universität innsbruck

The Aldous–Hoover theorem in Categorical Probability

Joint work with Leihao Chen, Tobias Fritz, Tomáš Gonda and Andreas Klinger

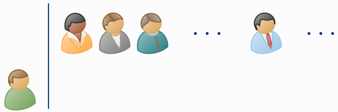
Antonio Lorenzin, University of Innsbruck

ACT conference 2024, Oxford

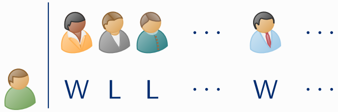
Contents

Rough idea

De Finetti theorem



De Finetti theorem



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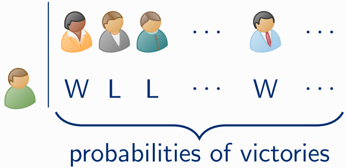

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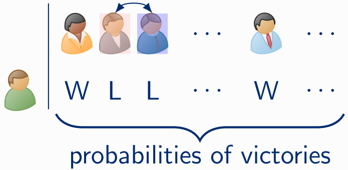

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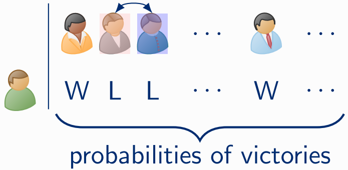
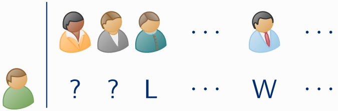
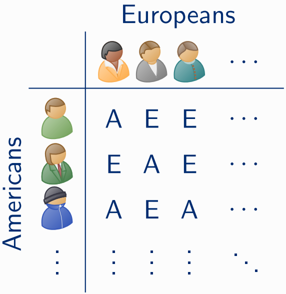


Then

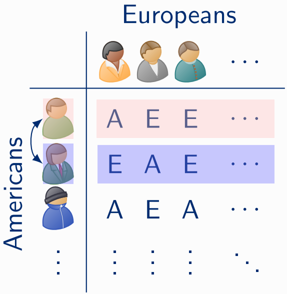


$\Longrightarrow$ *Independent predictions* on  and 

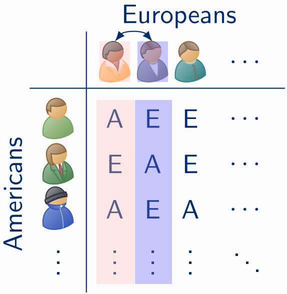
Aldous–Hoover theorem, 1



Aldous–Hoover theorem, 1



Aldous–Hoover theorem, 1



Aldous–Hoover theorem, 2

Then

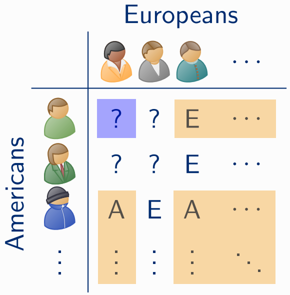
Europeans

Americans				
	?	?	E	$\cdots$
	?	?	E	$\cdots$
	A	E	A	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\Longrightarrow$ *Independent predictions* on the four missing games

Aldous–Hoover theorem, 2

Then



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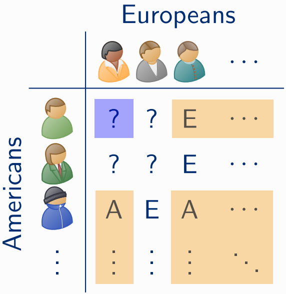
Aldous–Hoover theorem, 2

Then

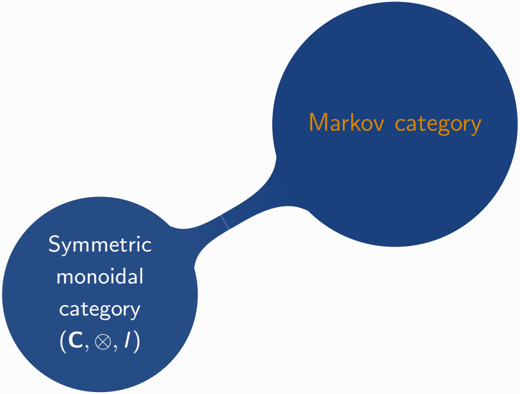


$\Longrightarrow$ *Independent predictions* on the four missing games

Tail: *general difference in skills*

Markov categories

Definition of a Markov category



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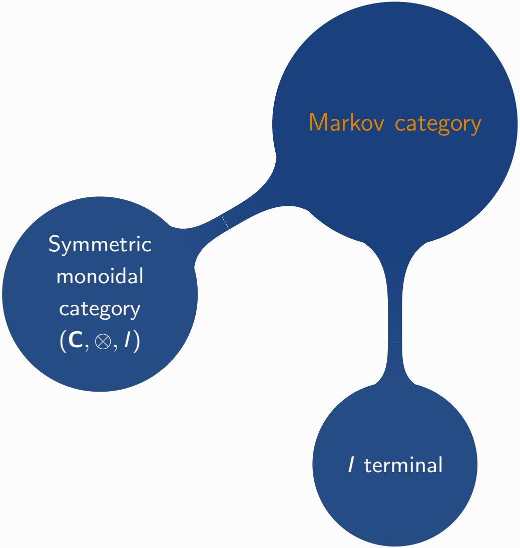

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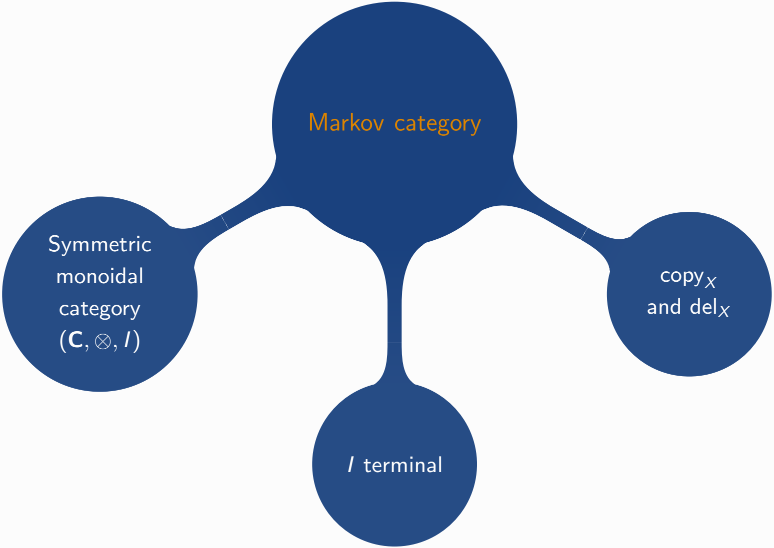

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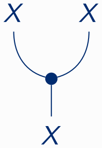
$\text{copy}_X : X \to X \otimes X \qquad \text{del}_X : X \to I$



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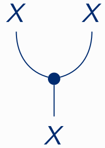
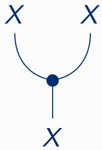


copy_X **copies the information of** X **<u>without</u> introducing randomness!**

Copy and delete

$\text{copy}_X \colon X \to X \otimes X \qquad \text{del}_X \colon X \to I$



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$p\colon I \to X$ are probabilities on X.

Toy example

Q. For finite sets, what is a probabilistic morphism?

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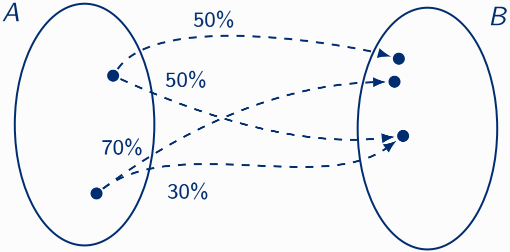
A. It's a Markov kernel:



Standard Borel spaces

BorelStoch: Standard Borel spaces with Markov kernels.

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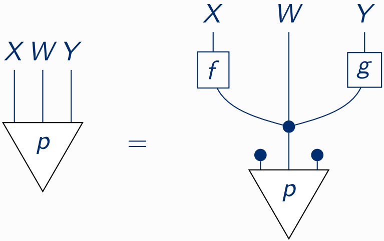
Kuratowski's theorem

Any standard Borel space is measurable isomorphic to $\mathbb{R}$, $\mathbb{Z}$, or a finite set.

Conditional independence

Fix a probability $p\colon I \to X \otimes W \otimes Y$.

Then X is conditionally independent of Y given W, in symbols $X \perp Y \mid W$, if



for some f, g.

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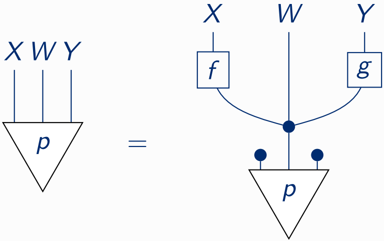

for some f, g. X and Y are identically distributed given W if $f = g$.

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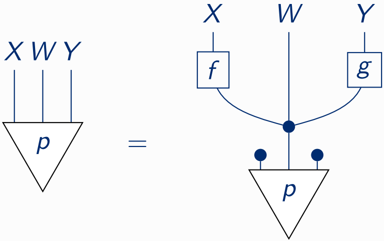

for some f, g. X and Y are identically distributed given W if $f = g$. Similarly, one can define $\perp_i X_i \mid W$.

Aldous–Hoover theorem

De Finetti theorem

Countable tensor product: $X^{\mathbb{N}}$, and we get $X^{\sigma} \colon X^{\mathbb{N}} \to X^{\mathbb{N}}$ for each finite permutation σ.

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De Finetti theorem

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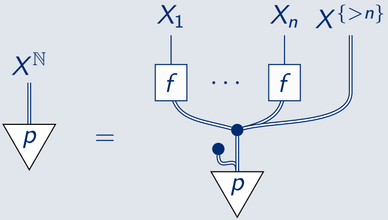
De Finetti theorem

Let p be an exchangeable probability $I \to X^{\mathbb{N}}$. For any $n \in \mathbb{N}$, $\perp_{i \leq n} X_i \mid X^{\{>n\}}$. Even more, they are identically distributed given the tail:



What do we need?

Synthetic proof of the de Finetti theorem under the following assumptions:

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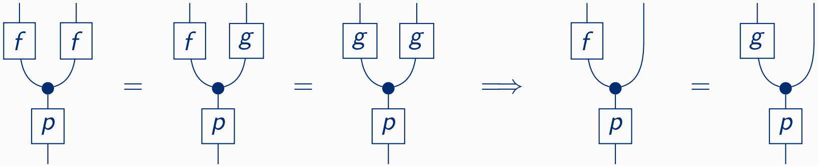
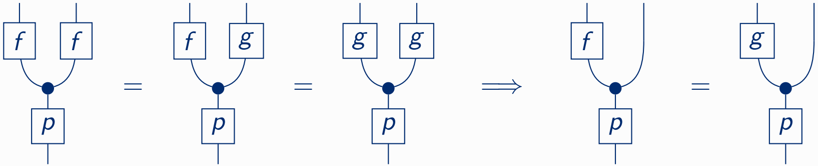
BorelStoch satisfies these assumptions.

Under the same hypotheses we can prove the Aldous–Hoover theorem!

The Cauchy–Schwarz axiom



The Cauchy–Schwarz axiom



Last equation: f is p-a.s. equal to g.

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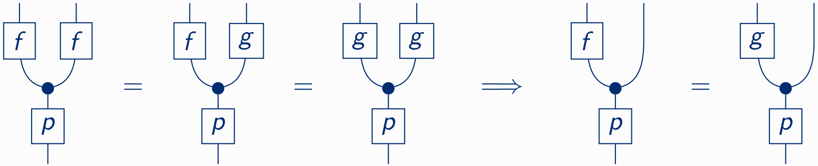
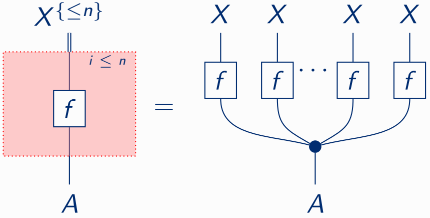


Last equation: f is p-a.s. equal to g.

This axiom controls almost sure equality!

Plate notation

To write the Aldous–Hoover theorem, we use the plate notation:



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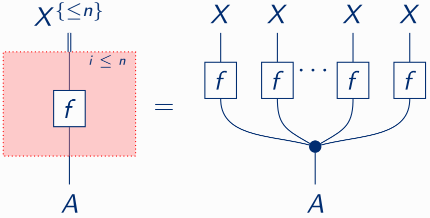
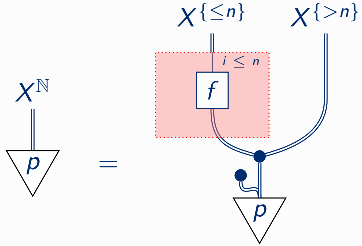


De Finetti theorem:



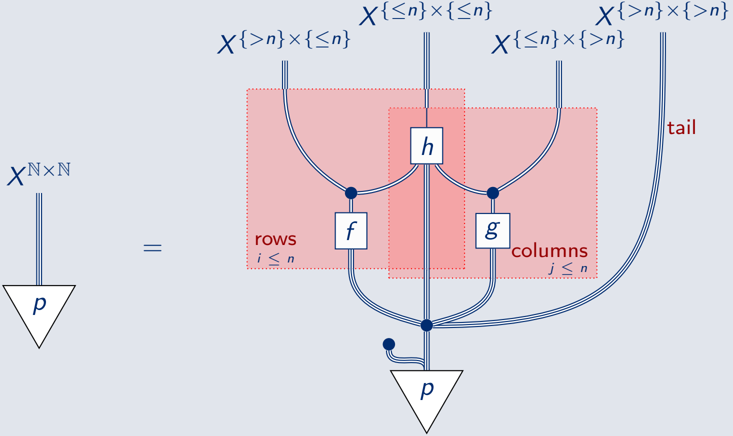
Main theorem

Aldous–Hoover theorem

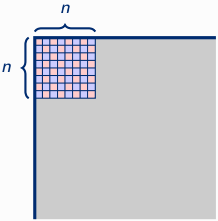
Let $p \colon I \to X^{\mathbb{N}\times\mathbb{N}}$ be row-and-column-exchangeable (for all finite permutations σ, $X^{\sigma\times \mathrm{id}}\, p = p = X^{\mathrm{id}\times\sigma}\, p$).

Aldous–Hoover theorem

Let $p\colon I \to X^{\mathbb{N}\times\mathbb{N}}$ be row-and-column-exchangeable (for all finite permutations σ, $X^{\sigma\times\mathrm{id}}\,p = p = X^{\mathrm{id}\,\times\sigma}\,p$). Then, for any $n \in \mathbb{N}$,



Idea of the proof



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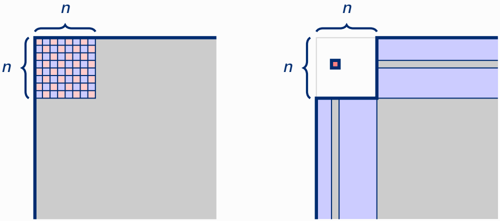

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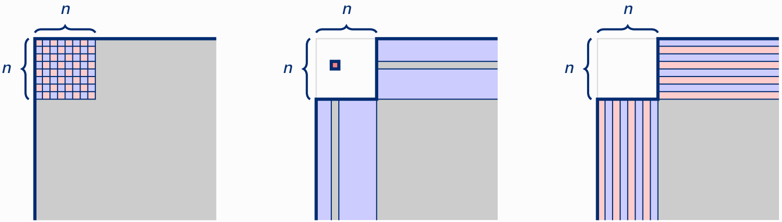

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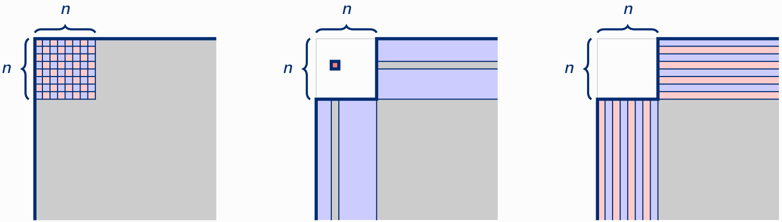

T. Fritz and A. Klinger. *The d-separation criterion in categorical probability*. J. Mach. Learn. Res., 24(46):1–49, 2023. arXiv:2207.05740.

THANK YOU FOR YOUR ATTENTION