# Universal recursive preference structures

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# Rationality and preference



In economics, the standard model of rationality endows each agent with a *preference order*: a *complete*, *transitive* binary relation $\succcurlyeq$ over the set of all possible outcomes.

If $x$ and $y$ are two outcomes, then $x \succcurlyeq y$ is usually interpreted

"The agent *prefers* $x$ to $y$."

Given a menu $\mathcal{Y}$ of outcomes, a rational agent chooses an element $x$ of $\mathcal{Y}$ that *maximizes* her preferences —i.e. such that $x \succcurlyeq y$ for all $y \in \mathcal{Y}$.

In some contexts (e.g. uncertainty or multiple time periods), "rational" preferences should take a specific form (e.g. expected utility maximization).

But in "simple" decisions, all possible preference orders are considered equally valid — *de gustibus non es disputandum*.

In such decisions, the standard view is that "rationality" just *is* preference-maximization.

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# The problem of freedom



Arguably, if someone has no choice over her preferences, then she is not truly autonomous.

Like Sisyphus, she is doomed to forever struggle up her preference gradient. But she never had any choice about what her preferences would be.

But is it even coherent to talk about “choosing” one’s own preferences? On what basis would a person make this choice?

Presumably, on the basis of her *second-order* preferences —that is, preferences *over* preferences.

And where do *these* preferences come from?

To be truly autonomous, she must also be able to chose her *second*-order preferences. This presumably involves *third*-order preferences.

At this point, we risk falling into an infinite regress.....

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**Idea.** Formalize these ideas by introducing *higher-order preferences.*

**Notation.** For any set $\mathcal{S}$, let $\mathrm{Prf}(\mathcal{S})$ be the set of all preference orders on $\mathcal{S}$.

Let $\mathcal{X}$ be a menu of possible "outcomes".

A *first order preference* is a preference order $\succeq_1$ on $\mathcal{X}$.

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But a rational agent doesn't form separate preferences over different aspects of her situation; she forms *holistic* preferences over her *entire* situation.

**Example.** Suppose Scott ($\succcurlyeq_S$) prefers scotch to bourbon. Meanwhile, Bob ($\succcurlyeq_B$) prefers bourbon to scotch. Thus

$$Lagavulin \succcurlyeq_S Buffalo\ Trace \quad \text{whereas} \quad Buffalo\ Trace \succcurlyeq_B Lagavulin.$$

But it is also meaningful to ask: Is it better to have preferences $\succcurlyeq_S$ with a tumbler of $Buffalo\ Trace$, or $\succcurlyeq_B$ with a lowball of $Lagavulin$?

Indeed it a complete preference description would be something like:

$$\begin{aligned}(Lagavulin, \succcurlyeq_S) \quad &\succ \quad (Buffalo\ Trace, \succcurlyeq_B) \\ &\succ \quad (Lagavulin, \succcurlyeq_B) \ \succ \ (Buffalo\ Trace, \succcurlyeq_S).\end{aligned}$$

Thus, $\succcurlyeq_2$ should not be a preference over $\mathrm{Prf}(\mathcal{X})$; it should be a preference over ordered pairs $(x, \succcurlyeq_1) \in \mathcal{X} \times \mathrm{Prf}(\mathcal{X})$.

By the same logic, $\succcurlyeq_3$ should be a preference over ordered triples $(x, \succcurlyeq_1, \succcurlyeq_2)$, where $x \in \mathcal{X}$, $\succcurlyeq_1 \in \mathrm{Prf}(\mathcal{X})$, and $\succcurlyeq_2 \in \mathrm{Prf}[\mathcal{X} \times \mathrm{Prf}(\mathcal{X})]$.

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**Problem 3.** We assumed *separate* preferences over $\mathcal{X}$ and $\text{Prf}(\mathcal{X})$.

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**Example.** Suppose Scott ($\succcurlyeq_S$) prefers scotch to bourbon. Meanwhile, Bob ($\succcurlyeq_B$) prefers bourbon to scotch. Thus

$$Lagavulin \succcurlyeq_S Buffalo\ Trace \quad \text{whereas} \quad Buffalo\ Trace \succcurlyeq_B Lagavulin.$$

But it is also meaningful to ask: Is it better to have preferences $\succcurlyeq_S$ with a tumbler of $Buffalo\ Trace$, or $\succcurlyeq_B$ with a lowball of $Lagavulin$?

Indeed it a complete preference description would be something like:

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Problems 2 and 3 can be addressed by using a (much) more complicated "hierarchical" model ...

But this presentation will instead focus on a recursive model.

In this model, the agent can choose from a menu of *types*.

Each type determines a preference order over type-outcome pairs.

In particular, this means each type determines preferences over other types.

Since types determine preferences, this means that each type implicitly determines a *preference over preferences*.

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# Part I

# Recursive preference structures

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As before, let $\mathcal{X}$ be a menu of "outcomes".

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A *simple preference structure* for $\mathcal{T}$ over $\mathcal{X}$ is a function $q : \mathcal{T} \longrightarrow \mathrm{Pf}(\mathcal{X})$.

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This is a common model in economic theory. But not what we want...

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# Example



Suppose that $\mathcal{X} = \{\bigstar, \blacklozenge, \blacktriangle\}$ and $\mathcal{T} = \{■, ■, ■, ■\}$.

Here is a recursive preference structure $p : \mathcal{T} \longrightarrow \mathrm{Pf}(\mathcal{T} \times \mathcal{X})$.



Here is the induced simple preference structure $q : \mathcal{T} \longrightarrow \mathrm{Pf}(\mathcal{X})$.

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Suppose that $\mathcal{X} = \{\bigstar, \blacklozenge, \blacktriangle\}$ and $\mathcal{T} = \{\blacksquare, \blacksquare, \blacksquare, \blacksquare\}$.

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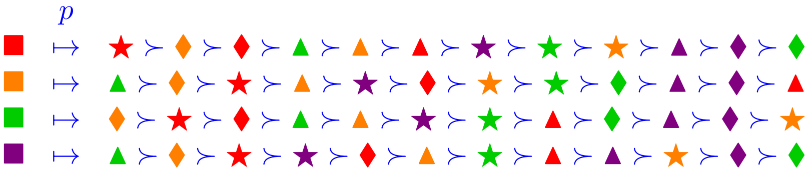

Here is the induced simple preference structure $q : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{X})$.

# Example



Suppose that $\mathcal{X} = \{\bigstar, \blacklozenge, \blacktriangle\}$ and $\mathcal{T} = \{\color{red}\blacksquare, \color{orange}\blacksquare, \color{green}\blacksquare, \color{purple}\blacksquare\}$.

Here is a recursive preference structure $p : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{T} \times \mathcal{X})$.

$$
\begin{array}{lcl}
 & p & \\
\color{red}\blacksquare & \mapsto & \color{red}\bigstar \succ \color{orange}\blacklozenge \succ \color{red}\blacklozenge \succ \color{green}\blacktriangle \succ \color{orange}\blacktriangle \succ \color{red}\blacktriangle \succ \color{purple}\bigstar \succ \color{green}\bigstar \succ \color{orange}\bigstar \succ \color{purple}\blacktriangle \succ \color{purple}\blacklozenge \succ \color{green}\blacklozenge \\
\color{orange}\blacksquare & \mapsto & \color{green}\blacktriangle \succ \color{orange}\blacklozenge \succ \color{red}\bigstar \succ \color{orange}\blacktriangle \succ \color{purple}\bigstar \succ \color{red}\blacklozenge \succ \color{orange}\bigstar \succ \color{green}\bigstar \succ \color{green}\blacklozenge \succ \color{purple}\blacktriangle \succ \color{purple}\blacklozenge \succ \color{red}\blacktriangle \\
\color{green}\blacksquare & \mapsto & \color{orange}\blacklozenge \succ \color{red}\bigstar \succ \color{red}\blacklozenge \succ \color{green}\blacktriangle \succ \color{orange}\blacktriangle \succ \color{purple}\bigstar \succ \color{green}\bigstar \succ \color{red}\blacktriangle \succ \color{green}\blacklozenge \succ \color{purple}\blacktriangle \succ \color{purple}\blacklozenge \succ \color{orange}\bigstar \\
\color{purple}\blacksquare & \mapsto & \color{green}\blacktriangle \succ \color{orange}\blacklozenge \succ \color{red}\bigstar \succ \color{purple}\bigstar \succ \color{red}\blacklozenge \succ \color{orange}\blacktriangle \succ \color{green}\bigstar \succ \color{red}\blacktriangle \succ \color{purple}\blacktriangle \succ \color{orange}\bigstar \succ \color{purple}\blacklozenge \succ \color{green}\blacklozenge
\end{array}
$$

Here is the induced simple preference structure $q : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{X})$.

$$
\begin{array}{lcl}
 & q & \\
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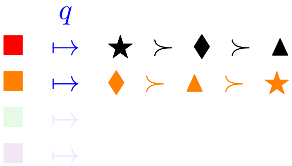

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# Example



Suppose that $\mathcal{X} = \{★, ♦, ▲\}$ and $\mathcal{T} = \{$🟥, 🟧, 🟩, 🟪$\}$.

Here is a recursive preference structure $p : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{T} \times \mathcal{X})$.

| | $p$ | |
|---|---|---|
| 🟥 | $\mapsto$ | ★ ≻ ♦ ≻ ♦ ≻ ▲ ≻ ▲ ≻ ▲ ≻ ★ ≻ ★ ≻ ★ ≻ ▲ ≻ ♦ ≻ ♦ |
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Here is the induced simple preference structure $q : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{X})$.

| | $q$ | |
|---|---|---|
| 🟥 | $\mapsto$ | ★ ≻ ♦ ≻ ▲ |
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# Example

Suppose that $\mathcal{X} = \{\bigstar, \blacklozenge, \blacktriangle\}$ and $\mathcal{T} = \{\textcolor{red}{\blacksquare}, \textcolor{orange}{\blacksquare}, \textcolor{green}{\blacksquare}, \textcolor{purple}{\blacksquare}\}$.

Here is a recursive preference structure $p : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{T} \times \mathcal{X})$.

$$
\begin{array}{ccl}
 & p & \\
\textcolor{red}{\blacksquare} & \mapsto & \textcolor{red}{\bigstar} \succ \textcolor{orange}{\blacklozenge} \succ \textcolor{red}{\blacklozenge} \succ \textcolor{green}{\blacktriangle} \succ \textcolor{orange}{\blacktriangle} \succ \textcolor{red}{\blacktriangle} \succ \textcolor{purple}{\bigstar} \succ \textcolor{green}{\bigstar} \succ \textcolor{orange}{\bigstar} \succ \textcolor{purple}{\blacktriangle} \succ \textcolor{purple}{\blacklozenge} \succ \textcolor{green}{\blacklozenge} \\
\textcolor{orange}{\blacksquare} & \mapsto & \textcolor{green}{\blacktriangle} \succ \textcolor{orange}{\blacklozenge} \succ \textcolor{red}{\bigstar} \succ \textcolor{orange}{\blacktriangle} \succ \textcolor{purple}{\bigstar} \succ \textcolor{red}{\blacklozenge} \succ \textcolor{orange}{\bigstar} \succ \textcolor{green}{\bigstar} \succ \textcolor{green}{\blacklozenge} \succ \textcolor{purple}{\blacktriangle} \succ \textcolor{purple}{\blacklozenge} \succ \textcolor{red}{\blacktriangle} \\
\textcolor{green}{\blacksquare} & \mapsto & \textcolor{orange}{\blacklozenge} \succ \textcolor{red}{\bigstar} \succ \textcolor{red}{\blacklozenge} \succ \textcolor{green}{\blacktriangle} \succ \textcolor{orange}{\blacktriangle} \succ \textcolor{purple}{\bigstar} \succ \textcolor{green}{\bigstar} \succ \textcolor{red}{\blacktriangle} \succ \textcolor{green}{\blacklozenge} \succ \textcolor{purple}{\blacktriangle} \succ \textcolor{purple}{\blacklozenge} \succ \textcolor{orange}{\bigstar} \\
\textcolor{purple}{\blacksquare} & \mapsto & \textcolor{green}{\blacktriangle} \succ \textcolor{orange}{\blacklozenge} \succ \textcolor{red}{\bigstar} \succ \textcolor{purple}{\bigstar} \succ \textcolor{red}{\blacklozenge} \succ \textcolor{orange}{\blacktriangle} \succ \textcolor{green}{\bigstar} \succ \textcolor{red}{\blacktriangle} \succ \textcolor{purple}{\blacktriangle} \succ \textcolor{orange}{\bigstar} \succ \textcolor{purple}{\blacklozenge} \succ \textcolor{green}{\blacklozenge}
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# Fully autonomous recursive preference structures 

Let $p : \mathcal{T} \longrightarrow \mathrm{Prf}(\mathcal{T} \times \mathcal{X})$ be a recursive preference structure.

Let $\succcurlyeq$ be a preference order over $\mathcal{T} \times \mathcal{X}$.

Say that $\succcurlyeq$ is *$p$-realizable* if there is some $t \in \mathcal{T}$ such that $p(t) = (\succcurlyeq)$.

**Problem.** The set of $p$-realizable preference orders could be very small.

In particular, they might all induce the *same* first-order preferences over $\mathcal{X}$.

But then a $p$-agent would not have much "autonomy"...

**Goal:** A recursive preference structure $p$ where the set of $p$-realizable preferences is very *large* —perhaps even includes *every* "feasible" (e.g. continuous) preference order over $\mathcal{T} \times \mathcal{X}$.

In this case, we could say that $p$ is *fully autonomous*.

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# Existence of fully autonomous RPS 

**Answer.** In some cases, yes.

Let $\mathcal{T}$ and $\mathcal{X}$ be metric spaces.

Let $\text{Prf}_c(\mathcal{T} \times \mathcal{X})$ be the set of *continuous* preference orders on $\mathcal{T} \times \mathcal{X}$.

Then $\text{Prf}_c(\mathcal{T} \times \mathcal{X})$ is itself a metric space, with the Hausdorff metric.

**Theorem A.** *For any compact metric space $\mathcal{X}$, there exists a metric space $\mathcal{T}$ with a continuous surjection $p : \mathcal{T} \longrightarrow \text{Prf}_c(\mathcal{T} \times \mathcal{X})$.*

Such an agent is *fully autonomous*:

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# Towards full autonomy



As a recipe for "full autonomy", Theorem A has several shortcomings.

- It only applies when $\mathcal{X}$ is a compact metric space.
- ....yet the type space $\mathcal{T}$ itself is *not* compact.
- The function $p : \mathcal{T} \longrightarrow \mathrm{Prf}_c(\mathcal{T} \times \mathcal{X})$ is surjective, but *not* injective.

  So many types map to the same preference order.
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- An agent should be able to adopt *incomplete* preferences if she wants.

  *Reason:* Some options may be *incommensurable* with others.
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We will now develop a theory of "universal" recursive preference structures.

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# Part II

# Universal recursive preference structures

# Local continuous partial orders



**Recall.** A (strict) partial order is a transitive, antisymmetric binary relation.

Any partial order on a set $\mathcal{X}$ corresponds to a subset of $\mathcal{X} \times \mathcal{X}$.

Now suppose that $\mathcal{X}$ is a topological space.

A partial order is *continuous* if it corresponds to an *open* subset of $\mathcal{X} \times \mathcal{X}$.

**Definition.** A *local continuous partial order* on $\mathcal{X}$ is a pair $(\mathcal{Y}, \succ)$, where:

- $\mathcal{Y} \subseteq \mathcal{X}$ is a closed subset; and
- $\succ$ is a continuous partial order on $\mathcal{Y}$.

Let $P(\mathcal{X}) := \{$all local continuous partial orders on $\mathcal{X}\}$.

**Wanted.** A natural topology on $P(\mathcal{X})$....

For any $(\mathcal{Y}, \succ)$ in $P(\mathcal{X})$, let $[\![\mathcal{Y}, \succ]\!] := \{(x, y) \in \mathcal{Y} \times \mathcal{Y};\ \ x \not\succ y\}$.

This is a closed subset of $\mathcal{Y} \times \mathcal{Y}$ (its complement is open because $\succ$ is continuous).

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(If $\mathcal{X}$ is a compact *metric* space, it is induced by the Hausdorff metric.)

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# Forward images of partial orders



**Wanted.** A notion of *morphism* for recursive preference structures.

This requires further mathematical preliminaries.....

Let $\mathcal{X}$ and $\mathcal{X}'$ be compact Hausdorff spaces.

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# Morphisms of recursive preference structures



Let $(\mathcal{T}_1, \phi_1)$ and $(\mathcal{T}_2, \phi_2)$ be two recursive preference structures over $\mathcal{X}$.

Let $\psi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be another continuous function.

Let $I_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X}$ be the identity function.

Define $\psi \times I_{\mathcal{X}} : \mathcal{T}_1 \times \mathcal{X} \longrightarrow \mathcal{T}_2 \times \mathcal{X}$ in the obvious way.

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Let $\psi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be another continuous function.

Let $I_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X}$ be the identity function.

Define $\psi \times I_{\mathcal{X}} : \mathcal{T}_1 \times \mathcal{X} \longrightarrow \mathcal{T}_2 \times \mathcal{X}$ in the obvious way.

Let $\psi^{\dagger} := (\psi \times I_{\mathcal{X}})^{\P} : P(\mathcal{T}_1 \times \mathcal{X}) \longrightarrow P(\mathcal{T}_2 \times \mathcal{X})$.

**Definition.** The function $\psi$ is a *morphism* of recursive preference structures if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_1 & \xrightarrow{\phi_1} & P(\mathcal{T}_1 \times \mathcal{X}) \\ {\scriptstyle \psi}\big\downarrow & & \big\downarrow{\scriptstyle \psi^{\dagger}} \\ \mathcal{T}_2 & \xrightarrow[\phi_2]{} & P(\mathcal{T}_2 \times \mathcal{X}) \end{array}$$

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# Recursive preference structures are co-algebras 

Let $\mathbf{CHS}$ = category of compact Hausdorff spaces and continuous maps.

Our earlier result says there is a endofunctor $P$ on $\mathbf{CHS}$ that:

- transforms every space $\mathcal{X}$ into the space $P(\mathcal{X})$ of local continuous partial orders on $\mathcal{X}$; and
- transforms every function $\psi : \mathcal{X} \longrightarrow \mathcal{Y}$ into $\psi^{\P} : P(\mathcal{X}) \longrightarrow P(\mathcal{Y})$.

Now fix a compact Hausdorff space $\mathcal{X}$.

Consider the endofunctor $(- \times \mathcal{X})$ on $\mathbf{CHS}$ that:

- transforms every space $\mathcal{T}$ into the product space $(\mathcal{T} \times \mathcal{X})$; and
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Now compose them to get the endofunctor $R_{\mathcal{X}} := P \circ (- \times \mathcal{X})$.

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# Universal recursive preference structures



Let $\mathcal{X}$ be a compact Hausdorff space, and let $(\widehat{\mathcal{T}}, \widehat{\phi})$ be an RPS over $\mathcal{X}$.

**Definition.** $(\widehat{\mathcal{T}}, \widehat{\phi})$ is a *universal* RPS over $\mathcal{X}$ if it is a terminal object in the category of

In other words: for any other $(\mathcal{T}, \phi)$, there is a *unique* morphism $\psi : \mathcal{T} \longrightarrow \widehat{\mathcal{T}}$.

**Proposition.** *Suppose that $(\widehat{\mathcal{T}}, \widehat{\phi})$ is a universal RPS over $\mathcal{X}$. Then:*

(a) (Lambek) *The function $\widehat{\phi} : \widehat{\mathcal{T}} \longrightarrow P(\widehat{\mathcal{T}} \times \mathcal{X})$ is a homeomorphism.*

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Part (a) means $(\widehat{\mathcal{T}}, \widehat{\phi})$ is fully autonomous, and has no redundant types.

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# Existence of Universal RPS



We now come to our main result.

**Theorem.** *For any compact Hausdorff space $\mathcal{X}$, there is a universal recursive preference structure over $\mathcal{X}$.*

*The type space of this universal RPS is a compact Hausdorff space.*

The type space of a universal RPS depends on the outcome space $\mathcal{X}$.

**Proposition.**
*Let $\mathcal{X}$ be a compact Hausdorff space. Let $(\widehat{\mathcal{T}}, \widehat{\phi})$ be its universal RPS.*

- *If $\mathcal{X}$ is metrizable, then so is $\widehat{\mathcal{T}}$.*
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Consider the chain $\mathcal{T}_0 \xleftarrow{\phi_0} \mathcal{T}_1 \xleftarrow{\phi_1} \mathcal{T}_2 \xleftarrow{\phi_2} \mathcal{T}_3 \xleftarrow{\phi_3} \cdots$,
where $\mathcal{T}_0$ is the one-point space (the terminal object of **CHS**),
$\mathcal{T}_1 := R_{\mathcal{X}}(\mathcal{T}_0)$, $\mathcal{T}_2 := R_{\mathcal{X}}(\mathcal{T}_1)$, $\mathcal{T}_3 := R_{\mathcal{X}}(\mathcal{T}_2)$, ...etc.
Meanwhile, $\phi_0 : \mathcal{T}_1 \longrightarrow \mathcal{T}_0$ is the (unique) terminal morphism;
$\phi_1 := R_{\mathcal{X}}(\phi_0) : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$, $\phi_2 := R_{\mathcal{X}}(\phi_1) : \mathcal{T}_3 \longrightarrow \mathcal{T}_2$,.... etc.

Let $\mathcal{T}_\infty := \lim_{n\to\infty} (\mathcal{T}_n, \phi_n)$.

The functor $R_{\mathcal{X}}$ preserves $\omega$-limits in **CHS** (Pivato, 2023). So there is an isomorphism $R_{\mathcal{X}}(\mathcal{T}_\infty) = R_{\mathcal{X}}\left(\lim_{n\to\infty} (\mathcal{T}_n, \phi_n)\right) \cong \lim_{n\to\infty} \left(R_{\mathcal{X}}(\mathcal{T}_n), R_{\mathcal{X}}(\phi_n)\right)$.

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To formalize this, I introduced recursive preference structures (RPS).

In a topological setting, there exist *fully autonomous* (i.e. surjective) RPS.

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