A model of linear logic using polynomials in Homotopy Type Theory

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Overview

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- Our contribution : polynomial functors in types form a model of linear logic.

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- Our contribution : polynomial functors in types form a model of linear logic.
- Linear logic : "$(A \implies B) \simeq (!A \multimap B)$"

Polynomials in sets

A polynomial in sets is an expression

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where B is a set and $(E_b)_{b \in B}$ a family of sets indexed by B.

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No coefficients, but repetitions allowed :

$$3 \times X^{\mathbb{2}} = X^{\mathbb{2}} \sqcup X^{\mathbb{2}} \sqcup X^{\mathbb{2}}$$

Polynomials in sets

$$(B, (E_b)_{b \in B}) \iff (E \to B)$$

so the data of a polynomial P can be encoded as a map of sets

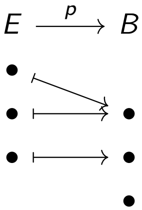
$$E \xrightarrow{p} B$$

- $b \in B$ represents a monomial
- $E_b := p^{-1}(b)$ is the exponent of the monomial b

It induces a **polynomial functor**

$$\mathbf{Set} \to \mathbf{Set}$$
$$X \mapsto \sum_{b \in B} X^{E_b}$$

Example

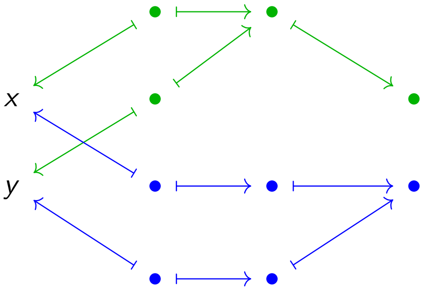


Induced functor:

$$\mathbf{Set} \to \mathbf{Set}$$
$$X \mapsto X^2 \sqcup X \sqcup \mathbb{1}$$

Polynomials with sorts/colors

$$\{x, y\} \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} \{0, 1\}$$



$$\mathbf{Set}^{\{x,y\}} \to \mathbf{Set}^{\{0,1\}}$$

$$(X, Y) \mapsto (XY, X + Y)$$

Composition of polynomials

Generally, a diagram of sets

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- Composition can already be defined at the level of diagrams, but it's only associative/unital up to isomorphism.
- Polynomials in sets thus form a **(2,1)-category**.

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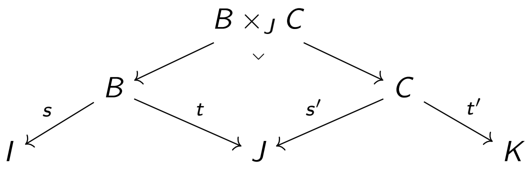
And they compose via pullbacks



A model of linear logic ?

Spans are linear polynomials. To get a model of LL, we would like

$$(A \implies B) \simeq (!A \multimap B)$$
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A polynomial functor $P : \mathbf{Set} \to \mathbf{Set}$ is determined by $B : \mathbf{Set}$, $E_{(-)} : B \to \mathbf{Set}$, so by a span

$$\mathbf{Set} \xleftarrow[E_{(-)}]{} B \longrightarrow \mathbb{1}$$

- This suggests $!(\mathbb{1}) \simeq \mathbf{Set}$, but there is not set of all sets.
- So we restrict the size of the sets E_b.

Finitary polynomials

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Examples:

- $X \mapsto X^3 + X + 1$ is finitary
- $X \mapsto \mathbb{N} \times X$ is finitary
- $(X_k)_{k\in\mathbb{N}} \mapsto ((X_k)^k)_{k\in\mathbb{N}}$ is finitary
- $X \mapsto X^{\mathbb{N}}$ is **not** finitary
- a linear polynomial is always finitary

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A finite set E is characterized by its cardinality $\#E \in \mathbb{N}$. Do we now have

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- Need to replace $\mathbb{N}$ by the groupoid Fin of finite sets and bijections.

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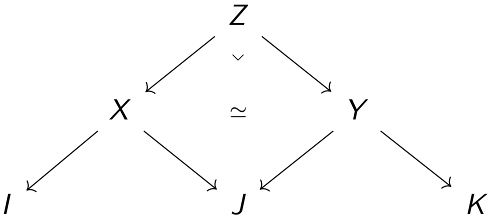
Working homotopically is essential : composition of spans of groupoids needs to be done by **homotopy pullback**.



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  - fibers are homotopy fibers, pullbacks are homotopy pullbacks. . .

Spans and polynomials in HoTT

- Polynomials and spans in HoTT : same as before, but replace "set" with "type".
- A polynomial is **finitary** if the E_b's are finite **sets** (seen as types).
- Write Fin for the type of finite sets and bijections.

$$\mathbf{Poly}_{\mathrm{Fin}}(\mathbb{1}, \mathbb{1}) \simeq \mathbf{Span}(\mathrm{Fin}, \mathbb{1})$$

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More generally, define $!I :=$ finite sets colored by I, i.e.

$$!I \simeq \sum_{E:\mathrm{Fin}} (E \to I)$$

Theorem

$$\mathbf{Poly}_{\mathrm{Fin}}(I, J) \simeq \mathbf{Span}(!I, J)$$

Idea of proof

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

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Remark

Not all coherences can be stated in HoTT, but some can be proven meta-theoretically. For instance, we can prove a wild category has cartesian products, and know meta-theoretically that the induced monoidal structure is homotopy coherent.

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$(\mathbf{Span}, !)$ has a compatible symmetric monoidal structure, making it a **Seely category** (model of intuitionistic linear logic).
It is moreover compact closed, hence $\star$-autonomous, i.e. a model of full classical linear logic.

A full model of classical linear logic

Proposition

! defines a comonad on the wild category of spans of types.

Theorem

The Kleisli category $\mathbf{Span}_!$ is equivalent to $\mathbf{Poly}_{\mathrm{Fin}}$.

Theorem

$(\mathbf{Span}, !)$ has a compatible symmetric monoidal structure, making it a **Seely category** (model of intuitionistic linear logic).
It is moreover compact closed, hence $\star$-autonomous, i.e. a model of full classical linear logic.

Corollary

Since $(\mathbf{Span}, !)$ is monoidal closed, $\mathbf{Poly}_{\mathrm{Fin}}$ is cartesian closed.

Variations on size

We had to restrict to finitary polynomials.

Actually, any suitable notion of “small” gives **smallary** polynomials, with results analogous to everything before.

Given a universe of “big types” $\mathcal{U}$ and $\mathcal{V} : \mathcal{U}$ a universe of “small types” closed under suitable constructions :

- a polynomial is $\mathcal{V}$-ary if the E_b's are equivalent to types in $\mathcal{V}$
- define $!_{\mathcal{V}} I := \sum_{E:\mathcal{V}} (E \to I)$

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Theorem

$$\mathbf{Poly}_{\mathcal{V}}(I, J) \simeq \mathbf{Span}(!_{\mathcal{V}} I, J)$$

and we get a model of classical linear logic.

- $\mathcal{V} = \mathrm{Fin}$, $\mathcal{U} =$ groupoids
- $\mathcal{V} = \mathbf{Set}$, $\mathcal{U} =$ large groupoids
- $\mathcal{V} =$ groupoids, $\mathcal{U} =$ all types

Examples of higher polynomials - 1

$$\mathrm{Fin}_* \xrightarrow{p} \mathrm{Fin}$$

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- Induced polynomial : $F(X) = \sum_{E:\mathrm{Fin}} X^E$
- This exactly the definition $!_{\mathrm{Fin}}$: the comonad is itself a polynomial functor.

Examples of higher polynomials - 2

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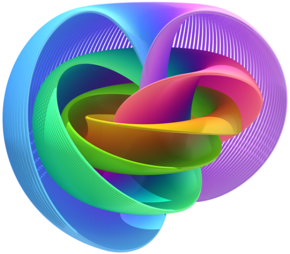
- $p^{-1}((n,\star)) \simeq \mathbb{Z}/n\mathbb{Z}$
- Induced polynomial : $F(X) = \sum_{n:\mathbb{N}} X^n /\!\!/ (\mathbb{Z}/n\mathbb{Z})$
- The type of cyclic lists over X
- Generally, summing over groupoids amounts to quotienting the summand

Examples of higher polynomials - The Hopf Fibration



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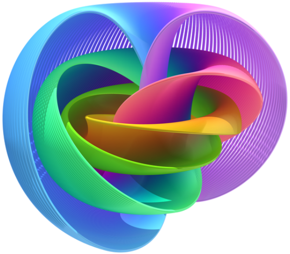

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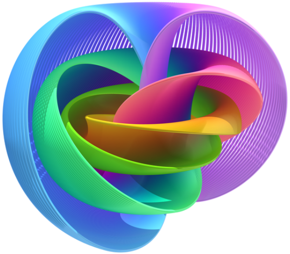

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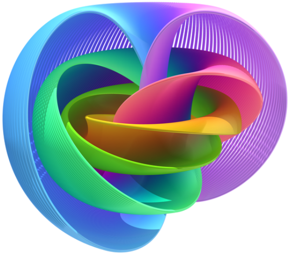

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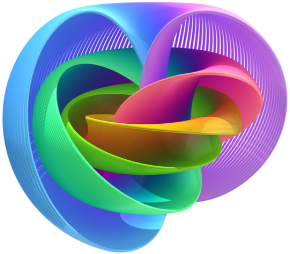

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- This locally looks like $S^2 \times X^{S^1}$, but in a globally twisted way.
- If you have any idea what this represents, please reach out !

Conclusion/future work

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- Comparison with other span-based models of linear logic by Mellies, Clairambault, Forest
- Comparison with generalized species of structure (Fiore, Gambino, Galal, Hyland, Paquet, Winskel)