \times U$ and $\vartheta_X(o) = \{o\}$. This induces a Kleisli lifting $\overline{B}$ as $\overline{B}X = BX$ and $\overline{B}f = \vartheta_Y \circ Bf$ (for every $f \in X \to Y \in \mathbf{K}\boldsymbol{\ell}(\mathcal{P})$). The latter coincides with the definition of $\overline{B}f$ given in Eq. (3).

Theorem 9 is now applicable and giving us a Kleisli lifting $\widehat{B} \colon \mathbf{K}\boldsymbol{\ell}(\mathcal{P}^G) \to \mathbf{K}\boldsymbol{\ell}(\mathcal{P}^G)$, which on objects is $\widehat{B}(X) = BX = \mathbb{A} \times X + O$ (for each $X \in \mathbf{Set}$) and on an arrow $f \colon \mathbf{K}\boldsymbol{\ell}(\mathcal{P}^G)(X, Y)$ is defined as follows:

$\widehat{B}(f)(k,o) = \{(k,o)\}$ and using Eq. (8) we get

$$\widehat{B}(f)(k,a,x) = TG\iota_{AY} \circ \widetilde{A}f(k,a,x) = \{(k',(a',y)) \mid (k',y) \in f(k,x)\}$$

The next proposition is a consequence of Theorem 10.

Proposition 27. *The Kleisli lifting $\overline{A}$ is locally continuous and the operation $\_ + f$ (for every fixed arrow $f \in \mathbf{K\ell}(\mathcal{P})$) on Kleisli homsets commutes with ω-directed joins. As a result, the initial algebra $L(\mu_B) = \mathbb{A}^\star \times O$ coincides with the final coalgebra of $\widehat{B}$ in $\mathbf{K\ell}(\mathcal{P}^G)$.*

Theorem 28. *Let $\alpha\colon X \to AX + O \in \mathbf{K\ell}(\mathcal{P}^G)$ be a coalgebra.*

(i) *If $O = 1$ then the unique coalgebra homomorphism is given by the mapping $(k,x) \mapsto \{k\} \times L(x,k)$.*

(ii) *If $O = \mathcal{P}\mathbb{A}$ and α models the ready set (resp. refusal set) of a state, i.e. $(k,o) \in \alpha(k,x)$ iff o is the set of actions enabled (resp. disabled) from the state x at condition k, then the unique coalgebra homomorphism is given by the mapping $(k,x) \mapsto \{k\} \times R(x,k)$ (resp. $(k,x) \mapsto \{k\} \times F(x,k)$).*

CTSs with upgrades when $C = \mathbf{Pos}$ and $T = \mathcal{P}_\downarrow$

We begin by recalling the downset monad $\mathcal{P}_\downarrow$ on the category **Pos** of posets that maps a poset to its downward closed subsets. In particular,

$$\mathcal{P}_\downarrow(X) = \{U \subseteq X \mid U = {\downarrow}U\} \qquad {\downarrow}U = \{x' \mid \exists_x\ x \in U \wedge x' \leq x\}.$$

On an arrow $f\colon X \to Y \in \mathbf{Pos}$, $\mathcal{P}_\downarrow$ maps a downward closed subset $U \subseteq X$ to the downward closed subset ${\downarrow}f(U)$. Moreover, the unit η_X maps a point x to its history ${\downarrow}\{x\}$ and the multiplication μ is given by union. Now a Kleisli arrow $f\colon X \to \mathcal{P}_\downarrow Y \in \mathbf{Pos}$ and the relation $\theta_{X,Y}(f) = \{(x,y) \mid y \in fx\}$. It is not hard to see that the relation is upward closed in X and downward closed in Y (or alternatively upward closed in the dual poset Y^o). Moreover, it is well known that upward closed sets of a poset X are in correspondence with poset maps of type $X \to 2$ where $2 = \{0,1\}$ and is ordered by $0 < 1$.

Due to these considerations, define an indexed category $\Phi\colon \mathbf{Pos}^{\mathrm{op}} \to \mathbf{Pos}$ of upward closed subsets of a poset and $I\colon \mathbf{Pos} \to \mathbf{Pos}$ to be the functor that maps a poset to its dual. Moreover, the reindexing f^* (for an arrow $f \in \mathbf{Pos}$) is given by the inverse image f^{-1} (since inverse image preserves upward closed subsets). Moreover, Φ has the bifibration structure since the left adjoint $\exists_f$ (for $f\colon X \to Y \in \mathbf{Pos}$) is given by the set $\exists_f(U) = \{y \in Y \mid \exists_x\ x \in U \wedge fx \leq y\}$, for each $U \in \Phi X$.

Proposition 29. *With the above definition of Φ, Assumption **A3.1** is valid. Moreover, Assumption **A3.2** is also valid when $F = A$.*

Proposition 30. *With the above definitions of I and Φ, a poset arrow $f\colon X \to \mathcal{P}_\downarrow Y$ is in correspondence with the relation $R \subseteq X \times Y$ satisfying: $\forall_{x,x' \in X, y,y' \in Y}\ (x\ R\ y \wedge x \leq x' \wedge y' \leq y) \implies x'\ R\ y'$. Moreover, Assumption **A3.3** is valid.*

We take the same predicate lifting σ as in the previous case since $\sigma_X(U) = \mathbb{A} \times U$ is upward closed whenever U is upward closed. So, Assumption **A3.4** is also valid. Now we again compute a relation lifting $\tilde{\sigma}_{X,Y}\colon \Phi(X \times IY) \to \Phi(AX \times IAY)$ to be (below $\uparrow$ denotes the upward closure of a subset):

$$\begin{aligned}
\tilde{\sigma}_{X,Y} R &= \exists_{\lambda_{X,IY}} \sigma_{X \times IY} R \\
&= \exists_{\lambda_{X,IY}} \{(a,x,y) \mid x\ R\ y \wedge a \in \mathbb{A}\} \\
&= {\uparrow}\{\lambda_{X,IY}(a,x,y) \mid x\ R\ y \wedge a \in \mathbb{A}\} \\
&= {\uparrow}\Big\{\big((a,x),(a,y)\big) \in AX \times IAY \mid x\ R\ y \wedge a \in \mathbb{A}\Big\} \\
&= \Big\{\big((a,x),(a,y)\big) \in AX \times IAY \mid x\ R\ y \wedge a \in \mathbb{A}\Big\}.
\end{aligned}$$

Proposition 31. *In this setting, the 'identity' relation $\Delta_X \subseteq X \times IX$ (for each $X \in \mathbf{Pos}$) is given by $\{(x,x') \mid x' \leq x \wedge x \in X\}$. The 'relational' composition $\odot$, in this setting, coincides with the usual relational* *composition of binary relations. Moreover, the relation lifting* $\tilde{\sigma}_{X,Y}\colon \Phi(X\times IY) \to \Phi(AX\times IAY)$ *preserves identity relations and relational compositions.*

Moreover, the relation lifting $\tilde{\sigma}$ on a component evaluates like in the case of **Set**, we obtain a $\mathbf{K}\boldsymbol{\ell}$-law $\vartheta\colon A\mathcal{P}_\downarrow \Rightarrow \mathcal{P}_\downarrow A$ given by $\vartheta_X(a,U) = \{(a,x) \mid x \in U\}$, for each $X \in \mathbf{Pos}$. So we get a Kleisli lifting $\overline{A}\colon \mathbf{K}\boldsymbol{\ell}(\mathcal{P}_\downarrow) \to \mathbf{K}\boldsymbol{\ell}(\mathcal{P}_\downarrow)$ which is defined exactly like in the previous case. Nevertheless, the Kleisli lifting $\widehat{B}\colon \mathbf{K}\boldsymbol{\ell}({\mathcal{P}_\downarrow}^G) \to \mathbf{K}\boldsymbol{\ell}({\mathcal{P}_\downarrow}^G)$ (due to Theorem 9) on an arrow $f\colon \mathbf{K}\boldsymbol{\ell}({\mathcal{P}_\downarrow}^G)(X,Y)$ is a bit different and it evaluates as follows: $\widehat{B}(f)(k,o) = \downarrow\{(k,o)\}$ and using Eq. (8) we get

$$\begin{aligned}\widehat{B}(f)(k,a,x) &= TG\iota_{AY} \circ \widetilde{A}f(k,a,x)\\ &= TG\iota_{AY}\{(k',(a,y)) \mid (k',y) \in f(k,x)\}\\ &= \downarrow\Big\{\big(k',(a,y)\big) \mid (k',y) \in f(k,x)\Big\}\\ &= \Big\{\big(k',(a,y)\big) \mid (k',y) \in f(k,x)\Big\}.\end{aligned}$$

Note that $\mathbf{K}\boldsymbol{\ell}(\mathcal{P}_\downarrow)$ is **Cppo**-enriched by taking the pointwise order, i.e. $f \leq g$ iff $\forall_{x\in X}\ fx \subseteq gx$ for any $f,g\colon X \to Y \in \mathbf{Pos}$, and the join is given by the union (since it preserves downward closed subsets).

Proposition 32. *The Kleisli lifting* $\overline{A}$ *is locally continuous and the operation* $\_+f$ *(for every* $f \in \mathbf{K}\boldsymbol{\ell}(\mathcal{P}_\downarrow)$*) on Kleisli homsets commutes with* ω*-directed joins. Moreover, the initial algebra of* B *is* $\mathbb{A}^\star \times O$*. As a result, the initial algebra* $L(\mu_B) = \mathbb{A}^\star \times O$ *coincides with the final coalgebra of* $\widehat{B}$ *in* $\mathbf{K}\boldsymbol{\ell}({\mathcal{P}_\downarrow}^G)$*.*

Theorem 33. *Let* $\alpha\colon X \to AX + O \in \mathbf{K}\boldsymbol{\ell}({\mathcal{P}_\downarrow}^G)$ *be a coalgebra.*

(i) *If* $O = 1$ *then the unique coalgebra homomorphism is given by* $(k,x) \mapsto \bigcup_{k'\leq k}\{k'\} \times L(x,k')$*.*

(ii) *If* $O = \mathcal{P}(\mathbb{A})$ *and* α *models the ready set of a state, i.e.* $(k',o) \in \alpha(k,x)$ *iff* o *is the set of actions enabled from the state* x *at condition* $k' \leq k$*, then the unique coalgebra homomorphism is given by* $(k,x) \mapsto \bigcup_{k'\leq k}\{k'\} \times R(x,k')$*.*

6 Conclusion and future work

In this paper, we developed a coalgebraic framework to describe CTSs with and without upgrades that allowed us to synthesise language, failure and ready equivalences through our main theorem, Theorem 10. The crucial assumption for this main theorem turned out to be the lifting property to the Kleisli category, which is equivalent to defining a $\mathbf{K}\boldsymbol{\ell}$-law. Under certain assumptions, we characterised a $\mathbf{K}\boldsymbol{\ell}$-law internally living in the fiber of an indexed category. We demonstrated how these assumptions can be easily checked in both cases—with and without upgrades.

Based on the development in Section 4, perhaps in the future, it would be worthwhile to investigate the sufficient conditions that guarantee the existence of initial algebra and final coalgebra for the Kleisli lifting $\overline{F}\colon \mathbf{K}\boldsymbol{\ell}(T) \to \mathbf{K}\boldsymbol{\ell}(T)$ constructed via the $\mathbf{K}\boldsymbol{\ell}$-law $\vartheta\colon FT \to TF$. Moreover, if both exist, a question of particular interest will be whether they coincide. A major contribution would be, if one could attack this question by weakening the assumption of a celebrated result to this effect by Freyd [12] for **Cppo**-enriched categories, a set-up we alluded to. To this end, it would be also fruitful to seek conditions when the Kleisli lifting constructed in Eq. (8) is actually locally continuous.

Another fruitful direction would be to develop coinductive characterisation of failure equivalence in the presence of upgrades and consider other monads than the powerset monad in modelling quantitative extensions of CTSs. Such an extension may enrich the transitions with real-time [9], probability [26] or even weights from a semiring modelling some resource usage [25,28]. Probabilistic extensions are among the upmost exciting one—specially so-called "parametric Markov models"—see [3, Def. 3.6], and also [20]—in which one considers the distribution monad and which originated from the formal verification of probabilistic systems.

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