Partial and Relational Algebraic Theories

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Goal: satisfying notions of theory with models in Par,Rel.

Exciting Novelty: String diagrams instead of classical terms.

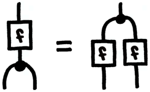
Plan:

- Algebraic theories recap.
- Partial Algebraic theories.
- Relational Algebraic Theories.

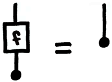
Three stories with the same shape. Legitimacy by analogy.

Categories with finite products are the same thing as symmetric monoidal categories with:

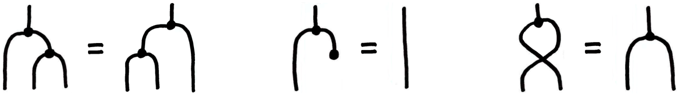
- Copying:



- Deleting:



Copying and deleting form a commutative comonoid:



And must satisfy a coherence condition.

An *algebraic theory* is a (small) category $\mathbb{X}$ with finite products.

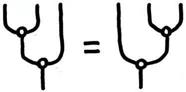
A *model* of $\mathbb{X}$ is a functor $F : \mathbb{X} \to \mathsf{Set}$ that preserves products.

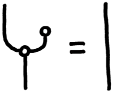
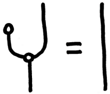
A *model morphism* is a natural transformation

For example, the theory of monoids is generated by:

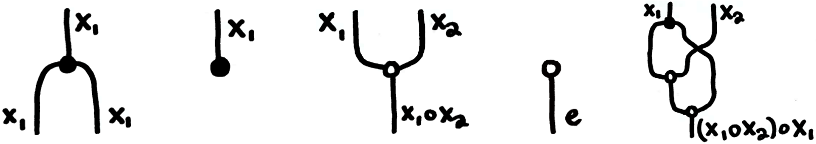
$\circ/_2 \quad \leftrightsquigarrow$  $\qquad e/_0 \quad \leftrightsquigarrow$ 

Subject to equations:

$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3) \quad \leftrightsquigarrow$ 

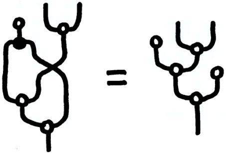
$x_1 \circ e = x_1 \quad \leftrightsquigarrow$  $\qquad e \circ x_1 = x_1 \quad \leftrightsquigarrow$ 

String correspond to (tuples of) terms.



Composition is substitution.

$$(x_1 \circ x_2) \circ x_1 \left[{}^{x_1 \mapsto e}_{x_2 \mapsto x_1 \circ x_2}\right] = (e \circ (x_1 \circ x_2)) \circ e$$



Models of the theory of monoids are monoids.

Model morphisms are monoid homomorphisms.

Categories of models and model morphisms are called *varieties*.

Two algebraic theories present equivalent varieties iff they have equivalent idempotent splitting completions (Karoubi envelope).

What about theories whose operations are *partial* functions?

That is, models in Par instead of Set.

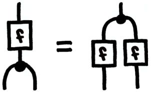
Classical Term Syntax $\leftrightsquigarrow$ Finite Products

Par does not have finite products . . . so classical term syntax isn't going to work!

What structure *does* Par have?

- Copying:

$$a \mapsto (a, a)$$



- Restriction:

$$a \mapsto *$$

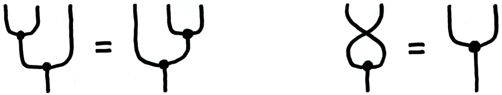
(not natural)

- Sameness:

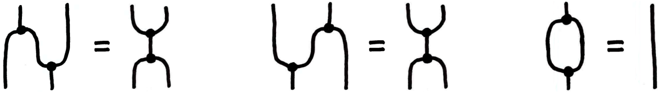
$$(a, b) \mapsto \begin{cases} a \text{ if } a = b \\ \uparrow \text{ otherwise} \end{cases}$$

Copying and restriction form a commutative comonoid.

Sameness is commutative and associative:



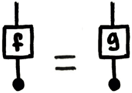
Copying and sameness are special Frobenius:



"Discrete Cartesian Restriction Categories"

The arrow

Corresponds to the domain of definition of f. Thus, the equation



Expresses that f and g have the same domain of definition.

Define a *partial algebraic theory* to be a (small) symmetric monoidal category $\mathbb{X}$ with this structure.

A *model* is a symmetric monoidal functor $\mathbb{X} \to \mathsf{Par}$ that preserves copying, restriction, and sameness.

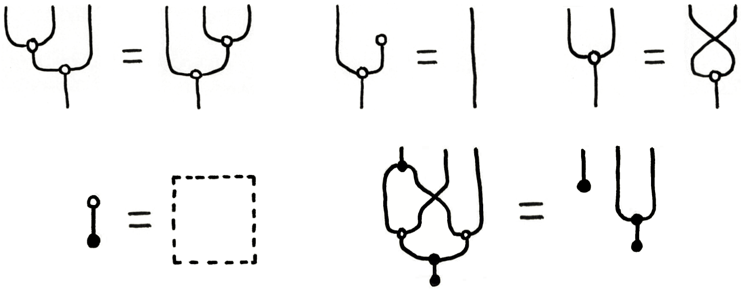
A *model morphism* is a monoidal lax transformation. That is, families $\alpha_X : FX \to GX$ such that for any $f : X \to Y$ we have:

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle F(f)}\downarrow & \leq & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array} \qquad \text{and} \qquad \begin{array}{c} \alpha_{X \otimes Y} = \alpha_X \otimes \alpha_Y \\ \alpha_I = 1_I \end{array}$$

where $\leq$ is the extension ordering on partial functions.

For example, the theory of *separation algebras* is generated by:

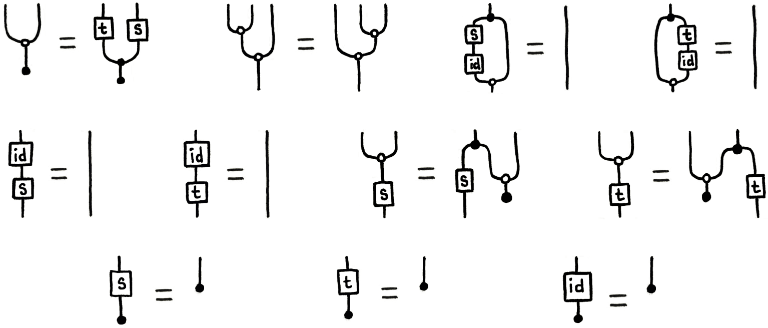
Subject to equations:



Bottom right is *cancellativity*: $a \circ c = b \circ c \;\Leftrightarrow\; a = b$.

The (2-sorted) theory of *categories* is generated by:

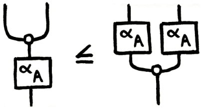
Subject to equations:



Models F, G are (small) categories.

Model morphisms $\alpha : F \to G$ are functors.

In particular, since α is a lax transformation we have:



If these were *equal*, our functor would *reflect composability*.

That is, $\alpha_A(f)\alpha_A(g)\downarrow$ would imply $fg\downarrow$. Functors don't do this.

Theorem

The categories that arise as the models and model morphisms of some partial algebraic theory are precisely the locally finitely presentable (LFP) categories.

Theorem

Two partial algebraic theories present equivalent LFP categories iff they have equivalent idempotent splitting completions.

Because: $\mathrm{DCR}_s \simeq \mathrm{Lex}$. Essentially algebraic theories.

What about theories whose operations are *relations*?

That is, models in Rel instead of Set.

Once again, classical syntax is unsuitable.

Again, we will use string diagrams instead.

What structure does Rel have?

- Comultiplication and Counit:

$$\{(a,(a,a)) \mid a \in A\}$$

$$\{(a,*) \mid a \in A\}$$

- Multiplication and Unit:

$$\{((a,a),a) \mid a \in A\}$$

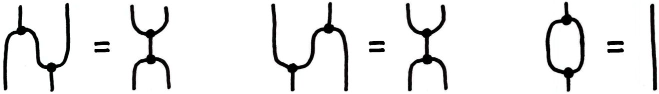
$$\{(*,a) \mid a \in A\}$$

“Carboni-Walters categories”. “Cartesian bicategories of relations”.

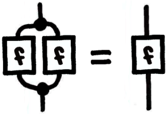
Comultiplication and counit form a commutative comonoid.

Multiplication and unit form a commutative monoid.

Multiplication and comultiplication are special frobenius.



Naturality properties are replaced by:

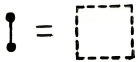


Define a *relational algebraic theory* to be a (small) symmetric monoidal category $\mathbb{X}$ with this structure.

A *model* is a symmetric monoidal functor $\mathbb{X} \to \mathrm{Rel}$ that preserves (co)multiplication and (co)unit.

Model morphisms are again monoidal lax transformations.

The (single-sorted) relational algebraic theory of *nonempty sets* is:



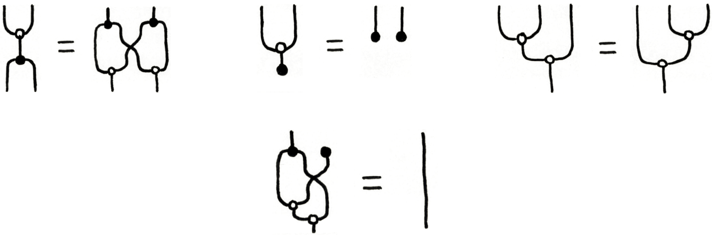
Models are nonempty sets.

$$\{(*, a) \mid a \in A\}\{(a, *) \mid a \in A\} = \{(*, *) \mid \exists a \in A\}$$

Model morphisms are (total) functions.

The theory of *regular semigroups* is generated by:

Subject to equations:



Regularity: $\forall a \in A . \exists b \in A . aba = a$

Theorem

The categories that arise as the models and model morphsims of some relational algebraic theory are precisely the definable categories (in the sense of Kuber and Rosický).

Theorem

Two relational algebraic theories present equivalent definable categories iff they have equivalent idempotent splitting completions.

Because: $\mathsf{CW}_s \simeq \mathsf{Ex}$.

Some References

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