# Two-dimensional Kripke Semantics

## II. Stability and Completeness

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# Kripke semantics vs. type theory

**Modal logic** is important in Computer Science:

- temporal logic
- epistemic logic
- dynamic logic
- Hennessy-Milner logic

In most cases, it is given a **Kripke semantics**.

But in type theory **proofs are important** (Curry-Howard-Lambek).

Type-theoretic **modalities** arise *everywhere*:

- 'logical' time
- proof-irrelevance
- globality
- information flow

How can we connect these two worlds?

# Models of intuitionistic logic

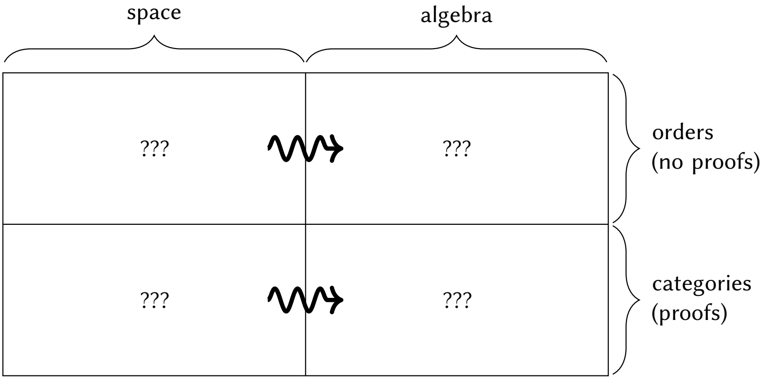
| space | algebra | |
|---|---|---|
| Kripke semantics (over a poset) ⇝ | Heyting algebras | orders (no proofs) |
| two-dimensional Kripke semantics (over a **category**) ⇝ presheaves | cartesian closed categories | categories (proofs) |

$$w \vDash \bot \overset{\text{def}}{\equiv} \text{never}$$

$$w \vDash \varphi \to \psi \overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi$$

**Monotonicity**: $w \vDash \varphi$ and $w \sqsubseteq v$ imply $v \vDash \varphi$

# Models of **intuitionistic** modal logic



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It has proven remarkably robust in modal type theory.

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# Prime algebraic lattices: from space to algebra

Let $(W, \sqsubseteq)$ be a **Kripke frame**, and $\mathbb{2} \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, \mathbb{2}]$ (= monotone maps $W \to \mathbb{2}$) has many curious properties:

- $[W, \mathbb{2}] \cong \mathsf{Up}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal upper set** embedding $\uparrow : W^{\mathrm{op}} \to [W, \mathbb{2}]$ given by $w \mapsto \{v \mid w \sqsubseteq v\}$ preserves meets and exponentials.
- An element is a **prime** ($p \sqsubseteq \bigsqcup_i d_i \Rightarrow \exists i.\ p \sqsubseteq d_i$) iff it is $\uparrow w$.
- Every upper set $S$ is a join of primes:

$$S = \bigsqcup\{P \mid P \text{ prime}, P \subseteq S\} = \bigsqcup\{\uparrow w \mid w \in S\}$$

In short: $[W, \mathbb{2}]$ is a **prime algebraic lattice** [Win09].

There is a **duality**: $\mathbf{Pos}^{\mathrm{op}} \simeq \mathbf{PrAlgLatt}$.

# Intuitionistic logic: from space to category

Play the same trick as before, but replace $\mathbb{2}$ by **Set** [Law73].

The category $[\mathcal{C}, \mathbf{Set}]$ of presheaves $\mathcal{C} \longrightarrow \mathbf{Set}$:

- is a **(co)complete cartesian closed category**
- The **Yoneda embedding** $\mathbf{y} : \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]$ given by $\mathbf{y}(w) \stackrel{\text{def}}{=} \mathrm{Hom}(w, -)$ preserves products and exponentials.
- A presheaf $P$ is **tiny** just if $\mathrm{Hom}(P, -)$ preserves colimits. All representables are tiny [and vice versa if $\mathcal{C}$ is Cauchy-complete].
- Every presheaf $P : \mathcal{C} \longrightarrow \mathbf{Set}$ is a colimit of tiny objects:

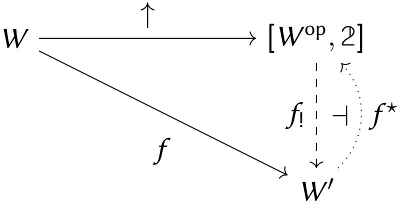
$$P = \varinjlim_{(w,x) \in \mathrm{el}\, P} \mathbf{y}(w)$$

There is a duality: $\mathbf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCat}$ (Bunge's theorem).

2D Kripke semantics = semantics in $[\mathcal{C}, \mathbf{Set}]$.

# Extensions

Let $W'$ be a **complete lattice**, and let $f : W \to W'$ be monotone.




$f_!$: the **unique join-preserving** map satisfying $f_!(\uparrow w) = f(w)$.

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup\{f(w) \mid w \in S\}$$

As both lattices are complete, this has a right adjoint $f^\star$. Explicitly:

$$f^\star(w') \stackrel{\text{def}}{=} \{w \mid f(w) \sqsubseteq w'\}$$

Then

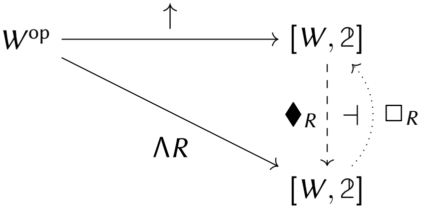
$$f_!(S) \sqsubseteq w' \quad \iff \quad S \subseteq f^\star(w')$$

# Bimodules and Extensions

Let $(W, \sqsubseteq)$ be a Kripke frame. $R \subseteq W \times W$ is a **bimodule** just if

$$w' \sqsubseteq w \, R \, v \sqsubseteq v' \implies w' \, R \, v'$$

Equivalently: $R : W^{\mathrm{op}} \times W \to \mathbb{2}$. Now extend $\Lambda R : W^{\mathrm{op}} \to [W, \mathbb{2}]$:




Concretely: $\begin{cases} \blacklozenge_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \exists v.\ v \, R \, w \text{ and } v \in S\} \\ \Box_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \forall v.\ w \, R \, v \text{ implies } v \in S\} \end{cases}$

Every such adjunction on $[W, \mathbb{2}]$ corresponds to a bimodule!

Duality: $\mathbf{EBimod}^{\mathrm{op}} \simeq \mathbf{PrAlgLattO}$.

# Lifting to categories

- Replace bimodules by **profunctors**
- Use **left Kan extension** along Yoneda

This leads to a duality $\mathbf{EProf}_{cc}^{op} \simeq \mathbf{PshCatO}$.

Modalities on presheaves $P : \mathcal{C} \longrightarrow \mathbf{Set}$:

$$(\blacklozenge P)(w) = \int^{v \in \mathcal{C}} R(v, w) \times P(v)$$

$$(\Box P)(w) = \int_{v \in \mathcal{C}} R(w, v) \to P(v) \cong \mathrm{Hom}_{[\mathcal{C}, \mathbf{Set}]}(R(w, -), [\![\varphi]\!])$$

## Theorem

*A two-dimensional Kripke semantics over $\mathcal{C}$ uniquely corresponds to*

$$[\mathcal{C}, \mathbf{Set}] \underset{\blacklozenge}{\overset{\Box}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [\mathcal{C}, \mathbf{Set}]$$

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# Completeness?

The developments so far only prove **relative completeness**:

- Suppose a formula is valid in all Heyting algebras.
- Then it is valid in all prime algebraic lattices.
- Then it is valid in all Kripke semantics

∴ the algebraic semantics is as complete as the Kripke semantics.

How to get the opposite direction?

The classic proof (Gehrke and van Gool [Gv24, §4.4]):

- Make a Kripke frame of **prime filters** of the algebra.
- Show relative completeness with respect to that.

For this logic: Dzik, Järvinen, and Kondo [DJK10, §5].

But this is **non-constructive**, and also not very nice.

# Stable semantics

Replace

- the poset of worlds by a **distributive lattice** $(W, \sqsubseteq)$
- upper sets by (non-prime) **filters**

$F \subseteq W$ is a **filter** just if it is an upper set and

$$1 \in F, \qquad x \in F \text{ and } y \in F \text{ imply } x \wedge y \in F$$

$$w \vDash p \stackrel{\text{def}}{\equiv} w \in V(p) \in \mathrm{Filt}(W)$$

$$w \vDash \bot \stackrel{\text{def}}{\equiv} (1 \leq w) \qquad \text{(i.e. } w = 1\text{)}$$

$$w \vDash \varphi \wedge \psi \stackrel{\text{def}}{\equiv} w \vDash \varphi \text{ and } v \vDash \psi$$

$$w \vDash \varphi \vee \psi \stackrel{\text{def}}{\equiv} \exists v_1, v_2.\ v_1 \wedge v_2 \sqsubseteq w \text{ and } v_1 \vDash \varphi, v_2 \vDash \psi$$

$$w \vDash \varphi \to \psi \stackrel{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi$$

This semantics is also sound and complete for intuitionistic logic!

# Spectral locales: from space to algebra

Let $(W, \sqsubseteq)$ be a **distributive lattice**, and $\mathbb{2} \overset{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, \mathbb{2}]_\wedge$ (= $\wedge$-preserving $W \to \mathbb{2}$) has many curious properties:

- $[W, \mathbb{2}]_\wedge \cong \mathrm{Filt}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal filter** embedding $\uparrow : W^{\mathrm{op}} \to [W, \mathbb{2}]_\wedge$ preserves finite meets, finite joins, and exponentials. Hence for any Heyting algebra $H$

$$H \hookrightarrow [H^{\mathrm{op}}, \mathbb{2}]_\wedge$$

- An elt. is **compact** ($p \sqsubseteq \bigsqcup^{\uparrow} X \Rightarrow \exists d \in X.\ p \sqsubseteq d$) iff it is $\uparrow w$.
- Every filter $F$ is a directed supremum of compact ones:

$$F = \bigsqcup^{\uparrow} \{S \mid S \text{ compact}, S \subseteq F\} = \bigsqcup^{\uparrow} \{\uparrow w \mid w \in F\}$$

In short: $[W, \mathbb{2}]$ is a **spectral locale** (or a **coherent frame**)
(= algebraic cHA whose compact elts form a sub-lattice).

# Prime algebraic lattices: from space to algebra

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# Dualities and modalities

The main duality is now

$$\mathbf{Stable}^{\mathrm{op}} \simeq \mathbf{Coh}$$

between

- **distributive lattices** and stable (= $\wedge$-preserving) maps
- **coherent frames** and Scott-continuous, $\prod$-preserving maps (**not** the usual category from Stone duality)

Then

> The stable semantics and the Heyting algebra semantics are **equi-complete**, **constructively**.

All previous work on modalities carries through, nearly verbatim.

# Categorifying the stable semantics

Let $\mathcal{C}$ be a category with finite products and coproducts, which is also a **co-distributive category**: $a + (c \times d) \cong (a + c) \times (a + d)$.

A two-dimensional stable semantics is a categorical semantics in a **category of algebras**.

**Why?** Because ‘filters’ are **product-preserving presheaves** over $\mathcal{C}$!

Seeing $\mathcal{C}$ as a Lawvere theory, the category of **product-preserving presheaves** $[\mathcal{C}, \mathbf{Set}]_\times \cong \mathrm{Sind}(\mathcal{C}^{\mathrm{op}})$ is that of **algebras over** $\mathcal{C}$.

**Fact**: $\mathcal{C}$ is co-distributive iff $[\mathcal{C}, \mathbf{Set}]_\times$ is cartesian closed.

For any bi-ccc $\mathcal{C}$ we have a bi-ccc functor $\mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_\times$. Hence

## Theorem

*The category $[\mathcal{C}, \mathbf{Set}]_\times$ of product-preserving presheaves over a co-distributive $\mathcal{C}$ is complete for typed $\lambda$-calculus with sums.*

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[Win09] Glynn Winskel. "Prime algebraicity". In: *Theoretical Computer Science* 410.41 (2009), pp. 4160–4168. DOI: 10.1016/j.tcs.2009.06.015 (cit. on pp. 9, 18).