On a fibrational construction for optics, lenses, and Dialectica categories

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Introduction

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Hence the question we started to ask is: **how are they related?**

This talk will attempt to answer that question.

References

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Fun with fibrations

Lenses

Let $\mathbf{C}$ be cartesian monoidal.

Definition

The category of **simple lenses** $\mathbf{Lens}_\times(\mathbf{C})$ has

- as objects, pairs (X, U) of objects of $\mathbf{C}$,
- as morphisms $(X, U) \leftrightarrows (Y, V)$, pairs

$$f : U \to V,$$
$$f^\sharp : U \times Y \to X$$

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A lens represents a back and forth dialogue: it answers to question coming from its left boundaries by asking questions to its right boundary.

Lenses

$$
\begin{array}{ccccc}
U \times X & \xleftarrow{f^\sharp} & U \times Y & \xrightarrow{f \times Y} & V \times Y \\
\Big\downarrow{\scriptstyle \pi_U} & & \Big\downarrow{\scriptstyle \pi_U} & \lrcorner & \Big\downarrow{\scriptstyle \pi_V} \\
U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dependent lenses

Let $\mathbf{C}$ be finitely complete

Definition

The **category of (dependent) lenses** $\mathbf{Lens}(\mathbf{C})$ has

- as objects, bundles $\begin{smallmatrix} X \\ \downarrow \\ U \end{smallmatrix}$ (i.e. morphisms) in $\mathbf{C}$,
- as morphisms $\begin{smallmatrix} X \\ \downarrow \\ U \end{smallmatrix} \leftrightarrows \begin{smallmatrix} Y \\ \downarrow \\ V \end{smallmatrix}$, pairs

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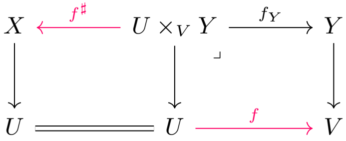
A dependent lens represents a back and forth dialogue *with strict rules on which type of answers we are allowed to give for a question*.

Dependent lenses



p-lenses

Let $p : \mathbf{E} \to \mathbf{C}$ be a fibration/let $p^{-1} : \mathbf{C}^{\mathsf{op}} \to \mathbf{Cat}$ be an indexed category.[1]

Definition

The category of p-lenses has

- as objects, p-bundles $\left(\begin{smallmatrix} X : p^{-1}U \\ U : \mathbf{C} \end{smallmatrix}\right)$ (i.e. objects of $\mathbf{E}$)
- as morphisms $\left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right) \leftrightarrows \left(\begin{smallmatrix} Y \\ V \end{smallmatrix}\right)$, morphisms:

$$\begin{aligned} f &: U \to V && : \mathbf{C} \\ f^\sharp &: f^*Y \to X && : p^{-1}U \end{aligned}$$

[1] To me every fibration is effectively cloven.

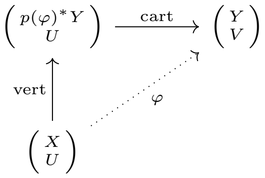
All lenses are p-lenses

	data
lenses	$s : \mathbf{S}(\mathbf{C}) \longrightarrow \mathbf{C}$
dependent lenses	$\mathrm{cod} : \mathbf{C}^{\downarrow} \longrightarrow \mathbf{C}$
p**-lenses**	$p : \mathbf{E} \longrightarrow \mathbf{C}$

Fibrations & vertical-cartesian factorization system

Definition

Any Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$ induces a factorization systems on $\mathbf{E}$ where the left morphisms are vertical morphisms ($p(f) = 1$) and the right morphisms are cartesian.



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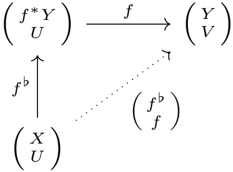

Idea: a fibred category is made of morphisms from the fibers (vertical morphisms) composed with morphisms from the base (cartesian).

p-lenses from dual fibrations

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Given a Grothendieck fibration $p : \mathbf{E} \to \mathbf{C}$, we can form its **dual** or **(fiberwise) opposite**

$$p^\vee : \mathbf{E}^\vee \longrightarrow \mathbf{C}$$

obtained by replacing each fiber with its opposite: $p^\vee = \int (p^{-1} \fatsemi (-)^{\mathsf{op}})$.

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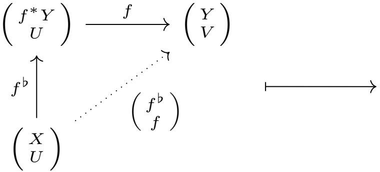

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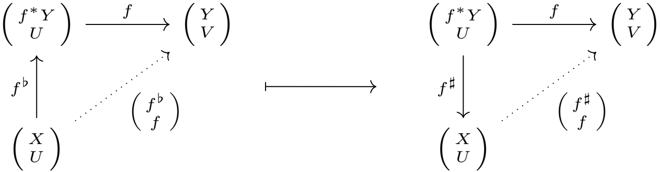

This is a p-lens!

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Hence $\mathbf{Dial}_\times(\mathbf{C})$ is a category of lenses 'augmented with predicates'.

(1) Can $\mathbf{Dial}_\times(\mathbf{C})$ be constructed in a similar way to $\mathbf{Lens}_\times(\mathbf{C})$?

(2) Can its shape be abstracted, like we did for p-lenses?

Dialectica, fibrationally

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We need a second fibration of predicates!

$$
\begin{array}{ccc}
\mathbf{P} & & `\alpha \subseteq \beta' \\
\big\downarrow{\scriptstyle q} & & \\
\mathbf{E} & & f^\sharp : f^*Y \to X \\
\big\downarrow{\scriptstyle p} & & \\
\mathbf{C} & & f : U \to V
\end{array}
$$

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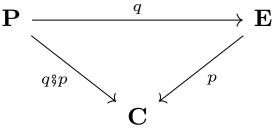
First, notice:



$$p \fatsemi q \xrightarrow{q} p : \mathbf{Fib}(\mathbf{C})$$

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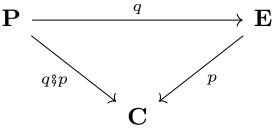

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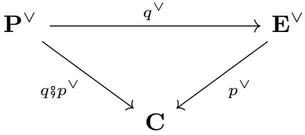
Then we notice $(-)^\vee$ is a functorial construction $\mathbf{Fib}(\mathbf{C}) \to \mathbf{Fib}(\mathbf{C})$, hence can be applied to the whole triangle:



Is this what we look for? Let's unpack.

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On $\mathbf{P}$ we had a more refined factorization system, since we have three kinds of arrows:

- q-**cartesian arrows**, which are cartesian lifts of $\mathbf{E}$-arrows, and come in two subcategories:
    - p-**cartesian arrows**, which are cartesian lifts of $\mathbf{C}$-arrows, hence cartesian $\mathbf{E}$-arrows,
    - p-**vertical arrows**, which are cartesian lifts of vertical $\mathbf{E}$-arrows,
- q-**vertical arrows**, which are in the fibers of q.

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  - p-**vertical arrows**, which are cartesian lifts of vertical $\mathbf{E}$-arrows,
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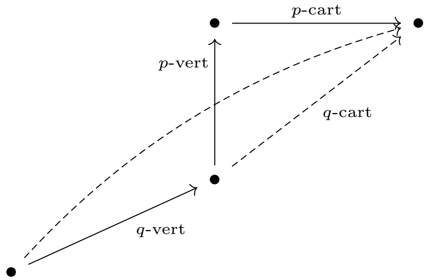
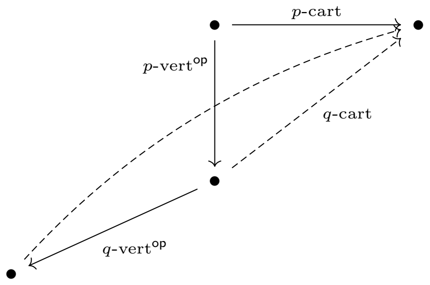
This forms a ternary factorization system **(q-vertical, p-vertical, p-cartesian)**:



Dialectica, fibrationally

When we turn around the fibers of $\mathbf{E}$, we swap q-vertical and p-vertical arrows:



Hence on $\mathbf{P}^{\vee}$ we end up with a ternary factorization system where p- and q-vertical arrows are swapped:

$$(p\text{-}\mathbf{vertical}^{\mathsf{op}},\ q\text{-}\mathbf{vertical}^{\mathsf{op}},\ p\text{-}\mathbf{cartesian})$$

Dialectica, fibrationally

We can understand this factorization system as arising from an **ambifibration** structure on $q^\vee$:

Definition

Let (L, R) be a factorization system on $\mathbf{D}$. An **ambifibration** $a : \mathbf{F} \to \mathbf{D}$ is a functor such that

- every arrow in L has an cocartesian lift (a is an opfibration on L)
- every arrow in R has a cartesian lift (a is a fibration on R)

This induces the ternary factorization system **(cocartesian, vertical, cartesian)** on $\mathbf{F}$:

$$\begin{array}{ccccccccc} \mathbf{F} & & A' & \xrightarrow[\text{opcart}]{} & \ell_* A' & \xrightarrow[\text{vert}]{} & r^* C' & \xrightarrow[\text{cart}]{} & C' \\ \big\downarrow a & & & & & & & & \\ \mathbf{D} & & A & \xrightarrow[\in L]{} & B & = & B & \xrightarrow[\in R]{} & C \end{array}$$

Dialectica, fibrationally

Then, recall the situation

$$\begin{array}{ccc} \mathbf{P}^\vee & \xrightarrow{q^\vee} & \mathbf{E}^\vee \\ & {\scriptstyle q ⨟ p^\vee}\searrow \quad \swarrow{\scriptstyle p^\vee} & \\ & \mathbf{C} & \end{array}$$

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- $q^\vee$ is a **fibration on the cartesian maps** of $\mathbf{E}$, given by q, and became an **opfibration on the vertical maps** because it acts like q^{op} there:

$$\begin{array}{c} \mathbf{P}^\vee \\ \big\downarrow {\scriptstyle q^\vee} \\ \mathbf{E}^\vee \end{array} \qquad \begin{array}{ccccccc} \begin{pmatrix} \alpha \\ X \\ U \end{pmatrix} & \xleftarrow{f^\sharp_\alpha} & \begin{pmatrix} (f^\sharp)^*\alpha \\ X \\ U \end{pmatrix} & \xrightarrow{f^{\times}} & \begin{pmatrix} f^*\beta \\ X \\ U \end{pmatrix} & \xrightarrow{f_\beta} & \begin{pmatrix} \beta \\ Y \\ V \end{pmatrix} \\ \\ \begin{pmatrix} X \\ U \end{pmatrix} & \xleftarrow{f^\sharp} & \begin{pmatrix} f^*Y \\ U \end{pmatrix} & = & \begin{pmatrix} f^*Y \\ U \end{pmatrix} & \xrightarrow{f} & \begin{pmatrix} Y \\ V \end{pmatrix} \end{array}$$

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This is very close! We have f, $f^\sharp$ and the correct boundaries for $f^\times$.

The iterated dual construction

Let $p = \mathbf{E}_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0$ be a sequence of n fibrations.

Definition

The **iterated dual construction** is defined inductively as follows:

- $n = 1$

$$\left(\mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee_1} = \mathbf{E}_1{}^{\vee_{\mathbf{E}_0}} \xrightarrow{p_1{}^{\vee_{\mathbf{E}_0}}} \mathbf{E}_0$$

- $n = k + 1$

$$\left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee_{k+1}} = \left(\mathbf{E}_{k+1} \xrightarrow{p_{k+1}} \cdots \mathbf{E}_1\right)^{\vee_{\mathbf{E}_0}\,\vee_k} \xrightarrow{p_1{}^{\vee_{\mathbf{E}_0}}} \mathbf{E}_0$$

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Example $\left(\mathbf{E}_2 \xrightarrow{p_2} \mathbf{E}_1 \xrightarrow{p_1} \mathbf{E}_0\right)^{\vee 2} = \left(\mathbf{E}_2{}^{\vee \mathbf{E}_0}\right)^{\vee \mathbf{E}_1} \xrightarrow{\left(p_2{}^{\vee \mathbf{E}_0}\right)^{\vee \mathbf{E}_1}} \mathbf{E}_1{}^{\vee \mathbf{E}_0} \xrightarrow{p_1{}^{\vee \mathbf{E}_0}} \mathbf{E}_0$

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Definition

A morphism in $\mathbf{E}_n{}^{\vee n}$ is called a p-**dialens of height** n.

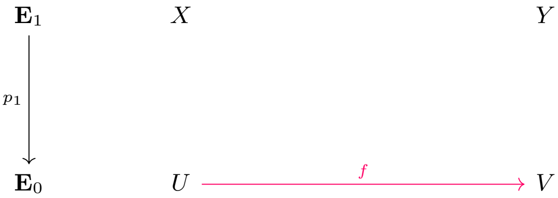
Dialens

Let's unpack the construction of a dialens of height 2:

$$\mathbf{E}_0 \qquad U \xrightarrow{\quad f \quad} V$$

Dialens

Let's unpack the construction of a dialens of height 2:



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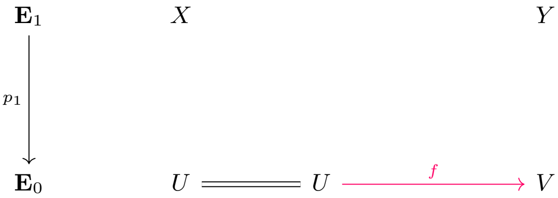

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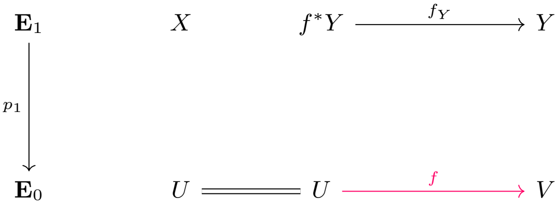

Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccc}
\mathbf{E}_1 & \qquad & X & \xrightarrow{f^\sharp} & f^*Y & \xrightarrow{\quad f_Y \quad} & Y \\
\Big\downarrow{\scriptstyle p_1} & & & & & & \\
\mathbf{E}_0 & & U & = & U & \xrightarrow{\quad f \quad} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccc}
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & \xrightarrow{f_Y} & Y \\
\Big\downarrow {\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & \\
\mathbf{E}_0 & & U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & & & & \beta \\
\Big\downarrow {\scriptstyle p_2{}^{\vee \mathbf{E}_0}} & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & \xrightarrow{f_Y} & Y \\
\Big\downarrow {\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & \\
\mathbf{E}_0 & & U & = & U & \xrightarrow{f} & V
\end{array}
$$

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\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & & & & & & \beta \\
{\scriptstyle p_2{}^{\vee \mathbf{E}_0}} \big\downarrow & & & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
{\scriptstyle p_1{}^{\vee \mathbf{E}_0}} \big\downarrow & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee\mathbf{E}_0} & & \alpha & & & & f^*\beta & \xrightarrow{f_\beta} & \beta \\
\Big\downarrow{\scriptstyle p_2{}^{\vee\mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_1{}^{\vee\mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
\Big\downarrow{\scriptstyle p_1{}^{\vee\mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dialens

Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & \xleftarrow{f^\sharp_\alpha} & (f^\sharp)^*\alpha & & f^*\beta & \xrightarrow{f_\beta} & \beta \\
\big\downarrow {\scriptstyle p_2{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
\big\downarrow {\scriptstyle p_1{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

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Let's unpack the construction of a dialens of height 2:

$$
\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee \mathbf{E}_0} & & \alpha & \xleftarrow{f^\sharp_\alpha} & (f^\sharp)^*\alpha & \xleftarrow{f^{\text{𝄪}}} & f^*\beta & \xrightarrow{f_\beta} & \beta \\
\Big\downarrow {\scriptstyle p_2{}^{\vee \mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_1{}^{\vee \mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
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\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$

Dialens

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$$
\begin{array}{ccccccccc}
\mathbf{E}_2{}^{\vee\mathbf{E}_0}{}^{\vee\mathbf{E}_1} & & \alpha & \xleftarrow{f^\sharp_\alpha} & (f^\sharp)^*\alpha & \xrightarrow{f^{\times}} & f^*\beta & \xrightarrow{f_\beta} & \beta \\
\downarrow{\scriptstyle p_2{}^{\vee\mathbf{E}_0}{}^{\vee\mathbf{E}_1}} & & & & & & & & \\
\mathbf{E}_1{}^{\vee\mathbf{E}_0} & & X & \xleftarrow{f^\sharp} & f^*Y & = & f^*Y & \xrightarrow{f_Y} & Y \\
\downarrow{\scriptstyle p_1{}^{\vee\mathbf{E}_0}} & & & & & & & & \\
\mathbf{E}_0 & & U & = & U & = & U & \xrightarrow{f} & V
\end{array}
$$


Genesis XXVIII c. 12 v

All things are p-dialenses

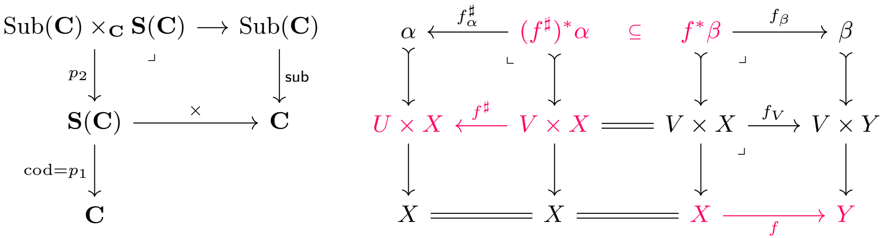
As expected, lenses are dialenses of height 1:

$$
p\text{-}\mathbf{lenses} \qquad \begin{array}{c} \mathbf{E} \\ \Big\downarrow{\scriptstyle p} \\ \mathbf{C} \end{array} \qquad \begin{array}{ccccc} U & \xleftarrow{f^\sharp} & f^*V & \xrightarrow{f_V} & V \\ \\ X & = & X & \xrightarrow[f]{} & Y \end{array}
$$

All things are p-dialenses

As expected, Dialectica morphisms are dialenses of height 2:

vanilla



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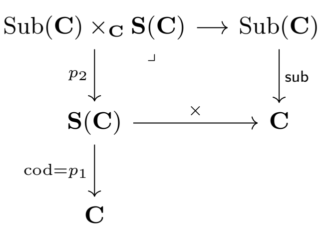
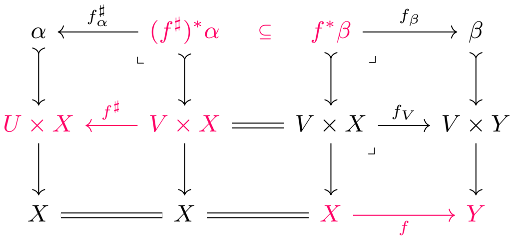
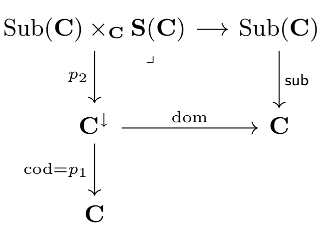
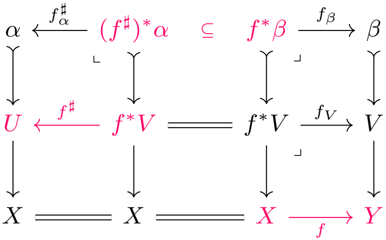




dependent





All things are p-dialenses

Optics are p-dialenses of height 2 (but using opfibrations instead of fibrations!):

(op)tics

$$
\begin{array}{ccc}
\mathbf{Para}(\circ) \times_{\mathbf{B}M} \mathbf{Para}(\bullet) & \longrightarrow & \mathbf{Para}(\circ) \\
\downarrow{\scriptstyle p_2} & \lrcorner & \downarrow{\scriptstyle \int \circ} \\
\mathbf{Para}(\bullet) & \xrightarrow{\int \bullet} & \mathbf{B}M \\
\downarrow{\scriptstyle \int \bullet = p_1} & & \\
\mathbf{B}M & &
\end{array}
$$

$$
\begin{array}{ccccccc}
U & \xrightarrow{m_U} & m \circ U & \xrightarrow{f^\sharp} & V & \overset{1_V}{=\!=\!=} & V \\
X & \xrightarrow{m_X} & m \bullet X & =\!=\!= & m \bullet X & \xleftarrow{f} & Y \\
* & \xrightarrow{m} & * & =\!=\!= & * & =\!=\!= & *
\end{array}
$$

Comparison with Hofstra's Dial monad

Hofstra defined a monad on fibrations that builds Dialectica-like categories by simple sum-product completion:

$$\mathbf{Dial}(p) = \mathbf{Fam}(\mathbf{Cofam}(p)) = \mathbf{Fam}(\mathbf{Fam}(p^{\vee})^{\vee})$$

where $p : \mathbf{P} \to \mathbf{C}$ is a fibration on $\mathbf{C}$ cartesian monoidal.

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where $p : \mathbf{P} \to \mathbf{C}$ is a fibration on $\mathbf{C}$ cartesian monoidal.

In $\mathbf{Dial}(p)$, objects are triples $(I, X, U : \mathbf{C}, \alpha : \mathbf{P}(I \times X \times U))$ and morphisms have **four** parts

$$f_0 : I \to J$$

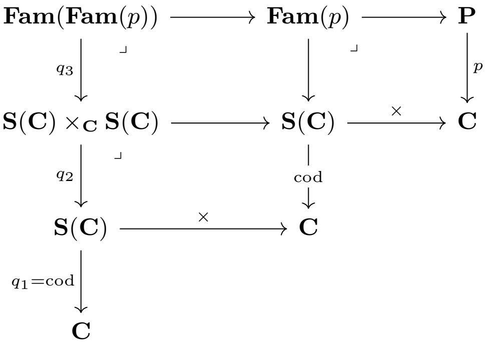
$$f : I \times X \to Y$$

$$f^\sharp : I \times X \times V \to U$$

given $i : I$, $x : X$, $v : V$, $f^{\times} : \alpha(i, x, f^\sharp(i, x, v)) \to \beta(f_0(i), f(i, x), v)$

Comparison with Hofstra's Dial monad

If we ignore the duals for a moment (they can be put back later), we see this is actually given by a sequence of **three** fibrations over $\mathbf{C}$:



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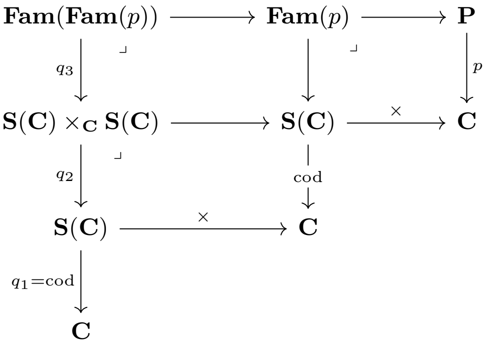

One can see $\mathbf{Dial}(p)$ is obtained by dualizing the top two:

$$\mathbf{Dial}(p) = (\mathbf{Fam}(\mathbf{Fam}(p)) \xrightarrow{q_3} \mathbf{S}(\mathbf{C}) \times_{\mathbf{C}} \mathbf{S}(\mathbf{C}) \xrightarrow{q_2} \mathbf{S}(\mathbf{C}))^{\vee_2} \xrightarrow{q_1} \mathbf{C}$$

Conclusions

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- how lenses can be constructed by dualizing fibrations

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- how lenses can be constructed by dualizing fibrations
- how Dialectica categories are augmented categories of lenses
- how Dialectica categories can be constructed as iterated duals of towers of fibrations

Thanks for your attention!