Learners are Almost Compact Closed

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For a category $\mathcal{C}$ with finite products,

Definition (Fong, Spivak, and Tuyéras 2019)

A learner $A \to B$ is an object of parameters P and maps

- $I : P \times A \to B$ (implementation)
- $U : P \times A \times B \to P$ (update)
- $r : P \times A \times B \to A$ (request)

up to isomorphism of P.

Why these maps? (Besides that it works!)

Why that quotient? (Fong and Johnson 2019)

$\mathsf{Learn}(\mathcal{C})$ is almost $\mathsf{Para}(\mathsf{Lens}(\mathcal{C}))$, per (Capucci, Gavranović, et al. 2022; Capucci, Ghani, et al. 2022)

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Munging Learners

$$\int^{P:\mathsf{Core}(\mathcal{C})} \mathcal{C}(P \times A, B) \times \mathcal{C}(P \times A \times B, P) \times \mathcal{C}(P \times A \times B, A)$$

$$\cong \int^{P:\mathsf{Core}(\mathcal{C})} \mathcal{C}(P \times A, B) \times \mathcal{C}(P \times A \times B, P \times A)$$

$$\cong \int^{P:\mathsf{Core}(\mathcal{C})} \int^{Q:\mathcal{C}} \mathcal{C}(P \times A, Q) \times \mathcal{C}(P \times A, B) \times \mathcal{C}(Q \times B, P \times A)$$

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$$
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Munging Learners

$$\int^{P:\mathcal{C}} \int^{Q:\mathcal{C}} \mathcal{C}(P \otimes A, Q \otimes B) \times \mathcal{C}(Q \otimes B, P \otimes A)$$

$$\int^{P:\mathcal{C}} \int^{Q:\mathcal{C}} \mathcal{C}(P \otimes A, Q \otimes B) \times \mathcal{C}(Q \otimes B', P \otimes A')$$

For a symmetric monoidal category $\mathcal{C}$,

Definition

A learner $(A, A') \to (B, B')$ is an element of

$$\begin{aligned}&\mathsf{Learn}_{\mathcal{C}}((A, A'), (B, B')) \\ &\quad := \int^{P,Q:\mathcal{C}} \mathcal{C}(P \otimes A, Q \otimes B) \times \mathcal{C}(Q \otimes B', P \otimes A')\end{aligned}$$

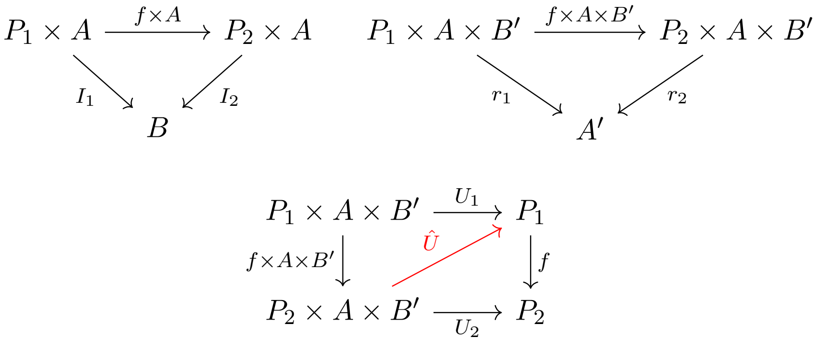
When $\otimes = \times$, this is almost the original definition.

Before: $(P_1, I_1, U_1, r_1) = (P_2, I_2, U_2, r_2)$ when there exists $f : P_1 \to P_2$ an isomorphism such that

$$\begin{array}{ccc} P_1 \times A & \xrightarrow{f \times A} & P_2 \times A \\ & {\scriptstyle I_1} \searrow \quad \swarrow {\scriptstyle I_2} & \\ & B & \end{array} \qquad \begin{array}{ccc} P_1 \times A \times B' & \xrightarrow{f \times A \times B'} & P_2 \times A \times B' \\ & {\scriptstyle r_1} \searrow \quad \swarrow {\scriptstyle r_2} & \\ & A' & \end{array}$$

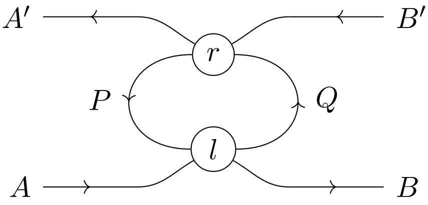
$$\begin{array}{ccc} P_1 \times A \times B' & \xrightarrow{U_1} & P_1 \\ {\scriptstyle f \times A \times B'} \downarrow & & \downarrow {\scriptstyle f} \\ P_2 \times A \times B' & \xrightarrow[U_2]{} & P_2 \end{array}$$

After: $(P_1, I_1, U_1, r_1) = (P_2, I_2, U_2, r_2)$ when there exists $f : P_1 \to P_2$ *any* map such that



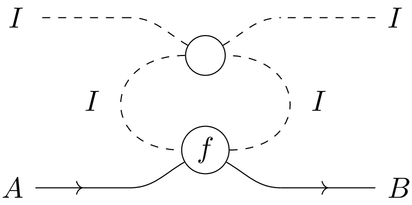
$$\int^{P,Q:\mathcal{C}} \mathcal{C}(P \otimes A, Q \otimes B) \times \mathcal{C}(Q \otimes B', P \otimes A')$$

A learner $(l \mid r) : (A, A') \to (B, B')$ is a *formal* diagram

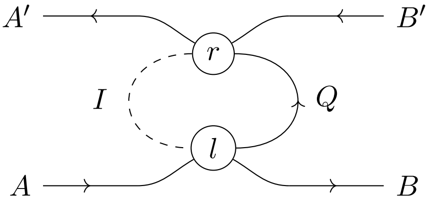


considered up to sliding.

$\iota : \mathcal{C} \to \mathsf{Learn}_{\mathcal{C}}$ is given on $f : A \to B$ by:



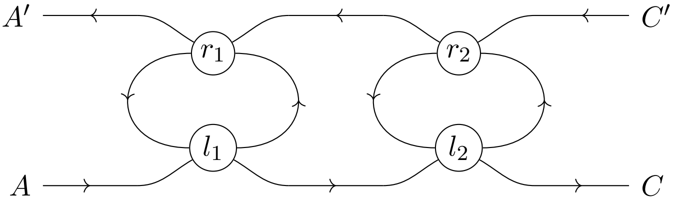
$\iota' : \mathsf{Optic}_{\mathcal{C}} \to \mathsf{Learn}_{\mathcal{C}}$ is given on $(l \mid r) : (A, A') \to (B, B')$ by:



Proposition

Learners form a symmetric monoidal category $\mathsf{Learn}_{\mathcal{C}}$.

Composition is roughly:



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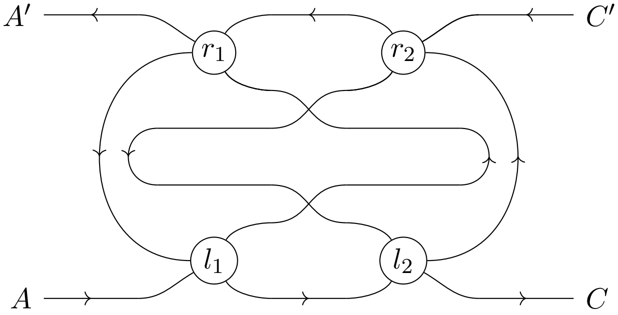
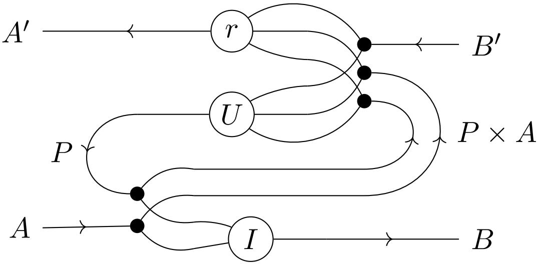
Composition is really:



A concrete learner (P, I, U, r) is represented as:

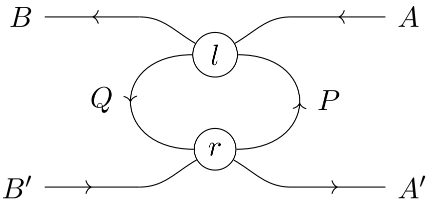


$$\int^{P,Q:\mathcal{C}} \mathcal{C}(P \otimes A, Q \otimes B) \times \mathcal{C}(Q \otimes B', P \otimes A')$$

Proposition

There is a (strict!) symmetric monoidal involution $(-)^*$ *with*

$$(A, A')^* := (A', A) \text{ and } (l \mid r)^* := (r \mid l)$$



For concrete learners:

$$
\begin{aligned}
&P \\
&I && && : P \times A \to B \\
&U && && : P \times A \times B' \to P \\
&r && && : P \times A \times B' \to A' \\
&P^* && := P \times A \\
&I^*((p, p_a), b') && := r(p, p_a, b') && : P^* \times B' \to A' \\
&U^*((p, p_a), b', a) && := (U(p, p_a, b'), a) && : P^* \times B' \times A \to P^* \\
&r^*((p, p_a), b', a) && := I(U(p, p_a, b'), a) && : P^* \times B' \times A \to B \\
&P^{**} && := P \times A \times B' \\
&I^{**}((p, p_a, p_{b'}), a) && := I(U(p, p_a, p_{b'}), a) && : P^{**} \times A \to B \\
&U^{**}((p, p_a, p_{b'}), a, b') && := (U(p, p_a, p_{b'}), a, b') && : P^{**} \times A \times B' \to P^{**} \\
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Duality

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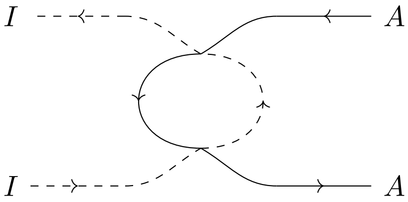
\end{aligned}
$$

Definition

For an object (A, I), define

$$\eta_{(A,I)} : (I, I) \to (A, I) \otimes (A, I)^* = (A \otimes I, A \otimes I)$$

as



and similarly for the cap $\varepsilon_{(A,I)}$.

So is $\mathsf{Learn}_{\mathcal{C}}$ compact closed?

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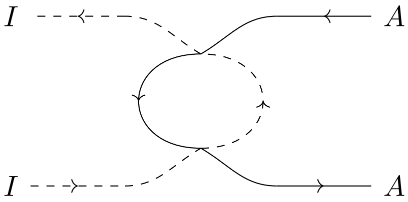

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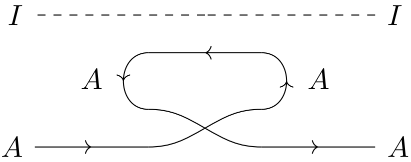
Composing the cup and the cap on (A, I) to form the snake yields:



No reason for this to be the identity!

$$\begin{aligned} P &:= A \\ I(p,a) &:= p && : P \times A \to A \\ U(p,a,b') &:= a && : P \times A \times 1 \to P \\ r(p,a,b') &:= \star && : P \times A \times 1 \to 1 \end{aligned}$$

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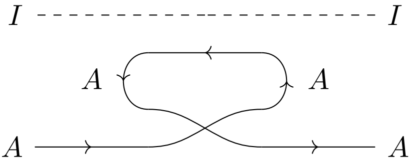

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\end{aligned}$$

Definition

The category $\mathsf{Atemp}_{\mathcal{C}}$ of *atemporal learners* is the quotient of $\mathsf{Learn}_{\mathcal{C}}$ that equates each snake with the identity.

Proposition

$\mathsf{Atemp}_{\mathcal{C}}$ *is compact closed.*

Proof (Idea).

All that remains is extranaturality of η and ε. For morphisms coming from $\mathcal{C}$ this is easy, the snake equations are enough to extend this to every morphism in $\mathsf{Atemp}_{\mathcal{C}}$. □

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For the 2-categories

- SymMon_g := symmetric monoidal categories, monoidal functors and monoidal natural *isomorphisms*,
- Comp := compact closed categories, monoidal functors and monoidal natural transformations.

Theorem

$\mathsf{Atemp}_{\mathcal{C}}$ *is the free compact closed category on a symmetric monoidal category* $\mathcal{C}$*. That is,* $\mathsf{Atemp} : \mathsf{SymMon}_g \to \mathsf{Comp}$ *assembles into a 2-functor that is left biadjoint to the inclusion* $\mathsf{Comp} \to \mathsf{SymMon}_g$.

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- $\mathsf{Comp} :=$ compact closed categories, monoidal functors and monoidal natural transformations.

Theorem

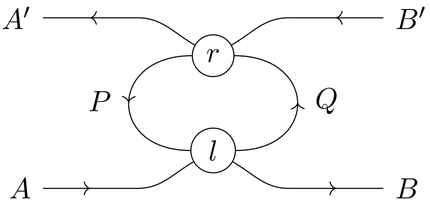
$\mathsf{Atemp}_{\mathcal{C}}$ *is the free compact closed category on a symmetric monoidal category* $\mathcal{C}$*. That is,* $\mathsf{Atemp} : \mathsf{SymMon}_g \to \mathsf{Comp}$ *assembles into a 2-functor that is left biadjoint to the inclusion* $\mathsf{Comp} \to \mathsf{SymMon}_g$.

Proof (Idea).

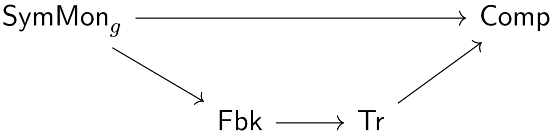
Our strategy is to show that precomposition by $\iota : \mathcal{C} \to \mathsf{Atemp}_{\mathcal{C}}$ gives an equivalence of categories

$$\mathsf{Comp}(\mathsf{Atemp}_{\mathcal{C}}, \mathcal{D}) \to \mathsf{SymMon}_g(\mathcal{C}, \mathcal{D}).$$

For this, every learner is equal to its formal diagram, when that diagram is interpreted in $\mathsf{Atemp}_{\mathcal{C}}$. Then once you decide where the morphisms from $\mathcal{C}$ go, everything is fixed. □



The free compact closed category has previously appeared:



SymMon $\to$ Tr from (Katis, Sabadini, and Walters 2002) and
Tr $\to$ Comp from (Joyal, Street, and Verity 1996)

What's Next?

- If we don't quotient the coend relation, does the resulting (bi/double)-category have a universal property?
- Does every feedback category embed faithfully in an "almost compact closed" category?
- Is there a "profunctor-style" formulation of learners?
- How can we decide equality of (atemporal) learners?
- Is remembering delay useful in practice?
- Can we mash learners into a Quipper/CHAD-style specification language?

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