Categorical Decision Theory

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Normative Decision Theory

Decision theory studies decision-making in situations of risk or uncertainty.

There are two branches: *normative* and *descriptive*.

- Normative DT: how should an "*ideal rational agent*" make decisions?
- Descriptive DT: how do *actual human beings* make decisions in reality?

This talk is about normative decision theory.

Question. How should a rational agent make decisions in a risky situation?

Answer. (Bernoulli, 1738) Choose actions that maximize *expected utility*.

Question. But why?

Answer. (von Neumann & Morgenstern, 1948) When choosing between lotteries (with objective, known probabilities), EU-maximization is the *only* procedure that satisfies certain axioms of "rationality" or "consistency".

Question. What about a situation *without* objective, known probabilities?

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  $\mathcal{S}$ represents all the information that is unknown to the agent.
- There is a set $\mathcal{X}$ of possible "**outcomes**" (e.g. consumption bundles). These are the things the agent ultimately cares about.
- Each alternative defines a function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$, called an **act**.
- If the agent chooses the act α, and the true state of nature turns out to be s, then she will obtain the outcome $\alpha(s)$.
- Let $\mathcal{X}^{\mathcal{S}}$ be the set of all logically possible acts.
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Savage's Theorem

Savage's Theorem. *Suppose $\succcurlyeq$ satisfies six axioms* (encoding various criteria of "consistency" or "rationality"). *Then there exists:*

- *a "cardinal utility" function $u : \mathcal{X} \longrightarrow \mathbb{R}$, and*
- *a* (finitely additive) *probability measure μ on $\mathcal{S}$,*

which provide a **subjective expected utility** (SEU) **representation** *for $\succcurlyeq$. In other words, given any acts $\alpha, \beta \in \mathcal{X}^{\mathcal{S}}$, we have*

$$\Big(\alpha \succcurlyeq \beta\Big) \iff \left(\int_{\mathcal{S}} u \circ \alpha \ \mathrm{d}\mu \geq \int_{\mathcal{S}} u \circ \beta \ \mathrm{d}\mu)\right).$$

Heuristically, u describes the agent's *desires* concerning outcomes in $\mathcal{X}$.

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There are several ways that we want to extend Savage's framework.

Let's start with two.

- Endow state space and outcome space with additional structure (e.g. topology or geometry), and require acts to respect this structure.
- Analyse decision problems *without* explicitly describing the state space and outcome space.

Idea. Reformulate Savage model using the tools of category theory, to obtain a model which satisfies these desiderata.

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We will use the following notational conventions.

If $\mathcal{C}$ is a category, then $\mathcal{C}^\circ$ denotes its set of objects.

For any objects $\mathcal{A}, \mathcal{B} \in \mathcal{C}^\circ$, $\vec{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ is the set of $\mathcal{C}$-morphisms from $\mathcal{A}$ to $\mathcal{B}$.

A functor from $\mathcal{C}$ to another category $\mathcal{D}$ is is indicated "$F : \mathcal{C} \Longmapsto \mathcal{D}$."

If $G : \mathcal{C} \Longmapsto \mathcal{D}$ is another functor, then a natural transformation from F to G is indicated "$\Phi : F \Longrightarrow G$."

For simplicity, this talk focuses on the category $\mathbf{Set}$ (sets & functions).

But the theory also applies to concrete categories like $\mathbf{Meas}$ (measurable spaces & measurable functions), $\mathbf{Top}$ (topological spaces & continuous maps), and $\mathbf{Diff}$ (differentiable manifolds & smooth maps), and also to abstract categories.

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For simplicity, this talk focuses on the category $\mathbf{Set}$ (sets & functions).

But the theory also applies to concrete categories like $\mathbf{Meas}$ (measurable spaces & measurable functions), $\mathbf{Top}$ (topological spaces & continuous maps), and $\mathbf{Diff}$ (differentiable manifolds & smooth maps), and also to abstract categories.

Notation

We will use the following notational conventions.

If $\mathcal{C}$ is a category, then $\mathcal{C}^\circ$ denotes its set of objects.

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Part I

Local SEU representations

Goal

Let $\mathcal{C}$ be a category. Let $\mathcal{S}, \mathcal{X} \in \mathcal{C}^{\circ}$ be two objects.

Interpretation:

- $\mathcal{S}$ = abstract "state space";
- $\mathcal{X}$ = abstract "outcome space";
- $\vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$ = set of abstract "acts".

Let $\succcurlyeq$ be a preference order on $\vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$.

Goal. Find a "subjective expected utility representation" for $\succcurlyeq$.

Problem. In an abstract category $\mathcal{C}$, what would this even mean?

- Objects in $\mathcal{C}^{\circ}$ do not necessarily have underlying sets. So we cannot represent beliefs by probability measures.
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Partially ordered vector spaces

A **partially ordered vector space** (**POVS**) is a (real) vector space $\mathcal{V}$ equipped with a partial order that is compatible with addition and scalar multiplication in the obvious way.

Examples. (a) $\mathbb{R}$ is a POVS with the obvious linear order.

(b) Let $\mathcal{S}$ be a set. The vector space $\mathbb{R}^{\mathcal{S}}$ of real-valued functions on $\mathcal{S}$ is a POVS with the pointwise dominance order.

An **order unit** for a POVS $\mathcal{V}$ is an element $u \in \mathcal{V}$ with $u > 0$, such that for any $v > 0$ there is some $r \in \mathbb{R}_+$ with $r\,u \geq v$.

A **unitary** partially ordered vector space (**UPOVS**) is a POVS equipped with an order unit.

Examples. (a) 1 is an order unit for $\mathbb{R}$, making $\mathbb{R}$ a unitary POVS.

(b) Let $\mathcal{S}$ be a set. Let $\ell^\infty(\mathcal{S})$ be the POVS of all *bounded* elements of $\mathbb{R}^{\mathcal{S}}$. This is a UPOVS: the constant function $\mathbf{1}_{\mathcal{S}}$ is an order unit for $\ell^\infty(\mathcal{S})$.

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Utility frames

An order-preserving linear transformation from a UPOVS $\mathcal{V}_1$ to a UPOVS $\mathcal{V}_2$ is **uniferent** if it sends the order unit of $\mathcal{V}_1$ to the order unit of $\mathcal{V}_2$.

Let $\mathbf{UPOVS}$ be the category of all unitary partially ordered vector spaces and all uniferent, order-preserving, linear transformations.

Let $\mathcal{C}$ be another category.

A **utility frame** on $\mathcal{C}$ is a contravariant functor $L : \mathcal{C}^{\text{op}} \Longrightarrow \mathbf{UPOVS}$.

Example. Suppose $\mathcal{C} = \mathbf{Set}$.

For any $\mathcal{S} \in \mathbf{Set}^{\circ}$, let $L(\mathcal{S}) := \ell^{\infty}(\mathcal{S})$ with order unit $\mathbf{1}_{\mathcal{S}}$.

For any $\mathcal{S}_1, \mathcal{S}_2 \in \mathbf{Set}^{\circ}$ and $\phi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$, define $L(\phi) : \ell^{\infty}(\mathcal{S}_2) \longrightarrow \ell^{\infty}(\mathcal{S}_1)$ by setting $L(\phi)[v] := v \circ \phi$ for all bounded functions $v : \mathcal{S}_2 \longrightarrow \mathbb{R}$.

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Let $\mathbf{UPOVS}$ be the category of all unitary partially ordered vector spaces and all uniferent, order-preserving, linear transformations.

Let $\boldsymbol{\mathcal{C}}$ be another category.

A **utility frame** on $\boldsymbol{\mathcal{C}}$ is a contravariant functor $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$.

Example. Suppose $\boldsymbol{\mathcal{C}} = \mathbf{Set}$.

For any $\mathcal{S} \in \mathbf{Set}^{\circ}$, let $L(\mathcal{S}) := \ell^{\infty}(\mathcal{S})$ with order unit $\mathbf{1}_{\mathcal{S}}$.

For any $\mathcal{S}_1, \mathcal{S}_2 \in \mathbf{Set}^{\circ}$ and $\phi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$, define $L(\phi) : \ell^{\infty}(\mathcal{S}_2) \longrightarrow \ell^{\infty}(\mathcal{S}_1)$ by setting $L(\phi)[v] := v \circ \phi$ for all bounded functions $v : \mathcal{S}_2 \longrightarrow \mathbb{R}$.

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Utility functionals: definition

Let $\mathcal{C}$ be a category, and fix a utility frame $L : \mathcal{C}^{\rm op} \Longmapsto \mathbf{UPOVS}$.

Let $F : \mathbf{UPOVS} \Longmapsto \mathbf{Set}$ be the forgetful functor.

Let $\underline{L} := F \circ L : \mathcal{C}^{\rm op} \Longmapsto \mathbf{Set}$. (This is a presheaf.)

Let $\mathcal{X} \in \mathcal{C}^\circ$. Let $\vec{\mathcal{C}}(\bullet, \mathcal{X}) : \mathcal{C}^{\rm op} \Longmapsto \mathbf{Set}$ be the contravariant hom functor.

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A **utility functional** for $\mathcal{X}$ is a natural transformation $U_\mathcal{X} : \vec{\mathcal{C}}(\bullet, \mathcal{X}) \Longrightarrow \underline{L}$.

In other words, $U_\mathcal{X} = (U_\mathcal{X}^\mathcal{C})_{\mathcal{C} \in \mathcal{C}^\circ}$, where for any object $\mathcal{C} \in \mathcal{C}^\circ$,
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Utility functionals: example

Suppose $\mathcal{C} = \mathbf{Set}$ and $L := \ell^\infty : \mathbf{Set}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$ as above.

Let $\mathcal{X}$ be a set. Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a bounded function.

For any $\mathcal{C} \in \mathbf{Set}^\circ$ and any function $\alpha : \mathcal{C} \longrightarrow \mathcal{X}$, define $U_{\mathcal{X}}^{\mathcal{C}}(\alpha) := u \circ \alpha : \mathcal{C} \longrightarrow \mathbb{R}$. Then $U_{\mathcal{X}}^{\mathcal{C}}(\alpha) \in \ell^\infty(\mathcal{C})$ —i.e. $U_{\mathcal{X}}^{\mathcal{C}}(\alpha) \in \underline{L}(\mathcal{C})$.

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Thus, $U_{\mathcal{X}} = (U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{C} \in \mathbf{Set}^\circ}$ is a utility functional.

Indeed, by an application of the Yoneda Lemma, *every* utility functional can be seen as a generalization of this example.

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Beliefs

Let $\mathcal{C}$ be a category, and let $L : \mathcal{C}^{\text{op}} \Longmapsto \mathbf{UPOVS}$ be a utility frame.

Regard $\mathbb{R}$ as a unitary POVS with order unit 1.

For any $\mathcal{C} \in \mathcal{C}^{\circ}$, a **belief about** $\mathcal{C}$ is a $\mathbf{UPOVS}$-morphism $\rho : L(\mathcal{C}) \longrightarrow \mathbb{R}$.

(i.e. ρ is an order-preserving linear function from $L(\mathcal{C})$ to $\mathbb{R}$ with $\rho(\mathbf{1}) = 1$.)

Example. Let $\mathcal{C} = \mathbf{Set}$ and let $L = \ell^{\infty} : \mathbf{Set}^{\text{op}} \Longmapsto \mathbf{UPOVS}$

Let $\mathcal{S}$ be a set and let μ be a probability measure the power set of $\mathcal{S}$.

Define $\rho : \ell^{\infty}(\mathcal{S}) \longrightarrow \mathbb{R}$ by setting $\rho(v) := \int_{\mathcal{S}} v \, \mathrm{d}\mu$ for all $v \in \ell^{\infty}(\mathcal{S})$.

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Local SEU representations

Let $\mathcal{S}$ and $\mathcal{X}$ be objects in $\mathcal{C}^\circ$. Let $\succcurlyeq$ be a weak order on $\vec{\mathcal{C}}(\mathcal{S},\mathcal{X})$.

Let $L : \mathcal{C}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$ be a utility frame.

A **local subjective expected utility representation** for $\succcurlyeq$ is a pair $(\rho, U_{\mathcal{X}})$, where ρ is a belief about $\mathcal{S}$ and $U_{\mathcal{X}}$ is a utility functional on $\mathcal{X}$, such that:

$$\text{for all } \alpha,\beta \in \vec{\mathcal{C}}(\mathcal{S},\mathcal{X}), \qquad \alpha \succcurlyeq \beta \iff \rho\left[U_{\mathcal{X}}^{\mathcal{S}}(\alpha)\right] \geq \rho\left[U_{\mathcal{X}}^{\mathcal{S}}(\beta)\right]. \quad (*)$$

Example. Suppose $\mathcal{C} = \mathbf{Set}$ and let $L = \ell^\infty : \mathbf{Set}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$.

Let $\mathcal{S}$ and $\mathcal{X}$ be sets. Let ρ be a belief on $\mathcal{S}$ defined by a prob. measure μ.

Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be bounded. Let $U_{\mathcal{X}} : \vec{\mathcal{C}}(\bullet,\mathcal{X}) \Longrightarrow \underline{L}$ be the utility functional such that, for any $\alpha \in \vec{\mathcal{C}}(\mathcal{S},\mathcal{X})$, we have $U_{\mathcal{X}}^{\mathcal{S}}(\alpha) = u \circ \alpha$.

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There are three more ways that we want to extend Savage's framework.

- Obtain a single SEU representation that applies to many different decision problems, with different state spaces and/or outcome spaces.
- Represent the same decision problem with different levels of "awareness", or access to different information sources.
- Represent "analogies" between different decision problems, or "internal symmetries" within a decision problem.

Wanted:
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Part II.

Decision environments

and

ex ante preferences

Let $\mathcal{C}$ be a category.

A *decision environment* on $\mathcal{C}$ is an ordered pair $(\mathcal{S}, \mathcal{X})$, where $\mathcal{S}$ and $\mathcal{X}$ are subcategories of $\mathcal{C}$.

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Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision environment in a category $\boldsymbol{\mathcal{C}}$.

For every $\mathcal{S}$ in $\boldsymbol{\mathcal{S}}^\circ$ and $\mathcal{X}$ in $\boldsymbol{\mathcal{X}}^\circ$, let $\succeq_{\mathcal{X}}^{\mathcal{S}}$ be a preference order on $\vec{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$, representing the agent's *ex ante* preferences over acts.

$\underline{\trianglerighteq}^{\text{xa}} := \{\succeq_{\mathcal{X}}^{\mathcal{S}};\ \mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ\}$ is an *ex ante preference structure* if:

(BP) For all $\mathcal{S}_1, \mathcal{S}_2 \in \boldsymbol{\mathcal{S}}^\circ$, $\phi \in \vec{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1, \mathcal{S}_2)$, $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, and $\alpha, \beta \in \vec{\boldsymbol{\mathcal{C}}}(\mathcal{S}_2, \mathcal{X})$,

$\alpha \succeq_{\mathcal{X}}^{\mathcal{S}_2} \beta$ if and only if $\alpha \circ \phi \succeq_{\mathcal{X}}^{\mathcal{S}_1} \beta \circ \phi$. (Idea: ϕ is "belief-preserving".)

(DP) For all $\mathcal{X}_1, \mathcal{X}_2 \in \boldsymbol{\mathcal{X}}^\circ$, $\phi \in \vec{\boldsymbol{\mathcal{X}}}(\mathcal{X}_1, \mathcal{X}_2)$, $\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$, and $\alpha, \beta \in \vec{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X}_1)$,

$\alpha \succeq_{\mathcal{X}_1}^{\mathcal{S}} \beta$ if and only if $\phi \circ \alpha \succeq_{\mathcal{X}_2}^{\mathcal{S}} \phi \circ \beta$. (Idea: ϕ is "desire-preserving".)

Ex ante preference structures: Definition

Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision environment in a category $\boldsymbol{\mathcal{C}}$.

For every $\mathcal{S}$ in $\boldsymbol{\mathcal{S}}^\circ$ and $\mathcal{X}$ in $\boldsymbol{\mathcal{X}}^\circ$, let $\succcurlyeq^{\mathcal{S}}_{\mathcal{X}}$ be a preference order on $\overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$, representing the agent's *ex ante* preferences over acts.

$\underline{\trianglerighteq}^{\mathrm{xa}} := \{\succcurlyeq^{\mathcal{S}}_{\mathcal{X}};\ \mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ\}$ is an *ex ante preference structure* if:

(BP) For all $\mathcal{S}_1, \mathcal{S}_2 \in \boldsymbol{\mathcal{S}}^\circ$, $\phi \in \overrightarrow{\boldsymbol{\mathcal{S}}}(\mathcal{S}_1, \mathcal{S}_2)$, $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, and $\alpha, \beta \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}_2, \mathcal{X})$,

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(DP) For all $\mathcal{X}_1, \mathcal{X}_2 \in \boldsymbol{\mathcal{X}}^\circ$, $\phi \in \overrightarrow{\boldsymbol{\mathcal{X}}}(\mathcal{X}_1, \mathcal{X}_2)$, $\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$, and $\alpha, \beta \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X}_1)$,

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Objectives

Objective 1. Define "subjective expected utility representation" for ex ante preference structures over decision environments in abstract categories.

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Part III.

Global SEU representations

Belief systems: definition

Let $\mathcal{C}$ be a category, and let $L : \mathcal{C}^{\text{op}} \Longmapsto \mathbf{UPOVS}$ be a utility frame.

Let $\mathcal{S}$ be a subcategory of $\mathcal{C}$ (e.g. state places in a decision environment).

Let $L_{|_\mathcal{S}} : \mathcal{S}^{\text{op}} \Longmapsto \mathbf{UPOVS}$ be the restriction of L to $\mathcal{S}$.

A **belief system** for $\mathcal{S}$ is a co-cone from $L_{|_\mathcal{S}}$ to $\mathbb{R}$ in category $\mathbf{UPOVS}$.

In other words: it is a collection of beliefs $\{\rho_\mathcal{S} : L(\mathcal{S}) \longrightarrow \mathbb{R}\}_{\mathcal{S} \in \mathcal{S}^\circ}$, such that, for any $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{S}^\circ$ and $\phi \in \overrightarrow{\mathcal{S}}(\mathcal{S}_1, \mathcal{S}_2)$, the next diagram commutes:

$$L(\mathcal{S}_1) \xleftarrow{\;L(\phi)\;} L(\mathcal{S}_2), \qquad \rho_{\mathcal{S}_1} : L(\mathcal{S}_1) \to \mathbb{R}, \qquad \rho_{\mathcal{S}_2} : L(\mathcal{S}_2) \to \mathbb{R}$$

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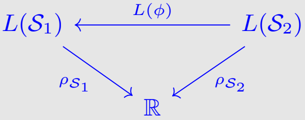

Proposition. *Let $L := \ell^\infty : \mathbf{Set}^{op} \Longrightarrow \mathbf{UPOVS}$.*

Let $\boldsymbol{\mathcal{S}}$ be a subcategory of $\mathbf{Set}$, and let $\{\rho_{\mathcal{S}}\}_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$ be a belief system.

For all $\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$, there is a unique finitely additive probability measure $\mu_{\mathcal{S}}$ on the power set of $\mathcal{S}$, such that $\rho_{\mathcal{S}} : \ell^\infty(\mathcal{S}) \longrightarrow \mathbb{R}$ is defined by

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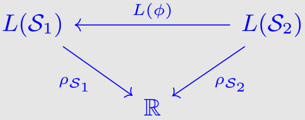

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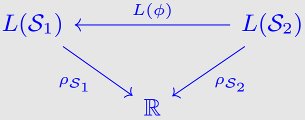

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The set of all positive affine transformations forms a group $\mathbf{Aff}$ under composition, which we can regard as a single-object category.

Let $\mathcal{V}$ be a unitary POVS with order unit $\mathbf{1}_{\mathcal{V}}$.

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Pos. Affine Transformations & Utility Systems

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A **positive affine transformation** is an increasing bijection $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ of the form $\phi(r) = a\,r + b$ for all $r \in \mathbb{R}$, where $a > 0$ and $b \in \mathbb{R}$ are constants.

The set of all positive affine transformations forms a group $\mathbf{Aff}$ under composition, which we can regard as a single-object category.

Let $\mathcal{V}$ be a unitary POVS with order unit $\mathbf{1}_{\mathcal{V}}$.

A **positive affine transformation** of $\mathcal{V}$ is an order-preserving bijection $\phi : \mathcal{V} \longrightarrow \mathcal{V}$ of the form $\phi(\mathbf{v}) = a\,\mathbf{v} + b\,\mathbf{1}_{\mathcal{V}}$ for all $\mathbf{v} \in \mathcal{V}$, where $a > 0$ and $b \in \mathbb{R}$ are constants.

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Example. Suppose $\boldsymbol{\mathcal{C}} = \mathbf{Set}$ and $L := \ell^\infty : \mathbf{Set}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$.

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such that for all $\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ$, all $\mathcal{X}, \mathcal{Y} \in \boldsymbol{\mathcal{X}}^\circ$ and all $\phi \in \vec{\boldsymbol{\mathcal{X}}}(\mathcal{X}, \mathcal{Y})$, this diagram commutes:

$$\begin{array}{ccc} \vec{\boldsymbol{\mathcal{C}}}(\mathcal{C}, \mathcal{X}) & \xrightarrow{\ \vec{\phi}\ } & \vec{\boldsymbol{\mathcal{C}}}(\mathcal{C}, \mathcal{Y}) \\ {\scriptstyle U_{\mathcal{X}}^{\mathcal{C}}}\big\downarrow & & \big\downarrow{\scriptstyle U_{\mathcal{Y}}^{\mathcal{C}}} \\ \underline{L}(\mathcal{C}) & \xrightarrow[\ \widehat{\phi}_{L(\mathcal{C})}\ ]{} & \underline{L}(\mathcal{C}) \end{array}$$

(where $\widehat{\phi} := A(\phi)$, and $\widehat{\phi}_{L(\mathcal{C})}$ is the automorphism of $\underline{L}(\mathcal{C})$ obtained from $\widehat{\phi}$.)

Example. Suppose $\boldsymbol{\mathcal{C}} = \mathbf{Set}$ and $L := \ell^\infty : \mathbf{Set}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$.

For all $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, let $u_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathbb{R}$, and define the utility functional $U_{\mathcal{X}} = (U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ} : \vec{\boldsymbol{\mathcal{C}}}(\bullet, \mathcal{X}) \Longrightarrow \underline{L}$ as before.

Let $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$ be a functor.

For all $\mathcal{X}, \mathcal{Y} \in \boldsymbol{\mathcal{X}}^\circ$ and $\phi \in \vec{\boldsymbol{\mathcal{X}}}(\mathcal{X}, \mathcal{Y})$, suppose that $u_{\mathcal{Y}} \circ \phi = \widehat{\phi} \circ u_{\mathcal{X}}$, where $\widehat{\phi} := A(\phi)$ (an affine function from $\mathbb{R}$ to itself).

Then the collection $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}$ together with A is a utility system on $\boldsymbol{\mathcal{X}}$.

An (L-valued) **utility system** on $\boldsymbol{\mathcal{X}}$ is an ordered pair (U, A), where

- $A : \boldsymbol{\mathcal{X}} \Longmapsto \mathbf{Aff}$ is a functor; and
- $U = (U_{\mathcal{X}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$, where $U_{\mathcal{X}} : \vec{\boldsymbol{\mathcal{C}}}(\bullet, \mathcal{X}) \Longrightarrow \underline{L}$ is a utility functional for each $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$;

such that for all $\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ$, all $\mathcal{X}, \mathcal{Y} \in \boldsymbol{\mathcal{X}}^\circ$ and all $\phi \in \vec{\boldsymbol{\mathcal{X}}}(\mathcal{X}, \mathcal{Y})$, this diagram commutes:

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Global SEU Representation: definition

Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision environment in a category $\boldsymbol{\mathcal{C}}$.

Let $\underline{\triangleright}^{\mathrm{xa}}$ be an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

A **global subjective expected utility representation** for $\underline{\triangleright}^{\mathrm{xa}}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
- A belief system $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}}$; and
- A utility system given by $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^{\circ}}$ and $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$;

such that for all $\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}$, and all $\alpha, \beta \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$

$$\alpha \succsim_{\mathcal{X}}^{\mathcal{S}} \beta \quad \iff \quad \rho_{\mathcal{S}}\left[U_{\mathcal{X}}^{\mathcal{S}}(\alpha)\right] \geq \rho_{\mathcal{S}}\left[U_{\mathcal{X}}^{\mathcal{S}}(\beta)\right].$$

Global SEU Representation: definition

Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision environment in a category $\boldsymbol{\mathcal{C}}$.

Let $\unrhd^{\mathrm{xa}}$ be an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

A **global subjective expected utility representation** for $\unrhd^{\mathrm{xa}}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
- A belief system $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$; and
- A utility system given by $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}$ and $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$;

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Let $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$ be a decision environment in a category $\boldsymbol{\mathcal{C}}$.

Let $\trianglerighteq^{\mathrm{xa}}$ be an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

A **global subjective expected utility representation** for $\trianglerighteq^{\mathrm{xa}}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
- A belief system $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$; and
- A utility system given by $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}$ and $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$;

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$$\alpha \succeq_{\mathcal{X}}^{\mathcal{S}} \beta \quad \iff \quad \rho_{\mathcal{S}} \left[U_{\mathcal{X}}^{\mathcal{S}}(\alpha) \right] \geq \rho_{\mathcal{S}} \left[U_{\mathcal{X}}^{\mathcal{S}}(\beta) \right].$$

A **global subjective expected utility representation** for $\underline{\rhd}^{\mathrm{xa}}$ consists of:

- A utility frame $L : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
- A belief system $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}}$; and
- A utility system given by $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^{\circ}}$ and $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$; such that

$\forall\ \mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}$, $\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}$, and $\alpha, \beta \in \vec{\mathcal{C}}(\mathcal{S}, \mathcal{X})$, $\quad \alpha \succeq_{\mathcal{X}}^{\mathcal{S}} \beta \Leftrightarrow \rho_{\mathcal{S}}\left[U_{\mathcal{X}}^{\mathcal{S}}(\alpha)\right] \geq \rho_{\mathcal{S}}\left[U_{\mathcal{X}}^{\mathcal{S}}(\beta)\right].\quad (*)$

Example. Suppose $\mathcal{C} = \mathbf{Set}$ and $L := \ell^{\infty} : \mathbf{Set}^{\mathrm{op}} \Longrightarrow \mathbf{UPOVS}$.

Let $\{\mu_{\mathcal{S}}\}_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}}$ be a set of prob. measures defining belief system $\{\rho_{\mathcal{S}}\}_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}}$.

For all $\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}$, let $u_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathbb{R}$ be a bounded function.

Let $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$ be a functor.

Define a utility system $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^{\circ}}$ using $(u_{\mathcal{X}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}}$ and A as before.

For all $\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}$, define an order $\succeq_{\mathcal{X}}^{\mathcal{S}}$ on $\overrightarrow{\mathbf{Set}}(\mathcal{S}, \mathcal{X})$ via $(*)$.

The system $\underline{\rhd}^{\mathrm{xa}} := (\succeq_{\mathcal{X}}^{\mathcal{S}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}}^{\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}}$ is an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

The data L, $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^{\circ}}$, $(U_{\mathcal{X}}^{\mathcal{C}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^{\circ}}^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^{\circ}}$ and A yield a global SEU repr. for $\underline{\rhd}^{\mathrm{xa}}$.

A **global subjective expected utility representation** for $\underline{\triangleright}^{\mathrm{xa}}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
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For all $\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, define an order $\succcurlyeq^{\mathcal{S}}_{\mathcal{X}}$ on $\overrightarrow{\mathbf{Set}}(\mathcal{S}, \mathcal{X})$ via $(*)$.

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A **global subjective expected utility representation** for $\trianglerighteq^{\mathrm{xa}}$ consists of:

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The data L, $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$, $(U^{\mathcal{C}}_{\mathcal{X}})^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ and A yield a global SEU repr. for $\trianglerighteq^{\mathrm{xa}}$.

A **global subjective expected utility representation** for $\underline{\rhd}^{\mathrm{xa}}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
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Let $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$ be a functor.

Define a utility system $(U^{\mathcal{C}}_{\mathcal{X}})^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ using $(u_{\mathcal{X}})_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ and A as before.

For all $\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$ and $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, define an order $\succcurlyeq^{\mathcal{S}}_{\mathcal{X}}$ on $\overrightarrow{\mathbf{Set}}(\mathcal{S}, \mathcal{X})$ via $(*)$.

The system $\underline{\rhd}^{\mathrm{xa}} := (\succcurlyeq^{\mathcal{S}}_{\mathcal{X}})^{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ is an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

The data L, $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$, $(U^{\mathcal{C}}_{\mathcal{X}})^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ and A yield a global SEU repr. for $\underline{\rhd}^{\mathrm{xa}}$.

A **global subjective expected utility representation** for $\underline{\trianglerighteq}^{\rm xa}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\rm op} \longrightarrow \mathbf{UPOVS}$;
- A belief system $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$; and
- A utility system given by $(U^{\mathcal{C}}_{\mathcal{X}})^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ and $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$; such that

$\forall\ \mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$, $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, and $\alpha, \beta \in \vec{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$, $\quad \alpha \succcurlyeq^{\mathcal{S}}_{\mathcal{X}} \beta \Leftrightarrow \rho_{\mathcal{S}}\left[U^{\mathcal{S}}_{\mathcal{X}}(\alpha)\right] \geq \rho_{\mathcal{S}}\left[U^{\mathcal{S}}_{\mathcal{X}}(\beta)\right].\quad (*)$

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Let $A : \boldsymbol{\mathcal{X}} \longrightarrow \mathbf{Aff}$ be a functor.

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The system $\underline{\trianglerighteq}^{\rm xa} := (\succcurlyeq^{\mathcal{S}}_{\mathcal{X}})^{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ is an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

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A **global subjective expected utility representation** for $\trianglerighteq^{\mathrm{xa}}$ consists of:

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The system $\trianglerighteq^{\mathrm{xa}} := (\succcurlyeq^{\mathcal{S}}_{\mathcal{X}})^{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ is an ex ante preference structure on $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{X}})$.

The data L, $(\rho_{\mathcal{S}})_{\mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ}$, $(U^{\mathcal{C}}_{\mathcal{X}})^{\mathcal{C} \in \boldsymbol{\mathcal{C}}^\circ}_{\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ}$ and A yield a global SEU repr. for $\trianglerighteq^{\mathrm{xa}}$.

A **global subjective expected utility representation** for $\underline{\triangleright}^{\mathrm{xa}}$ consists of:

- A utility frame $L : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{UPOVS}$;
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$\forall\, \mathcal{S} \in \boldsymbol{\mathcal{S}}^\circ$, $\mathcal{X} \in \boldsymbol{\mathcal{X}}^\circ$, and $\alpha, \beta \in \overrightarrow{\boldsymbol{\mathcal{C}}}(\mathcal{S}, \mathcal{X})$, $\quad \alpha \succcurlyeq_{\mathcal{X}}^{\mathcal{S}} \beta \Leftrightarrow \rho_{\mathcal{S}}\left[U_{\mathcal{X}}^{\mathcal{S}}(\alpha)\right] \geq \rho_{\mathcal{S}}\left[U_{\mathcal{X}}^{\mathcal{S}}(\beta)\right].\quad (*)$

Example. Suppose $\boldsymbol{\mathcal{C}} = \mathbf{Set}$ and $L := \ell^\infty : \mathbf{Set}^{\mathrm{op}} \Longmapsto \mathbf{UPOVS}$.

Let $\{\mu_{\mathcal{S}}\}_{\mathcal{S}\in\boldsymbol{\mathcal{S}}^\circ}$ be a set of prob. measures defining belief system $\{\rho_{\mathcal{S}}\}_{\mathcal{S}\in\boldsymbol{\mathcal{S}}^\circ}$.

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Question. Under what conditions does an ex ante preference structure have a global SEU representation?

Answer. Using an approach inspired by Anscombe & Aumann (1963), we prove a theorem giving necessary & sufficient conditions for an ex ante preference structure to have a global SEU representation.

Furthermore, in the category **Top**, we can ensure that the utility functions are continuous, and beliefs are represented by Borel probability measures.

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Positive Affine Transformations
Utility systems
Global SEU Representation
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