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Two-dimensional Kripke Semantics II: Stability and Completeness[⋆]

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Abstract

We revisit the duality between Kripke and algebraic semantics of intuitionistic and intuitionistic modal logic. We find that there is a certain mismatch between the two semantics, which means that not all algebraic models can be embedded into a Kripke model. This leads to an alternative proposal for a relational semantics, the stable semantics. Instead of an arbitrary partial order, the stable semantics requires a distributive lattice of worlds. We constructively show that the stable semantics is exactly as complete as the algebraic semantics. Categorifying these results leads to a 2-duality between two-dimensional stable semantics and categories of product-preserving presheaves, i.e. models of algebraic theories in the style of Lawvere.

Keywords: intuitionistic logic, modal logic, intuitionistic modal logic, Kripke semantics, algebraic semantics, duality, filters, presheaves, sifted colimits, product-preserving presheaves, Lawvere theories

1 Introduction

In a previous paper I revisited the relationship between the Kripke and algebraic semantics of intuitionistic logic and (intuitionistic) modal logic [35]. Kripke frames (i.e. partial orders) correspond to a certain class of complete Heyting algebras, the *prime algebraic lattices*. This entails a *duality* $\mathbf{Pos}^{\mathsf{op}} \simeq \mathbf{PrAlgLatt}$, which may be refined to 'truth-preserving' morphisms on one side, and implication-preserving on the other.

What is curious is that this duality can be reproduced at the level of categories, which model *proofs*. Replacing a Kripke frame by a category leads to an evident definition of a proof-relevant *two-dimensional Kripke semantics*. This amounts to taking presheaves over the category, yielding a bicartesian closed category, i.e. a model of intuitionistic proofs. The interpretation of formulas is then a direct categorification of Kripke semantics. This is a duality $\mathbf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCat}$ between Cauchy-complete categories (qua two-dimensional Kripke frames) and presheaf categories (qua prime algebraic lattices).

Moreover, this story can be adapted to *intuitionistic modal logic*. There is no widespread agreement on what the latter is. However, in [35] I showed that a relation that is compatible with a partial order, i.e. a *bimodule*, canonically induces two adjoint modalities $\blacklozenge \dashv \Box$ by Kan extension. This provides a canonical proposal as to what an intuitionistic modal logic should be. Its corresponding Kripke semantics is

$$w \vDash \blacklozenge\varphi \overset{\text{def}}{\equiv} \exists v.\ v\,R\,w \text{ and } v \vDash \varphi \qquad\qquad w \vDash \Box\varphi \overset{\text{def}}{\equiv} \forall v.\ w\,R\,v \text{ implies } v \vDash \varphi \tag{1}$$

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While $\Box$ is indeed the expected modality, $\blacklozenge$ uses R in the *opposite* variance to the more common $\Diamond$ modality. Conversely, any adjunction $\blacklozenge \dashv \Box$ on a prime algebraic lattice uniquely corresponds to a bimodule, giving a duality $\mathbf{EBimod}^{\mathsf{op}} \simeq \mathbf{PrAlgLattO}$ between bimodules on a partial order and prime algebraic lattices L equipped with an *operation*, i.e. a meet-preserving $\Box : L \to L$.

The modal picture can also be categorified, by replacing bimodules with *profunctors*. Left Kan extension then induces an adjunction on $[\mathcal{C}, \mathbf{Set}]$. By unfolding the definition of these adjoints we obtain a remarkable proof-relevant version of (1). This amounts to a duality $\mathbf{EProf}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCatO}$ between profunctors on a Cauchy-complete category, and presheaf categories equipped with a continuous $\Box : [\mathcal{C}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$, which automatically has a left adjoint $\blacklozenge$ (by local finite presentability). This is consistent with what we have come to regard in the last few years as the categorical semantics of modal logic, i.e. an adjunction on a bicartesian closed category [14].

Completeness

These dualities also come with theorems relating validity in the Kripke semantics to validity in the induced algebraic and categorical semantics. Consequently, we are able to use them to prove completeness of the algebraic semantics from completeness of the Kripke semantics. Suppose that a formula of intuitionistic logic is valid in all Heyting algebras; it is then valid in all prime algebraic lattices, and hence valid in all Kripke frames. Therefore, if the Kripke semantics is complete, this formula must be provable. As a result, completeness of the Kripke semantics implies completeness of the algebraic semantics.

Surprisingly, the converse implication is also provable. An old construction, whose origins we can trace at least as far as the book by Fitting [23, §1.6], gives a recipe for inducing a Kripke semantics from a general Heyting algebra, by taking all *prime filters*. The resulting structure is richer than an ordinary frame: it is a *descriptive frame* [13, §8.4]. This is part of a duality between Heyting algebras and descriptive frames, which is known as *Esakia duality* [22]. It is then possible to relate validity in the descriptive frame to validity in the Heyting algebra. A categorical version of this construction for coherent toposes has been shown by Joyal: see [45, Theorem 6.3.5]. When simplified, Joyal's result amounts to an embedding of every Heyting algebra into a prime algebraic lattice that preserves all connectives [28, §3.2] [44] [25].

However, the part of this result that relates validity in the descriptive frame to validity in the Heyting algebra requires the *prime filter existence theorem* [17, §10] [32, §I.2.3], which is a weak form of the axiom of choice. Similarly, the result of Joyal quoted above uses highly non-constructive reasoning.

This paper is about trying to avoid this particular reasoning step. Unlike other streams of work [11], this is not due to a predilection for constructivity. Instead, we are looking for a construction that we can *categorify*, so that it applies to models of intuitionistic (modal) proofs as well. This will in turn provide interesting information about the completeness of various classes of models of typed (modal) λ-calculi.

Stable semantics

However, relating Kripke and algebraic semantics appears impossible without using prime filters. In an attempt to overcome this I will introduce a new relational semantics for intuitionistic logic, which I call *stable semantics*. The essence can summarised as replacing upper sets, which play a central rôle in Kripke semantics [35], with *filters*. This inescapably leads to the use of distributive lattices as 'Kripke frames,' as well as a different interpretation of disjunction, which is reminiscent of Beth semantics [9] and Kripke-Joyal semantics [37, §II.9] [43, §VI.6] [12, §6.6]. The attendant duality, which is now between distributive lattices and *spectral locales* (aka coherent frames), is already well-known from Stone duality [32, §II.3.2]. Furthermore, the coherent semantics can be straightforwardly extended to modalities.

The advantage of stable semantics is that we can constructively show an *equi-completeness* result. Every Heyting algebra is a distributive lattice, and so can be used as the 'possible worlds' of a stable semantics. Moreover, every distributive lattice can be interpreted into a complete Heyting algebra—in fact a spectral locale—in a way that preserves all logical structure. Thus, the completeness of the stable semantics directly follows from the completeness of the algebraic semantics.

Two-dimensional stable semantics and algebraic theories

Categorifying the above story engenders a pleasant surprise. The most technically expedient categorification of the filter construction for our purposes is the *sifted colimit completion*. The very same completion plays a decisive rôle in the *algebraic theories* in the style of Lawvere [4]: every category of algebras is a sifted completion of the opposite of its *theory*, which is just a cartesian category (i.e. has finite products).

If we assume that the opposite of a theory is a *distributive category* [15], the results on stable semantics can be directly categorified. This shows that the class of product-preserving presheaves (cf. filters) on that distributive category (cf. stable frame) is a complete model of the typed λ-calculus with sums and an empty type. These results can be readily adapted to the proofs of intuitionistic modal logic.

Thus, the results herein bear a striking kinship with those of categorical algebra. I am not yet certain what the long-term impact of this observation is, but it seems far too compelling to ignore.

Roadmap

In §2 I discuss what it means to regard a Heyting algebra as a set of possible worlds, as well as the technical issues that arise when we try to embed that representation into a prime algebraic lattice. This leads to the introduction of stable semantics in §3, which is proved equi-complete with Heyting algebras. Moreover, the relevant duality is discussed. In §4 I show that the stable semantics can be effortlessly adapted to interpret adjoint modalities. Then, in §5 I categorify them; this requires a recap of the elements of Lawvere's approach to algebraic theories. I give an an equi-completeness proof, and discuss the resultant syntax-semantics duality. Finally, this approach is extended to intuitionistic modal proofs in §6.

2 Heyting algebras as possible worlds

It is a folklore fact that every Kripke frame $(W, \sqsubseteq)$ induces a prime algebraic lattice $[W, 2]$, which consists of its upper sets ordered by inclusion; see e.g. [35, §2]. Looking at this lattice as a Heyting algebra, i.e. an algebraic semantics for intuitionistic logic, we see that every formula φ is interpreted as the set $[\![\varphi]\!] \subseteq W$ of worlds in which it is true. This set is upper because Kripke semantics is monotonic: $w \sqsubseteq v$ can be read as saying that world v has potentially more information than world w. Thus, the passage from w to v may force more formulas to be true, but will not invalidate formulas that were previously known to be true.

It is interesting to consider a Heyting algebra H in the capacity of a Kripke frame itself. The most evident way of doing so is by taking the opposite of its order, yielding the partial order $(H^{\mathsf{op}}, \sqsubseteq)$, where $\sqsubseteq$ is just $\geq$ in H. Thinking of H as a Tarski-Lindenbaum algebra of an intuitionistic theory, we see that

$$\varphi \sqsubseteq \psi \quad \text{iff} \quad \psi \leq \varphi \quad \text{iff} \quad \text{“}\psi \vdash \varphi\text{”}$$

Roughly, each element $\varphi \in H$ can be thought of as a formula that specifies what we currently know. The relation $\varphi \sqsubseteq \psi$ holds just when ψ implies φ, i.e. when ψ potentially contains more information.

The order-embedding $\uparrow : H \to [H^{\mathsf{op}}, 2]$ then takes $\varphi \in H$ to $\{\psi \in H \mid \psi \leq \varphi\}$, i.e. the set of formulas that imply φ. It is well-known that $\uparrow$ preserves finite meets and exponentials, so that

$$\uparrow \top = H \qquad \uparrow(\varphi \wedge \psi) = \uparrow\varphi \wedge \uparrow\psi \qquad \uparrow(\varphi \Rightarrow \psi) = \uparrow\varphi \Rightarrow \uparrow\psi$$

However, $\uparrow$ famously does *not* preserve disjunction: sometimes $\uparrow(\varphi \vee \psi) \neq \uparrow\varphi \vee \uparrow\psi$. Thus, we can only embed the $(\wedge \to)$ fragment of the logic into a prime algebraic lattice in this manner.

These facts are perhaps better known at the two-dimensional level. Suppose that $\mathcal{C}$ is a bicartesian closed category, i.e. a model of intuitionistic proofs. It is a basic fact of category theory that the Yoneda functor $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is an *embedding*, i.e. full, faithful, and injective on objects. It is also well-known that $\mathbf{y}$ preserves finite products and exponentials [7], i.e. that

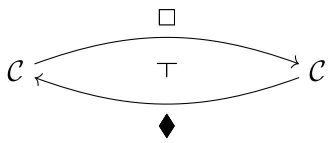
$$\mathbf{y}(\mathbf{1}) \cong \mathbf{1} \qquad \mathbf{y}(c \times d) \cong \mathbf{y}(c) \times \mathbf{y}(d) \qquad \mathbf{y}(c \Rightarrow d) \cong \mathbf{y}(c) \Rightarrow \mathbf{y}(d)$$

For a totally unrelated purpose, Dana Scott [49] noticed that this induces a useful isomorphism:

Lemma 2.1 (Scott) *If φ uses neither disjunction nor falsity then $[\![\varphi]\!]_{[\mathcal{C}^{\mathsf{op}},\mathbf{Set}]} \cong \mathbf{y}([\![\varphi]\!]_{\mathcal{C}})$.*

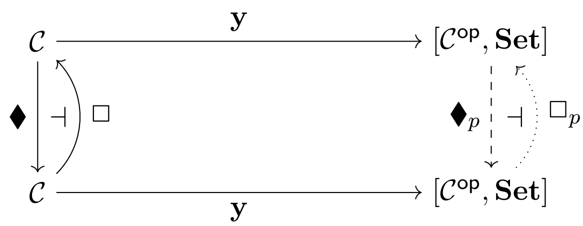
Here $[\![\varphi]\!]_{\mathcal{C}}$ is the interpretation of φ as an object of $\mathcal{C}$, and $[\![\varphi]\!]_{[\mathcal{C}^{\mathsf{op}},\mathbf{Set}]}$ is the interpretation of φ as an object of the category of presheaves $[\mathcal{C}^{\mathsf{op}},\mathbf{Set}]$, both defined following the respective cartesian closed structure. In the second instance basic propositions p are interpreted by the representable $\mathbf{y}([\![p]\!]_{\mathcal{C}})$.

It is not difficult to extend this to the categorical models of modal logic. Following the work of Clouston on Fitch-style λ-calculi [14], these are generally understood to be endo-adjunctions



(2)

on a bicartesian closed category $\mathcal{C}$. Given such a model, take the left Kan extension of $\blacklozenge \circ \mathbf{y}$ along Yoneda:



(3)

$\blacklozenge_p$ is then a colimit-preserving functor on the presheaf category, and has a right adjoint $\Box_p$. Thus, we obtain a categorical model of modal logic on the presheaf category. It is easy to calculate that the action of these adjoint functors on representables is essentially the same as that of $\blacklozenge$ and $\Box$, in that (3) commutes:

$$\begin{aligned}
&\blacklozenge_p(\mathbf{y}(c)) \stackrel{\text{def}}{=} \mathbf{y}(\blacklozenge c)\\
&\Box_p(\mathbf{y}(c)) \stackrel{\text{def}}{=} \mathrm{Hom}_{[\mathcal{C}^{\mathsf{op}},\mathbf{Set}]}\big(\mathbf{y}(\blacklozenge(-)),\mathbf{y}(c)\big) \cong \mathrm{Hom}_{\mathcal{C}}(\blacklozenge(-),c) \cong \mathrm{Hom}_{\mathcal{C}}(-,\Box c) = \mathbf{y}(\Box c)
\end{aligned}$$

Consequently, Scott's lemma directly extends to the categorical semantics of the $(\wedge \to \blacklozenge\Box)$ fragment of intuitionistic modal logic. Notice that the diagram (3) witnesses $\mathbf{y}$ as a (weak) morphism of categorical models of modal logic without disjunction: $\mathbf{y}$ is a cartesian closed functor that preserves the adjunction. Of course, this result can be de-categorified to one for Heyting algebras equipped with an adjunction.

This leaves the mystery of disjunction. One might think that sheaves are the answer. However, we will do something far more radical instead.

3 Stable semantics of intuitionistic logic

Given an arbitrary Kripke frame, i.e. a partial order $(W, \sqsubseteq)$, Kripke semantics interprets every formula as an *upper set* of worlds, i.e. a set $S \subseteq W$ for which $w \in S$ and $w \sqsubseteq v$ implies $v \in S$. The stable semantics will instead revolve around the notion of a *filter* over W.

Definition 3.1 A *filter* over $(W, \sqsubseteq)$ is a non-empty subset $F \subseteq W$ which is

- *upper*, in that $w \in F$ and $w \sqsubseteq v$ implies $v \in F$; and
- *filtered*, in that whenever $w, v \in F$ there exists a $z \in F$ with $z \sqsubseteq w$ and $z \sqsubseteq v$.

We write $\mathsf{Filt}(W)$ for the set of filters over W. $\mathsf{Filt}(W)$ is a poset under inclusion—in fact it is a *directed complete partial order* (dcpo) (without a bottom element) [27, §O-2.8].

When W has more structure the definition of a filter can be somewhat simplified.

Proposition 3.2 *Let $(W, \sqsubseteq)$ be a meet-semilattice. A subset $F \subseteq W$ is a filter if and only if it is*

- upper*, in that $w \in F$ and $w \sqsubseteq v$ implies $v \in F$; and*
- *a* sub-meet-semilattice*, in that $1 \in F$ and $w, v \in F$ implies $w \wedge v \in F$*

A *stable frame* is a partial order $(W, \sqsubseteq)$ which is a *distributive lattice*. This means that it has both finite joins and meets, and that they also satisfy the distributive law $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$. Consequently, they also satisfy the dual law $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ [32, §I.1.5], which we will use heavily.

A stable frame has much more structure than a good old fashioned Kripke frame. To begin, any two worlds $w, v \in W$ have a meet $w \wedge v$ and a join $w \vee v$ (we use the same notation as the logic, but rely on context for disambiguation). If we think of each world as containing information—in particular about which variables have become true—then these two operators tell us that it is possible to find least and greatest upper bounds of information. The fact the distributive law holds means that the interpretation of these bounds as 'intersection of information' and 'union of information' is tenable.

Furthermore, W has a bottom element 0 and a top element 1. The bottom element 0 represents the *baseline level of information*, i.e. the fewest facts we may regard as true. The top element 1 represents a *supernova of information*. This amount of information implies all formulae—even false ones.

A *stable model* $\mathfrak{M} = (W, \sqsubseteq, V)$ consists of a stable frame $(W, \sqsubseteq)$ and a function $V : \mathsf{Var} \to \mathsf{Filt}(W)$. The *valuation* V assigns to each propositional variable $p \in \mathsf{Var}$ a filter $V(p) \subseteq W$, to be thought of as the set of worlds in which p is true. The fact this is a filter leads to the following intuitions:

Upper set A proposition that is true remains true as information increases.

Top element Every proposition is true at the supernova world 1.

Meets If $w, v \in V(p)$ then both w and v contain the information that p is true. Therefore, their greatest lower bound should also contain that information, so that $w \wedge v \in V(p)$.

Notice that if $0 \in V(p)$ then $V(p) = W$, as filters are upper sets. Thus, a variable that is true at the baseline world 0 is true throughout a stable frame.

The *stable semantics* is defined through a relation $\mathfrak{M}, w \vDash \varphi$ with the meaning that φ is true in world w of model $\mathfrak{M}$. When it is clear which model we are using we will skip it, writing simply $w \vDash \varphi$. The clauses for $\mathfrak{M}, w \vDash \varphi$ are much like those for the Kripke semantics, with the characteristic clause for $\to$:

$$\mathfrak{M}, w \vDash \varphi \to \psi \overset{\text{def}}{\equiv} \forall w \sqsubseteq v.\ \mathfrak{M}, v \vDash \varphi \text{ implies } \mathfrak{M}, v \vDash \varphi$$

The only clauses that change are the ones for falsity and disjunction: [2]

$$\begin{aligned} \mathfrak{M}, w \vDash \bot \quad &\overset{\text{def}}{\equiv} (w = 1) \\ \mathfrak{M}, w \vDash \varphi \vee \psi &\overset{\text{def}}{\equiv} \exists v_1, v_2.\ v_1 \wedge v_2 \sqsubseteq w \text{ and } \mathfrak{M}, v_1 \vDash \varphi \text{ and } \mathfrak{M}, v_2 \vDash \psi \end{aligned}$$

There are a number of things to notice about this definition.

First, the falsity $\bot$ can now be a true formula; but it is only true at $1 \in W$, which is top element for the information order $\sqsubseteq$. In fact, every formula is true at 1. In that sense, 1 is a world that contains so much information that it forces everything—even falsity!—to be true. A similar concept of *exploding*, *fallible* or *inconsistent worlds* is common in the context of intuitionistically-provable completeness proofs for intuitionistic logic and associated realizability models [51,20,50,28,40].

Second, the clause for the disjunction $\varphi \vee \psi$ at world w requires that *both* φ and ψ are true at some worlds v_1 and v_2 respectively. However, the common information between v_1 and v_2, i.e. $v_1 \wedge v_2$, must be less than the information at w. When $v_1 \wedge v_2 \sqsubseteq w$ we say that w *fans into* v_1 and v_2. But what if one of the two formulas is a contradiction? This is not a cause for worry, due to the existence of the supernova world: as $w \wedge 1 = w$, we have that $w \vDash \varphi \vee \bot$ if and only if $w \vDash \varphi$.

Third, note that the definition does not mention the joins $w \vee v$ of worlds of W, even though such joins exist. While joins do not appear explicitly in the semantic clauses, they are used in the following lemma to show that the stable semantics is monotonic—in particular in the case of implication. Strictly speaking, we do not need joins: we could just as well reproduce this result even in the weaker setting of a *distributive semilattice*, i.e. a meet-semilattice with the property that $a \wedge b \sqsubseteq x$ implies that $x = a' \wedge b'$ for some $a \sqsubseteq a'$ and $b \sqsubseteq b'$. However, we prefer having joins, as their categorification is well-understood.

[2] One of the reviewers pointed out that the clause for disjunction can be simplified to $w = v_1 \wedge v_2$. However, that version of the clause does not seem immediately amenable to categorification, unlike this one.

Lemma 3.3 (Filtering)

(i) *$\mathfrak{M}, w \vDash \varphi$ and $w \sqsubseteq v$ imply $\mathfrak{M}, v \vDash \varphi$.*

(ii) *$\mathfrak{M}, 1 \vDash \varphi$ for any φ.*

(iii) *$\mathfrak{M}, w_1 \vDash \varphi$ and $\mathfrak{M}, w_2 \vDash \varphi$ imply $\mathfrak{M}, w_1 \wedge w_2 \vDash \varphi$.*

Proof. We prove (iii), and only show the cases for implication and disjunction.

Suppose that $w_1 \vDash \varphi \to \psi$, $w_2 \vDash \varphi \to \psi$, $w_1 \wedge w_2 \sqsubseteq v$ and $v \vDash \varphi$. As $w_i \sqsubseteq w_i \vee v$ and $v \sqsubseteq w_i \vee v$ we know that $w_i \vee v \vDash \varphi \to \psi$ and $w_i \vee v \vDash \varphi$ by (i), and hence that $w_i \vee v \vDash \psi$, for $i \in \{1, 2\}$. Hence, by the IH, $(w_1 \vee v) \wedge (w_2 \vee v) \vDash \psi$. But $v = (w_1 \wedge w_2) \vee v = (w_1 \vee v) \wedge (w_2 \vee v)$ by distributivity.

Suppose that $w_1 \vDash \varphi_1 \vee \varphi_2$ and $w_2 \vDash \varphi_1 \vee \varphi_2$. Then there exist v_{ij} with $v_{i1} \wedge v_{i2} \sqsubseteq w_i$ and $v_{ij} \vDash \varphi_j$. Then $v_{1j} \wedge v_{2j} \vDash \phi_j$, with $(v_{11} \wedge v_{21}) \wedge (v_{21} \wedge v_{22}) = (v_{11} \wedge v_{12}) \wedge (v_{12} \wedge v_{22}) \sqsubseteq w_1 \wedge w_2$. ☐

Both (i) and (iii) of this lemma require the existence of disjunctions; in fact, they make essential use of the *dual* distributive law $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$.

It now remains to show how the stable semantics induce an algebraic semantics. Given a stable frame $(W, \sqsubseteq)$ consider the set $[W, 2]_\wedge$ of monotonic functions $p : W \to 2$ which preserve finite meets. This is a partial order under the pointwise order. This poset has a number of curious properties.

First, the monotonicity of $p : W \to 2$ implies that if $p(w) = 1$ and $w \sqsubseteq v$, then $p(v) = 1$. Hence, the subset $U = p^{-1}(1)$ of W is an upper set. As $p(\top) = 1$, we know that $\top \in U$. Moreover, if $p(w) = 1$ and $p(v) = 1$, then $p(v \wedge w) = p(v) \wedge p(w) = 1$, so U is closed under finite meets. In short, U is a filter. It is not difficult to show that every filter $F \subseteq W$ gives rise to a map $p_F : W \to 2$ which is monotonic and finite-meet-preserving. Consequently, there is an order-bijection $\mathsf{Filt}(W) \cong [W, 2]_\wedge$. I will keep using the somewhat cumbersome notation $[W, 2]_\wedge$ for reasons that will become clear later.

Second, the poset $[W, 2]_\wedge$ is a *complete lattice*, with meets given by intersection [27, §O-1.15, O-2.8]. The bottom element is $\{\top\}$, while the binary join is $F_1 \vee F_2 = \uparrow\{a \wedge b \mid a \in F_1, b \in F_2\}$ [27, §O-1.15]. Infinitary joins $\bigsqcup F_i$ are given by $\uparrow\{a_{n_1} \wedge \ldots \wedge a_{n_j} \mid a_{n_k} \in F_{n_k}\}$. In fact, as W is distributive, infinite joins and finite meets satisfy the *infinite distributive law*, making $[W, 2]_\wedge$ a *locale*, or *complete Heyting algebra* (cHA) [32, §II.2.11]. The exponential is given by $F_1 \Rightarrow F_2 = \{w \in W \mid \forall w \sqsubseteq v.\ v \in F_1 \text{ implies } v \in F_2\}$, which one can readily check is a filter whenever F_1 and F_2 are—as long as W is distributive.

Third, given any $w \in W$ its *principal filter* $\uparrow w$ is $\{v \in W \mid w \sqsubseteq v\} \in [W, 2]_\wedge$. As $w \sqsubseteq v$ iff $\uparrow v \subseteq \uparrow w$, this gives an *order-embedding* $\uparrow : W^{\mathsf{op}} \to [W, 2]_\wedge$. The key to this paper is the following lemma.

Lemma 3.4 *$\uparrow : W^{\mathsf{op}} \to [W, 2]_\wedge$ preserves finite and infinite meets, **finite joins**, and exponentials.*

(The dual of) most of this lemma can be found in [27, §O-2.15]; the rest is elementary—at least if one notices that the domain of $\uparrow$ is the *opposite* of W.

Fourth, the principal upper sets $\uparrow w$ are special, as they are *compact*. Let L be a dcpo. An element $d \in L$ is compact just if $d \sqsubseteq \bigsqcup^{\uparrow} X$ implies that $d \sqsubseteq x$ for some $x \in X$, for any directed set X. Like (completely) prime elements, this says that d contains a small, indivisible fragment of information: as soon as it approximates a *directed* supremum, i.e. a 'recursively defined element,' it must approximate some 'finite unfolding' of it. We write $\mathsf{K}(L)$ for the set of compact elements of L. It is not hard to show that the compact elements of $[W, 2]_\wedge$ are exactly the principal upper sets $\uparrow w$ for some $w \in W$ [27, §I-4.10] [1, Prop. 2.2.2]. Due to the finitary cases of Lemma 3.4 this means that the sub-poset of compact elements $\mathsf{K}([W, 2]_\wedge)$ is in fact a *sub-lattice* of $[W, 2]_\wedge$. This fact will prove very important.

Fifth, the complete lattice $[W, 2]_\wedge$ is *algebraic* [27, §I-4.10] [1, §2.2]. This means that all its elements can be reconstructed as directed suprema of compact ones. In symbols, L is algebraic just if

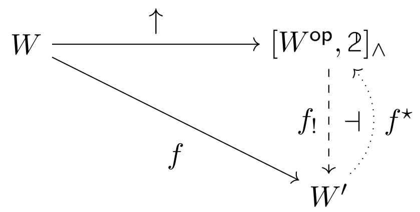
$$\text{for any } d \in L,\ \mathsf{K}_d \stackrel{\text{def}}{=} \{c \in \mathsf{K}(L) \mid c \sqsubseteq d\} \text{ is directed and } d = \bigsqcup\nolimits^{\uparrow} \mathsf{K}_d$$

In summary, if W is distributive, $[W, 2]_\wedge$ is a frame which is (i) algebraic, and (ii) whose compact elements form a sub-lattice. Such lattices are referred to as *spectral locales* (or *coherent frames*), and play an important rôle in Stone duality. In fact, every such locale arises as the partial order of filters of a distributive lattice [32, §II.3.2]:

Theorem 3.5 *A frame is coherent iff it is isomorphic to $[W, 2]_\wedge$ for a distributive lattice W.*

This theorem says that a coherent frame L is isomorphic to the filters $\mathsf{Filt}(\mathsf{K}(L))$ of its compact elements.

Finally, the fact that every element can be reconstructed as a supremum of compact elements means that it is possible to canonically extend any monotonic $f : W \to W'$ that *preserves finite joins* to a monotonic, join-preserving $[W^{\mathsf{op}}, 2] \to W'$, as long as W' is a complete lattice. Diagrammatically, in the situation



(4)

there exists a unique $f_!$ which preserves all joins and satisfies $f_!(\uparrow w) = f(w)$. It is given by

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup \{ f(w) \mid w \in S \}$$

We call $f_!$ the *Scott-continuous extension* of f along $\uparrow$. This follows from a much more general theorem: if W is merely a poset and W' a dcpo, then $[W^{\mathsf{op}}, 2]_\wedge$ is a dcpo, and there is a unique Scott-continuous $f_!$ that makes (4) commute [1, Prop. 2.2.24]. However, if W already has finite joins, then $[W^{\mathsf{op}}, 2]$ is a complete lattice. The reason is every join can be written as a directed supremum of non-empty finite ones. Then, if f preserves finite joins, $f_!$ preserves all of them. As $[W^{\mathsf{op}}, 2]_\wedge$ is complete, it has a right adjoint $f^\star$, by the adjoint functor theorem [17, §7.34] [32, §I.4.2].

Suppose then that we start with a stable model $(W, \sqsubseteq, V)$. By taking its filters we then obtain a spectral locale $[W, 2]_\wedge$. Defining $[\![p]\!] = V(p)$ we obtain a model of intuitionistic logic which interprets every formula φ as a filter $[\![\varphi]\!] \in [W, 2]_\wedge$, namely the filter of worlds in which it is true:

Proposition 3.6 *$w \vDash \varphi$ if and only if $w \in [\![\varphi]\!]$.*

In view of this proposition,

Theorem 3.7 (Soundness) *The stable semantics is sound for intuitionistic logic.*

3.1 Completeness

Revisiting the remarks of §2 we can prove that completeness of the stable semantics implies completeness of the algebraic semantics and vice versa. One direction works exactly as it would for Kripke semantics:

Theorem 3.8 *Completeness of the stable semantics implies completeness of the algebraic semantics.*

Proof. Suppose $[\![\varphi]\!]_H = 1$ for every Heyting algebra H and any interpretation $[\![p]\!]_H \in H$ of the propositions. Hence, given any stable model $(W, \sqsubseteq, V)$ we have that $[\![\varphi]\!]_{[W,2]_\wedge} = 1 = W$ where $[\![p]\!]_{[W,2]_\wedge} = V(p)$. But Prop. 3.6 then implies that $w \vDash \varphi$ for all $w \in W$. By completeness of the stable semantics, $\vdash \varphi$. □

However, the filter construction enables a proof of the other direction as well. We have an embedding

$$\uparrow_{H^{\mathsf{op}}} : H \to [H^{\mathsf{op}}, 2]_\wedge$$

of any Heyting algebra H into the cHA of filters of H^{op} which is a Heyting homomorphism by Lemma 3.4. It is worth pausing for a moment to ponder that a filter on H^{op} is in fact an *ideal* of H, viz. a lower set that is a sub-join-semilattice. Thus, $\uparrow_{H^{\mathsf{op}}}$ sends $x \in H$ to the *principal ideal* $\{y \in H \mid y \leq x\}$ of x in H.

Suppose we have a Heyting algebra H, and some interpretation of propositional variables $[\![p]\!]_H \in H$. Define an interpretation into $[H^{\mathsf{op}}, 2]_\wedge$ starting from $[\![p]\!]_{[H^{\mathsf{op}},2]_\wedge} \stackrel{\text{def}}{=} \uparrow_{H^{\mathsf{op}}}([\![p]\!]_H)$. Then, by Lemma 3.4,

Proposition 3.9 $[\![\varphi]\!]_{[H^{\mathsf{op}},2]_\wedge} = \uparrow_{H^{\mathsf{op}}} [\![\varphi]\!]_H = \{y \in H \mid y \leq [\![\varphi]\!]_H\}$

We are now in a position to prove that

Theorem 3.10 *Completeness of the algebraic semantics implies completeness of the stable semantics.*

Proof. Suppose φ is valid in every stable model. By completeness of the algebraic semantics, it suffices to show that $[\![\varphi]\!]_H = 1$ for any Heyting algebra H, and any interpretation $[\![p]\!]_H \in H$, as this implies $\vdash \varphi$. Consider then the stable model $(H^{\mathsf{op}}, \sqsubseteq, V)$ where $V(p) = \uparrow_{H^{\mathsf{op}}}([\![p]\!]_H)$. As φ is valid in this model, $x \vDash \varphi$ for every $x \in H$. By Proposition 3.6 and Proposition 3.9 we get that $H = [\![\varphi]\!]_{[H^{\mathsf{op}},2]_\wedge} = \{y \in H \mid y \leq [\![\varphi]\!]_H\}$.□

3.2 Morphisms

We briefly consider what it means to have a morphism $f : W \to W'$ of stable frames. We would like such morphisms to induce a map $f^* : [W', 2]_\wedge \to [W, 2]_\wedge$ by $f^*(F) = \{v \in W' \mid f(v) \in F\}$. To conclude that $f^*(F)$ is a filter we need to know that f is monotonic, and that it preserves finite meets. Such maps warrant their own name, which we borrow from the literature on stable domain theory [8]:

Definition 3.11 A monotonic map $f : W \to W'$ is *stable* just if it preserves finite meets. We define **Stable** to be the category of distributive lattices and stable maps.

Unlike the category **DLatt** of distributive lattices, the morphisms of **Stable** need not preserve disjunctions.

It is straightforward to show that when f preserves finite meets, f^* is Scott-continuous and preserves arbitrary meets. This defines a functor $[-, 2]_\wedge : \mathbf{Stable}^{\mathsf{op}} \longrightarrow \mathbf{Coh}$, where **Coh** is the category of coherent frames and Scott-continuous, meet-preserving morphisms. Note that this is *not* the usual category that is used in Stone duality, whose morphisms are frame maps that preserve compact elements [32, §II.3.3].

It is not difficult to see that $[-, 2]_\wedge$ is an equivalence. On objects this is guaranteed by Theorem 3.5. On morphisms, it suffices to spot that every $f^* : [W', 2]_\wedge \to [W, 2]_\wedge$ preserves meets, and hence has a left adjoint $f_!$ by the adjoint functor theorem. It is then simple to show that left adjoints preserve compact elements, so that f can be extracted by restricting $f_!$ to $\mathsf{K}([W, 2]_\wedge) \cong W$. This leads to a duality

$$\mathbf{Stable}^{\mathsf{op}} \simeq \mathbf{Coh} \tag{5}$$

Weaker versions of this duality are well-known, see e.g. [27, §IV-1.16] for a duality between meet-semilattices and algebraic lattices, as well as references to it in the literature.

However, stable morphisms do not preserve truth. For that, we need to refine the above duality to maps that are stable, open, surjective, and also *L-morphisms* in the sense of Bezhanishvili et al. [10, §2], which appropriately preserve disjunction. The details are similar to those in [35].

4 Stable semantics of modal logic

In [35] I argued that a canonical Kripke semantics for intuitionistic modal logic is given by a *bimodule*, i.e. a monotonic function $R : W^{\mathsf{op}} \times W \to 2$ over a Kripke frame $(W, \sqsubseteq)$. In this section we adapt this to the case where $(W, \sqsubseteq)$ is a stable frame.

Definition 4.1 A *stable bimodule on* W is a bimodule $R : W^{\mathsf{op}} \times W \to 2$ that additionally satisfies the following *stability conditions*:

(i) $w \, R \, v_1$ and $w \, R \, v_2 \Longrightarrow w \, R \, (v_1 \wedge v_2)$

(ii) $w \, R \, 1$

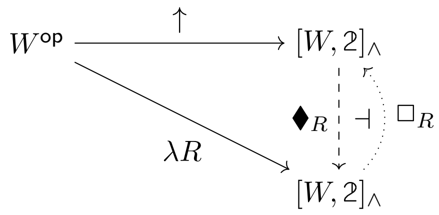
(iii) $(w_1 \wedge w_2) \, R \, v \Longrightarrow \exists v_1, v_2.\ v_1 \wedge v_2 \sqsubseteq v$ and $w_1 \, R \, v_1, w_2 \, R \, v_2$

(iv) $1 \, R \, v \iff v = 1$

A bimodule is a relation $R \subseteq W \times W$ with the property that $w' \sqsubseteq w \, R \, v \sqsubseteq v'$ implies $w' \, R \, v'$. This automatically implies the converses of (i) and (iii). Condition (ii) is redundant, as it is implied by (iv) and the bimodule conditions, but we keep it for symmetry. A *modal stable frame* $(W, \sqsubseteq, R)$ comprises a stable frame $(W, \sqsubseteq)$ and a stable bimodule R.

Stability conditions (i) and (ii) ensure that abstracting the second variable yields a monotonic map $\Lambda R : W^{\mathsf{op}} \to [W, 2]_\wedge$. Moreover, stability conditions (iii) and (iv) ensure that ΛR preserves finite joins.

Then λR induces the following adjunction by Scott-continuous extension:



(6)

Like in the Kripke case [35] it can be shown that these maps are given by

$$\blacklozenge_R(F) \stackrel{\text{def}}{=} \{w \in W \mid \exists v.\ v\, R\, w \text{ and } v \in F\} \qquad \Box_R(F) \stackrel{\text{def}}{=} \{w \in W \mid \forall v.\ w\, R\, v \text{ implies } v \in F\}$$

It is easy to show that both $\blacklozenge_R(F)$ and $\Box_R(F)$ are filters whenever F is; the proof of the first uses stability conditions (i) and (iv), and the second uses stability conditions (iii) and (iv).

This directly leads to the following clauses of a stable semantics of the two modalities:

$$\mathfrak{M}, w \vDash \blacklozenge\varphi \stackrel{\text{def}}{\equiv} \exists v.\ v\, R\, w \text{ and } \mathfrak{M}, v \vDash \varphi \qquad \mathfrak{M}, w \vDash \Box\varphi \stackrel{\text{def}}{\equiv} \forall v.\ w\, R\, v \text{ implies } \mathfrak{M}, v \vDash \varphi$$

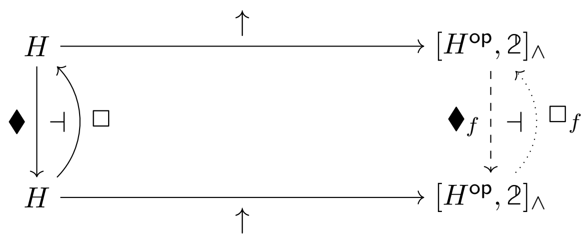
to which Proposition 3.6 readily extends. I have neglected to mention what a *modal stable model* $\mathfrak{M} = (W, \sqsubseteq, R, V)$ is: $(W, \sqsubseteq, R)$ is a modal stable frame, and the valuation V maps propositions into filters.

4.1 Completeness

In [35] I argued that applying Kan extension to a bimodule inescapably leads us to an intuitionistic modal logic with two adjoint modalities, $\blacklozenge$ and $\Box$, as studied by Dzik et al. [21]. The two clauses of the stable semantics are identical to the Kripke semantics in *loc. cit.* But is the logic the same? To answer that we have to reach for its algebraic models, which are Heyting algebras H equipped with two operators $\blacklozenge, \Box : H \to H$ that form an adjunction $\blacklozenge \dashv \Box$. We have just seen that stable bimodules on W correspond precisely to such adjunctions on the cHA $[W, 2]_\wedge$. As Proposition 3.6 remains true if we include $\blacklozenge$ and $\Box$, we have that the stable semantics is sound for the logic of Dzik et al. Furthermore,

Theorem 4.2 *Completeness of the modal stable semantics implies completeness of the modal algebraic semantics.*

The proof is the same as that of Theorem 3.8. For the other direction we have to combine our work from intuitionistic logic, and the argument from §2. Given a Heyting algebra H and an adjunction on it, the map $\uparrow \circ \blacklozenge$ preserves finite joins, as both $\uparrow$ and $\blacklozenge$ do. Take its Scott-continuous extension, $\blacklozenge_f$:



(7)

The map $\uparrow \circ \blacklozenge$ corresponds to a stable $R_\blacklozenge : H \times H^{\mathsf{op}} \to 2$, which maps (x, y) to 1 iff $y \leq \blacklozenge x$ in H. This is by definition stable, but in any case that is easy to verify manually—as long as one is careful about variance. For example, to prove (iii), we need to show that whenever $y \leq \blacklozenge x_1 \vee \blacklozenge x_2$ there exist y_1, y_2 with $y \leq y_1 \vee y_2$ and $y_1 \leq \blacklozenge x_1$ and $y_2 \leq \blacklozenge x_2$. It suffices to take $y_i = y \wedge \blacklozenge x_i$ and use distributivity.

Diagram (7) commutes. For $\blacklozenge$ we have that $\uparrow \circ \blacklozenge = \blacklozenge_f \circ \uparrow$ by definition. For the $\Box$ we have

$$\Box_f(\uparrow z) = \{x \in H \mid \forall y.\, y \leq \blacklozenge x \text{ implies } y \leq z\} = \{x \in H \mid \blacklozenge x \leq z\} = \{x \in H \mid x \leq \Box z\} = \uparrow \Box z$$

Proposition 3.9 extends to the modal case. We therefore have

Theorem 4.3 *Completeness of the modal algebraic semantics implies completeness of the modal stable semantics.*

The proof is also like that of Theorem 3.10. Thus, the modal stable semantics is sound and complete for the intuitionistic modal logic of Dzik et al. [21].

4.2 Morphisms

Like in [35], the duality (5) of §3.2 can be restricted to a duality

$$\mathbf{SBimod}^{\mathrm{op}} \simeq \mathbf{CohO}$$

The category on the left has distributive lattices equipped with a stable bimodule as objects; and stable morphisms that preserve the bimodule as morphisms. The category on the right has coherent frames L equipped with a meet-preserving operation $\Box_L : L \to L$ as objects; and Scott-continuous, meet-preserving maps $h : L \to L'$ for which $h\Box_L \sqsubseteq \Box_{L'}h$. In analogy with previous results this can be further refined to a duality where the morphisms preserve truth on the left, and the operator and implication on the right.

5 Two-dimensional stable semantics of intuitionistic logic

Following the programme of [35], we look for categorifications of the stable semantics. Thus, we exchange stable frames $(W, \sqsubseteq)$ for arbitrary categories $\mathcal{C}$ with finite products; we could call these *stable categories.* The first thing we must categorify is the notion of *filter.* Surprisingly, there are two possible choices:

(i) the *Ind-completion* $\mathsf{Ind}(\mathcal{C})$, which adds all *filtered colimits* to $\mathcal{C}$ [32, §VI.1] [4, §4.17]; and

(ii) the *Sind-completion* $\mathsf{Sind}(\mathcal{C})$, which adds all *sifted colimits* to $\mathcal{C}$ [3,5,4].

All filtered colimits are sifted, so the latter involves adding strictly more colimits to $\mathcal{C}$. However, for a poset W we have that $\mathsf{Ind}(W^{\mathsf{op}}) \cong \mathsf{Sind}(W^{\mathsf{op}}) \cong \mathsf{Filt}(W)$ [3, §2.3]. Thus, these two completions are *indistinguishable* at the order-theoretic level. As an aside, note that the former completion is related to *essentially algebraic theories* [2], while the latter to simpler *algebraic theories* of Lawvere [4].

We will work with the sifted completion, for several reasons. The most important one is that, when $\mathcal{C}$ has finite coproducts, $\mathsf{Sind}(\mathcal{C})$ is cocomplete. This is just enough to allow us to embed any bicartesian closed category $\mathcal{C}$ (which has coproducts) into a cocomplete category $\mathsf{Sind}(\mathcal{C})$, adapting the story of §3. The cocompleteness is absolutely essential in the semantics of modalities given in §6. Second, the conditions required on $\mathcal{C}$ for $\mathsf{Sind}(\mathcal{C})$ to be a cartesian closed category—and hence a model of intuitionistic proofs—are rather weak. Third, there is an analogy to working with filters as elements of $[W, \mathbb{2}]_\wedge$: the classic Lawverean move of replacing $\mathbb{2}$ by $\mathbf{Set}$ [39] leads us to consider product-preserving presheaves $[\mathcal{C}, \mathbf{Set}]_\times$, which coincide with $\mathsf{Sind}(\mathcal{C}^{\mathsf{op}})$ whenever $\mathcal{C}$ has products, mirroring Proposition 3.2.

The following proposition collects various well-known facts about the sifted completion [3,4]. These are analogous to well-known facts about presheaf categories [35], and mirror those of $\mathsf{Filt}(W)$ given in §3.

Proposition 5.1 *Let* $\mathsf{Sind}(\mathcal{C})$ *be the sifted completion of* $\mathcal{C}$.

(i) *If* $\mathcal{C}$ *has coproducts then* $\mathsf{Sind}(\mathcal{C})$ *is isomorphic to the category* $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ *of* product-preserving presheaves *and natural transformations. It is a complete and cocomplete category.*

For the rest of this proposition we assume that $\mathcal{C}$ *has coproducts.*

(ii) *Representable presheaves are product-preserving, so* $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ *is an embedding.*

(iii) $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ *preserves products and* ***coproducts****.*

(iv) $\mathbf{y}(c)$ *is* perfectly presentable, *i.e.* $\mathrm{Hom}(\mathbf{y}(c), -) : [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times \to \mathbf{Set}$ *preserves sifted colimits.*

(v) *A category is equivalent to* $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ *for some* $\mathcal{C}$ *if and only if it is cocomplete and has a strong generator consisting of perfectly presentable objects. Moreover, there is a unique idempotent-complete category* $\mathcal{C}$ *for which this is true (up to equivalence): the subcategory of perfectly presentable objects.*

Proof. (i) is shown in [3, §2.8] [4, §1.22, 4.5, 4.13]. (ii) is shown in [4, §1.12] and (iii) in [4, §1.13]. (iv) follows from the fact representables are *tiny*, and that $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ is closed under sifted colimits in presheaves [4, §5.5]. (v) is shown in [3] [4, §6.9, 8.12]. ☐

Categories satisfying (v) above are called *algebraic categories* by Lawvere [38]. They are essentially categories of models of algebraic theories: see the textbook by Adamek et al. [4].

One might wonder whether $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ preserves exponentials. It would, were $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ to have them; and it has them exactly when $\mathcal{C}$ is a *distributive category*, i.e. whenever the canonical morphism $(a \times b) + (a \times c) \to a \times (b + c)$ is an isomorphism [15]. The following result is quoted on the nLab.

Proposition 5.2 *Let $\mathcal{C}$ have both products and coproducts. Then, the following are equivalent:*

(i) *$\mathcal{C}$ is distributive.*

(ii) *$P \times \mathbf{y}(a + b) \cong P \times \mathbf{y}(a) + P \times \mathbf{y}(b)$ in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$*

(iii) *$\mathsf{Sind}(\mathcal{C}) \cong [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ is cartesian closed.*

In that case $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ preserves exponentials.

Proof. (i) ⇒ (ii): Write $P \cong \varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c)$ as a colimit of representables. As P is product-preserving, its category of elements $\mathsf{el}\, P$ is sifted [4, §4.2]. Hence, it does not matter if this colimit is in presheaves or product-preserving presheaves, as the latter are closed under sifted colimits within the former [4, §2.5]. Noticing also that $\times$ is the same operation in both $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ and $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ we may calculate

$$\begin{aligned}
P \times \mathbf{y}(a+b) &\cong \left(\varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c)\right) \times \mathbf{y}(a+b) && \text{now in presheaves} \\
&\cong \varinjlim_{(c,x) \in \mathsf{el}\, P} (\mathbf{y}(c) \times \mathbf{y}(a+b)) && \text{as } - \times \mathbf{y}(a+b) \text{ is left adjoint} \\
&\cong \varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c \times (a+b)) && \text{now back in } [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times \\
&\cong \varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}((c \times a) + (c \times b)) && \\
&\cong \varinjlim_{(c,x) \in \mathsf{el}\, P} (\mathbf{y}(c) \times \mathbf{y}(a)) + (\mathbf{y}(c) \times \mathbf{y}(b)) && \text{where } + \text{ is now in } [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times \\
&\cong \left(\varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c) \times \mathbf{y}(a)\right) + \left(\varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c) \times \mathbf{y}(b)\right) && \text{as colimits commute with colimits} \\
&\cong \left(\varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c)\right) \times \mathbf{y}(a) + \left(\varinjlim_{(c,x) \in \mathsf{el}\, P} \mathbf{y}(c)\right) \times \mathbf{y}(b) && \\
&\cong P \times \mathbf{y}(a) + P \times \mathbf{y}(b)
\end{aligned}$$

(ii) ⇒ (iii): We only need to prove that the usual exponential $(P \Rightarrow Q)(c) \stackrel{\text{def}}{=} \mathrm{Hom}_{[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times}(P \times \mathbf{y}(c), Q)$ is a product-preserving presheaf. But this easily follows from the observation that $\mathbf{y}(\mathbf{0}) \cong \mathbf{0}$ and (ii).

(iii) ⇒ (i): Then $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ is a bicartesian closed category, and hence it is distributive. But $\mathbf{y}$ is an embedding that preserves products and coproducts, so the subcategory $\mathcal{C}$ is distributive as well. ☐

Tracing the origins of the result that was just proven appears challenging. The claim (i) ⇒ (iii) appears to be due to Younesse Kaddar [34]. The presentation here simplifies Kaddar's calculation by using (ii). (iii) ⇒ (i) is stated without proof on the nLab, and appears to be due to Sam Staton.

This puts us in a good place to introduce a *two-dimensional stable semantics*. This amounts to replacing the stable frame $(W, \sqsubseteq)$ by a category $\mathcal{C}$ with products and coproducts for which $\mathcal{C}^{\mathsf{op}}$ is distributive. This means that the somewhat unusual isomorphism $a + (b \times c) \cong (a + b) \times (a + c)$ holds in $\mathcal{C}$. This is known to hold in a number of categories of algebras, including distributive lattices and commutative rings [18]. I have yet to grasp the meaning of this for a general Lawvere theory. Perhaps previous work by Johnstone [33] and Garner [26] on when categories of varieties are cartesian closed might give some insight. Finally, note that—unlike distributive lattices—distributive categories are *not self-dual*, so this is all we get.

By Propositions 5.1 and 5.2, the category $[\mathcal{C}, \mathbf{Set}]_\times$ is a bicartesian closed category. The two-dimensional stable semantics is then dictated by the bicartesian closed structure. The results in this section mean that these follow the two-dimensional Kripke semantics given in [35]. Thus, every formula φ

is interpreted as a product-preserving presheaf

$$[\![\varphi]\!] : \mathcal{C} \to_\times \mathbf{Set}$$

Writing $[\![\varphi]\!]_w$ to mean $[\![\varphi]\!](w)$ for any $w \in \mathcal{C}$ and $f \cdot x \in [\![\varphi]\!]_v$ for $[\![\varphi]\!](f)$ and $f : w \to v$, the clauses are the expected proof-relevant categorifications of the stable semantics of §3. Some are the same as in [35]:

$$\begin{aligned} [\![\varphi \wedge \psi]\!]_w &\stackrel{\text{def}}{=} [\![\varphi]\!]_w \times [\![\psi]\!]_w \\ [\![\varphi \to \psi]\!]_w &\stackrel{\text{def}}{=} \{F : (v : \mathcal{C}) \to \mathrm{Hom}_\mathcal{C}(w, v) \to [\![\varphi]\!]_v \to [\![\psi]\!]_v \mid \forall g.\ g \cdot F(v)(f)(x) = F(v')(g \circ f)(g \cdot x)\} \end{aligned}$$

However, the interpretation of $\varphi \vee \psi$ is not immediately evident, as it is a coproduct in $[\mathcal{C}, \mathbf{Set}]_\times$. Adamek et al. [4, §4.5] prove the existence of such coproducts abstractly, by decomposing presheaves as sifted colimits of representables and using the fact $\mathbf{y}(a) + \mathbf{y}(b) = \mathbf{y}(a \times b)$. To enable a direct comparison with the stable semantics of disjunction of §3, we need to describe them in a more direct way.

Theorem 5.3 *Let $\mathcal{C}$ have finite products. Then the coproduct in $[\mathcal{C}, \mathbf{Set}]_\times$ is given by the coend*

$$(P + Q)(c) \stackrel{def}{=} \int^{c_1, c_2 \in \mathcal{C}} \mathrm{Hom}_\mathcal{C}(c_1 \times c_2, c) \times P(c_1) \times Q(c_2)$$

An element of $(P + Q)(c)$ essentially consists of tuples (c_1, c_2, f, x, y) where $f : c_1 \times c_2 \to c$ is a 'decomposition' of c, $x \in P(c_1)$, and $y \in Q(c_2)$. If we think of $\mathcal{C}$ as an algebraic theory, f can be thought of as a term of sort c in terms of two variables of sorts c_1 and c_2; and the elements of $P(c_1)$ and $Q(c_2)$ can be considered as elements of the algebra at sorts c_1 and c_2 respectively.

This is evidently a direct categorification of the stable semantics of disjunction given in §3. However, as this is now a coend, these data have to be appropriately quotiented: for any $g : c_1' \to c_1$, $h : c_2' \to c_2$, $t_1' \in P(c_1')$ and $t_2' \in P(c_2')$, the tuples $(c_1, c_2, f, g \cdot t_1', h \cdot t_2')$ and $(c_1', c_2', f \circ (g \times h), t_1', t_2')$ should be identified. This guarantees that the choice of decomposition is 'minimal.' It is easy to prove that this object has the right universal property. However, a conceptual proof that it is product-preserving eludes me.

Finally, notice that this is essentially the 'free' product of algebras, as expected. It is also clearly a version of the *Day convolution product* on presheaves [41, §6.2]. It is in fact a known result of higher algebra that the Day convolution is the coproduct of commutative algebra objects over a symmetric monoidal ∞-category: see Lurie's book [42, Lemma 3.2.4.7].

5.1 Completeness

We are now able to show completeness results for the categorical semantics of intuitionistic logic, i.e. bicartesian closed categories: if $\mathcal{C}$ is a bicartesian closed category then it is distributive, and $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ is an embedding that preserves the bicartesian closed structure of $\mathcal{C}$. Lemma 2.1 extends to disjunction and falsity on objects, but also to *proofs*. The latter can be represented as terms of the typed λ-calculus with sums and an empty type up to $\beta\eta$ equality. We refer to [16,37] for background on the categorical semantics of the typed λ-calculus.

Lemma 5.4 *Let $\mathcal{C}$ be a bicartesian closed category.*

(i) *There is an isomorphism $\theta_A : [\![A]\!]_{[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times} \cong \mathbf{y}([\![A]\!]_\mathcal{C})$ for any type A of the simply-typed λ-calculus.*

(ii) *If $\Gamma \vdash M : A$ is a term of the typed λ-calculus, then the following diagram commutes:*

$$\begin{array}{ccc} [\![\Gamma]\!]_{[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times} & \xrightarrow{[\![M]\!]} & [\![A]\!]_{[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times} \\ {\scriptstyle i \circ \prod_{(x:A) \in \Gamma} \theta_A} \downarrow & & \downarrow {\scriptstyle \theta_A} \\ \mathbf{y}([\![\Gamma]\!]_\mathcal{C}) & \xrightarrow[\mathbf{y}([\![M]\!])]{} & \mathbf{y}([\![A]\!]_\mathcal{C}) \end{array}$$

where $[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{(x:A)\in\Gamma}[\![A]\!]$*, and* $i : \prod_{(x:A)\in\Gamma}\mathbf{y}([\![A]\!]_{\mathcal{C}}) \xrightarrow{\cong} \mathbf{y}(\prod_{(x:A)\in\Gamma}[\![A]\!]_{\mathcal{C}})$ *arises from the fact* $\mathbf{y}$ *preserves finite products.*

Then, assuming that bicartesian closed categories are complete for the typed λ-calculus:

Theorem 5.5 *The subclass of models consisting of* $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ *over a distributive* $\mathcal{C}$ *is complete for equational theory of the typed* λ*-calculus with sums and an empty type.*

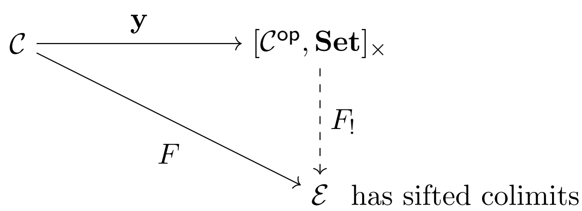
Proof. Let $\Gamma \vdash M, N : A$ be two terms with $[\![M]\!] = [\![N]\!]$ when interpreted in any product-preserving presheaf category $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ with $\mathcal{C}$ distributive. Pick any bicartesian closed $\mathcal{C}$. By Lemma 5.4 we have $\mathbf{y}([\![M]\!]) = \mathbf{y}([\![N]\!])$, where the interpretation is now in $\mathcal{C}$. But $\mathbf{y}$ is faithful, so $[\![M]\!] = [\![N]\!]$ in every bicartesian closed category $\mathcal{C}$. Then $\Gamma \vdash M = N : A$ by the completeness of bicartesian closed categories. □

There is of course a converse, which shows that completeness of this class of models implies completeness of the class of bicartesian closed categories. It is similar in spirit to Theorem 3.8.

5.2 Morphisms

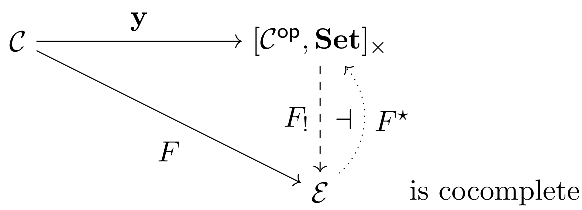
Unlike most the previous dualities we have presented, the one in this section has been carefully studied [6,4]. However, the terminology is different: instead of *stable categories* they speak of *algebraic categories*; and instead of *stable functors*, i.e. functors preserving finite products, they speak of *morphisms of algebraic theories*. In fact, the duality required here is a *syntax-semantics duality* for Lawvere's algebraic theories.

To sketch this duality we must first look at the extension properties of $\mathsf{Sind}(\mathcal{C})$. Given any $F : \mathcal{C} \longrightarrow \mathcal{E}$, where $\mathcal{E}$ is a category with sifted colimits, there is a unique $F_! : [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times \longrightarrow \mathcal{E}$ that extends F and preserves sifted colimits, as in the following commuting diagram:



(8)

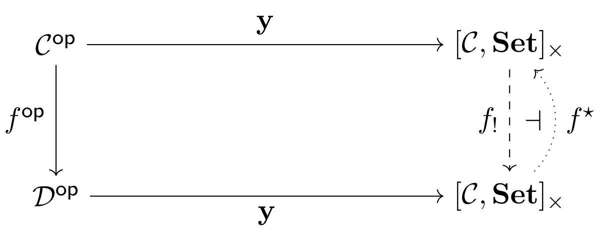
This property is exactly what it means for $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ to be the sifted colimit completion [4, §4.9]. However, we can also get a slightly more refined extension property. Suppose that $F : \mathcal{C} \longrightarrow \mathcal{E}$ also **preserves coproducts**, and that $\mathcal{E}$ is cocomplete. Then $F_!$ preserves **all colimits** and has a right adjoint:



(9)

The reason is that the usual functor $F^\star(e) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{E}}(F-, e)$ is then valued in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$, and can readily be shown to be right adjoint to $F_!$ [4, §4.15].

Then, given any stable $f : \mathcal{C} \longrightarrow \mathcal{D}$ take the extension of $\mathbf{y} \circ f^{\mathsf{op}}$ as in (9)



(10)

We have that $f^\star(P) = \mathrm{Hom}(\mathbf{y}(f(-)), P) \cong P \circ f$ acts by precomposition. It clearly preserves limits; it also preserves sifted colimits, as they are computed pointwise in $[\mathcal{C}, \mathbf{Set}]_\times$ [4, §9.3]. Such a functor is called an *algebraic functor* [4, §9.4]. We thus obtain a (strict) 2-functor

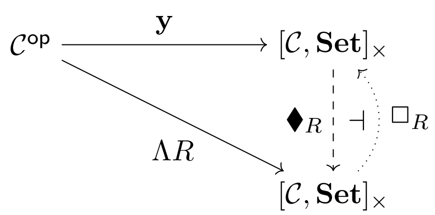
$$[-, \mathbf{Set}]_\times : \mathbf{Cat}^{\mathrm{op}}_{\text{cc, stable}} \longrightarrow \mathbf{AlgCat}$$

from the (strict) 2-category of *Cauchy-complete stable categories*, stable functors, and natural transformations to the (strict) 2-category of *algebraic categories*, algebraic functors, and natural transformations. This functor is a *biequivalence*, and hence a *2-duality*; more details can be found in [4, §9]. Finally, this can be further refined to 2-dualities that 'preserve truth' in terms of frames.

6 Two-dimensional stable semantics for modal logic

Definition 6.1 A *stable profunctor* on $\mathcal{C}$ is a profunctor $R : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$ which preserves products in its second argument, and for which $\Lambda R : \mathcal{C}^{\mathsf{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]_\times$ preserves coproducts.

This corresponds precisely to an adjunction on $[\mathcal{C}, \mathbf{Set}]_\times$, by the universal property (9):



These functors are more directly expressed as follows:

$$[\![\blacklozenge\varphi]\!]_w \stackrel{\text{def}}{=} \blacklozenge_R [\![\varphi]\!]_w = \int^{v \in \mathcal{C}} [\![\varphi]\!]_v \times R(v, w) \qquad [\![\Box\varphi]\!]_w \stackrel{\text{def}}{=} \Box_R [\![\varphi]\!]_w = \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]_\times}(R(w, -), [\![\varphi]\!])$$

The expression for $\Box$ follows from (9); it evidently preserves products. The expression for $\blacklozenge$ is the coend formula for the left Kan extension along Yoneda to *all presheaves* [41, §2.3]. It is still the right expression, by the uniqueness of adjoints. However, it is not easy to see that it preserves products in w: to see that, write $[\![\varphi]\!]$ as a sifted colimit and use the rules of coends to show that this set is isomorphic to $\varinjlim_{(v,x) \in \mathsf{el}[\![\varphi]\!]} R(v, w)$. The latter clearly preserves products in w: $R(v, -)$ does, and the colimit is sifted.

As in [35], these are the expected categorifications of the semantics of $\blacklozenge$ and $\Box$.

6.1 Completeness

We can now show another completeness result like that of §5, which applies to *intuitionistic modal proofs*. These are bicartesian closed categories $\mathcal{C}$ equipped with an adjunction $\blacklozenge \dashv \Box$. They can be represented syntactically by Clouston's *Fitch-style λ-calculus* which is sound and complete for such models [14]. Then $\mathbf{y} : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ is an embedding that preserves all this structure: Scott's lemma 2.1 extends to $\blacklozenge$ and $\Box$, following exactly the proof in §2. Then, a result similar to Lemma 5.4 holds, leading to the

Theorem 6.2 *The subclass of models consisting of categories $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ over a distributive $\mathcal{C}$ equipped with an adjunction $\blacklozenge \dashv \Box$ on $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]_\times$ is complete for equational theory of intuitionistic modal proofs.*

6.2 Morphisms

Following the lead of [35], the 2-duality of §5.2 can be restricted to a 2-duality

$$\mathbf{SProf}^{\mathrm{op}}_{\mathrm{cc}} \simeq \mathbf{AlgCatO}$$

The (strict) 2-category on the left has Cauchy-complete categories equipped with a stable profunctor as 0-cells; stable functors that preserve the profunctor as 1-cells; and natural transformations natural transformations. The (strict) 2-category on the right has algebraic categories $\mathcal{A}$ equipped with an *operation* $\Box_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ that preserves limits and sifted colimits as 0-cells; algebraic functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ equipped with natural transformations $F\Box_{\mathcal{A}} \Rightarrow \Box_{\mathcal{B}}F$ as 1-cells; and natural transformations as 2-cells. This is essentially a direct categorification of the duality of §4.2. It can be further refined to a 2-duality where the morphisms preserve truth on the left, and the operator and implication on the right.

7 Related work

Completions

Much of the development in §3 was based on the *filter completion* of a distributive lattice. The dual notion of *ideal completion* is far more commonly encountered. It plays a significant rôle in domain theory, as the ideal completion of a preorder is the *free algebraic dcpo* over an arbitrary set of compact-elements-to-be [1, §2.2.6]. The category of algebraic dcpos and continuous maps is then equivalent to the category of preorders and *approximable relations*, which appear rather similar to stable bimodules. The ideal completion also plays a central rôle in Stone duality for distributive lattices [32, §II].

Choice-free dualities

It is well-known that many Stone-type dualities require the use of a choice-like principle, e.g. the existence of prime ideals. Choice is sometimes only necessary when connecting the localic viewpoint with the topological one; see e.g. [32, §II.4].

Avoiding this use of choice has been a rather active area of research in recent years, following the work of Bezhanishvili and Holliday [11]. A choice-free duality for Heyting algebras, as well as multiple references to recent literature, is given by Hartonas [29].

Other related work

Bezhanishvili et al. [10] present a positive modal logic. Their semantics uses a meet-semilattice as a frame. Every formula is interpreted as a filter over that, leading to the same falsity and disjunction clauses as the ones I use here. However, the lack of joins and distributivity means that they cannot handle implication. They also present some interesting links between their logic and logics of *independence* and *team semantics* [53,36,54] to which the results of this paper might be applicable.

De Groot and Pattison [19] study the $(\wedge\times)$ fragment of intuitionistic logic with a meet-preserving modality $\Box$. They give it a semantics in semilattices, relating it to filters. Their semantics for $\Box$ is based on relations which are extremely close to stable bimodules.

The coherent semantics appears rather close to the Kripke semantics of the separating conjunction of the BI logic of O'Hearn and Pym [47,48]. This is not surprising, as the Day convolution is one of their main monoidal products. However, the fact that their Kripke semantics can only be shown complete when falsity (interpreted as never true) is excluded [48, §4] suggests that there might be interesting connections with the results presented here.

Galal [24] explores a categorification of the Scott-continuous model of Linear Logic, which also consists of prime algebraic lattices (but with weaker morphisms than the ones used here) [30,31,46,52]. The key notion of directed-completeness is replaced by sifted colimits. No connection to Kripke semantics is made.

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