Toposes of Finitely Supported M-sets

Andrew Pitts

UNIVERSITY OF CAMBRIDGE

ACT-MFPS, Oxford, June 2024

Toposes, computer science and me

Oxford 1978 (sheaves and logic)

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Later...

- the mathematics of syntax involving binders
- semantics of univalent type theories

Toposes of finitely supported M-sets for various monoids M played an unexpected role.

What is a Topos?

Category $\mathcal{E}$ with finite limits [and a natural number object]
for which every object X has $\mathrm{Sub}_{\mathcal{E}}(X \times -) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ representable

What is a Topos?

Category $\mathcal{E}$ with finite limits [and a natural number object]
for which every object X has a

power object $\varepsilon_X \rightarrowtail X \times \mathrm{P}X$

for all $R \rightarrowtail X \times Y$, there is a unique $\chi_R : Y \to \mathrm{P}X$ such that

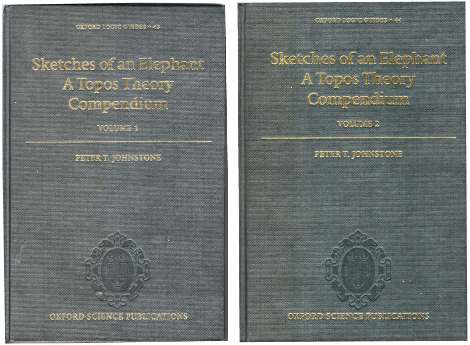
$$\begin{array}{ccc} R & \dashrightarrow & \varepsilon_X \\ \downarrow & \lrcorner & \downarrow \\ X \times Y & \xrightarrow[\mathrm{id}\times\chi_R]{} & X \times \mathrm{P}X \end{array}$$

Simple to state. Hard to satisfy!

Need a bit more for some applications, namely universes – which we are still learning about (see e.g. Gratzer, Shulmann & Sterling, *Strict Universes for Grothendieck Topoi* [arXiv:2202.12012]).

Four blind men

That definition of topos is number 2
of André Joyal's 7 answers to the question
"What is a topos?"



What is a Topos?

Logical aspect:

semantics of intuitionistic HOL / set theory / type theory

topos morphism = "logical functor" (functor preserving finite limits, NNO and powerobjects)

Geometric aspect:

toposes as generalised spaces

topos morphism = "geometric morphism" (functor with left exact left adjoint)

The category $\mathbf{Set}^{M}$

for a given monoid $\mathbf{M}$ (write operations multiplicatively).

Objects of $\mathbf{Set}^{M}$ are sets X equipped with (left-)action

$$m \in \mathbf{M}, x \in X \mapsto m \cdot x \in X$$
$$m' m \cdot x = m' \cdot (m \cdot x)$$
$$1 \cdot x = x$$

Morphisms are functions $f : X \to Y$ preserving action

$$f(m \cdot x) = m \cdot (f\, x)$$

Composition and identities as in $\mathbf{Set}$.

The topos $\mathbf{Set}^M$

[Special case of topos $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves on a small category $\mathbf{C}$, when $\mathbf{C}$ has one object.]

- Finite limits are created by $\mathbf{Set}^M \xrightarrow{\text{forget}} \mathbf{Set}$

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- Finite limits are created by $\mathbf{Set}^{\mathbf{M}} \xrightarrow{\text{forget}} \mathbf{Set}$
- Powerobject $\mathrm{P}X$ of $X \in \mathbf{Set}^{\mathbf{M}}$ consists of all subsets $S \subseteq \mathbf{M} \times X$ satisfying

$$m \Vdash x \in S \;\Rightarrow\; \forall m' \, (m'm \Vdash m' \cdot x \in S)$$

stands for $(m, x) \in S$
read as "m forces x to be in S"

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Action $m, S \mapsto m \cdot S$ is given by:

$$m' \Vdash x \in m \cdot S \;\Leftrightarrow\; m'm \Vdash x \in S$$

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Universal relation $\varepsilon_X \rightarrowtail X \times \mathrm{P}X$ is

$$\varepsilon_X = \{(x, S) \in X \times \mathrm{P}X \mid 1 \Vdash x \in S\}$$

Full transformation monoid, $\mathbf{T}_{\mathbb{A}}$

Given a set $\mathbb{A}$

$\mathbf{T}_{\mathbb{A}}$ = all functions $\mathbb{A} \to \mathbb{A}$, with monoid structure given by function composition and identity function

Support

Given submonoid $\mathbf{M} \subseteq \mathbf{T}_{\mathbb{A}}$ and given an $\mathbf{M}$-set X, define:

$x \in X$ is supported by $S \subseteq \mathbb{A}$ if

$$\forall m, m' \; (m|_S = m'|_S \Rightarrow m \cdot x = m' \cdot x)$$

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- In $\mathbb{A}$ ($\mathbf{M}$-set via function application), $a \in \mathbb{A}$ is supported by $\{a\}$.
- If S supports $x \in X$ and S' supports $x' \in X'$, then $S \cup S'$ supports (x, x') in the product $\mathbf{M}$-set $X \times X'$.
- Support in a powerobject in general has no simpler explanation than the definition.

The topos $\mathbf{Set}^{\boldsymbol{M}}_{\mathrm{fs}}$

Given submonoid $\boldsymbol{M} \subseteq \mathbf{T}_{\mathbb{A}}$

$\mathbf{Set}^{\boldsymbol{M}}_{\mathrm{fs}}$ is the full subcategory of $\mathbf{Set}^{\boldsymbol{M}}$ whose objects are the $\boldsymbol{M}$-sets X for which every $x \in X$ is supported by some finite subset $S \subseteq \mathbb{A}$

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A more honest notation would be $(\mathbf{Set}^{\mathbf{M} \subseteq \mathbf{T}_{\mathbb{A}}})_{\mathrm{fs}}$.

N.B. by the monoid version of Cayley's Theorem every monoid $\mathbf{M}$ is a submonoid of $\mathbf{T}_{\mathbb{A}}$ for some $\mathbb{A}$, namely $\mathbb{A} = \mathbf{M}$, but that might not give a useful notion of support.

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Inclusion $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}} \hookrightarrow \mathbf{Set}^{\mathbf{M}}$ creates finite limits, reflects isos and has right adjoint $(\_)_{\mathrm{fs}} : \mathbf{Set}^{\mathbf{M}} \to \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ given by $X_{\mathrm{fs}} \triangleq \{x \in X \mid x \text{ has a finite support}\}$.

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By a (co)monadicity theorem, $\mathbf{Set}^{\boldsymbol{M}}_{\mathrm{fs}}$ is equivalent to the category of coalgebras for the left exact comonad induced by the adjunction; and so by an old theorem of Lawvere & Tierney, it is a topos (with a geometric surjection from $\mathbf{Set}^{\boldsymbol{M}}$ to $\mathbf{Set}^{\boldsymbol{M}}_{\mathrm{fs}}$).

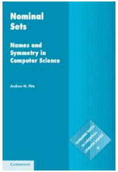
Finite limits as in $\mathbf{Set}^{\boldsymbol{M}}$; powerobject of $X \in \mathbf{Set}^{\boldsymbol{M}}_{\mathrm{fs}}$ is $(\mathrm{P}X)_{\mathrm{fs}}$.

Mathematics of syntax involving binders

If $\mathbb{A} = \mathbb{N}$ and

$\mathbf{M} \subseteq \mathbf{T}_{\mathbb{N}}$ is the symmetric group $\mathbf{S}_{\mathbb{N}}$ (or the subgroup of finite permutations, it makes no difference), then $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ is the Gabbay-AMP topos of nominal sets, equivalent to Schanuel's atomic topos classifying the geometric theory of an infinite decidable set.

Because elements of this $\mathbf{M}$ are invertible, $(\mathrm{P}X)_{\mathrm{fs}}$ simplifies to a subset of the usual powerset $\mathcal{P}X$ and $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ is a Boolean topos. It provides a rich and easily accessible and syntax-independent foundation for fresh names, name-binding, recursion and induction mod-α. Read the book.



Algorithms on slightly infinite data

The $M = \mathbf{S}_{\mathbb{N}}$ case of $\mathbf{Set}^{M}_{\mathrm{fs}}$ (and the associated Fraenkel-Mostowski cumulative hierarchy of sets) provides an instance of the notion of *slightly infinite* (orbit-finite) sets—data that is finite modulo symmetry with an interesting algorithmics.

Read the book: Mikołaj Bojańczyk, *Slightly Infinite Sets* (2019)
`mimuw.edu.pl/~bojan/paper/atom-book`

Mathematics of syntax involving binders

Categorical foundations for name-for-name substitution:

Sam Staton, *Name-Passing Process Calculi*, PhD thesis, Cambridge 2007.
Made use of a certain sheaf subcategory of presheaves on finite sets and functions.

Jamie Gabbay & Martin Hofmann, *Nominal Renaming Sets*, LPAR 2008.
Finitely supported $\mathbf{M}$-sets for $\mathbf{M} = \{m \in \mathbf{T}_{\mathbb{N}} \mid m(a) = a \text{ for all but finitely many } a \in \mathbb{N}\}$

Andrei Popescu, *Rensets and Renaming-Based Recursion for Syntax with Bindings*, IJCAR 2022.
Category of finitely supported "renaming sets" = sets X equipped with ternary operation $(\_ := \_)\_ : \mathbb{N} \times \mathbb{N} \times X \to X$ satisfying

$$
\begin{aligned}
(a := a)x &= x \\
a \neq c \Rightarrow (a := b)(a := c)x &= (a := c)x \\
(b := c)(a := b)x &= (a := c)(b := c)x \\
b \neq a' \neq a \neq b' \Rightarrow (a := b)(a' := b')x &= (a' := b')(a := b)x
\end{aligned}
$$

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Theorem. All three categories introduced above are equivalent to $\mathbf{Set}_{\mathrm{fs}}^{\mathrm{T}\mathbb{N}}$.

Proof is a corollary of work on "locally nameless sets" (AMP, POPL 2023), using some classic semigroup theory about full transformation monoids on finite sets to capture Popescu's notion of "renaming set".

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$\mathbf{S}_{\mathbb{N}} \hookrightarrow \mathbf{T}_{\mathbb{N}}$ induces a geometric morphism that makes $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{T}_{\mathbb{N}}}$ a topos defined over the topos of nominal sets. The *internal modal type theory of this relative topos* (whatever that means!) bears further investigation for applications to the mathematics of syntax.

Semantics of univalent type theories

The pursuit of models of Homotopy Type Theory (Martin-Löf Type Theory + univalence, higher inductive types, etc), especially ones with computational content, has involved [Quillen model structures on] pre-sheaf toposes.

Some of those pre-sheaf toposes turn out to be equivalent $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{M}}$ for various $\mathbb{A}$ and $\mathbf{M} \subseteq \mathbf{T}_{\mathbb{A}}$.

Thesis: developing the relevant structures and calculations may be easier "nominally" (e.g. the elements of $\mathbb{A}$ are named cartesian axes), compared with the usual possible-world Kripke-Joyal semantics for presheaves.

(**Anti-thesis**: working in the internal [modal] type theory of the topos proved to be even easier. See publications by AMP & Ian Orton.)

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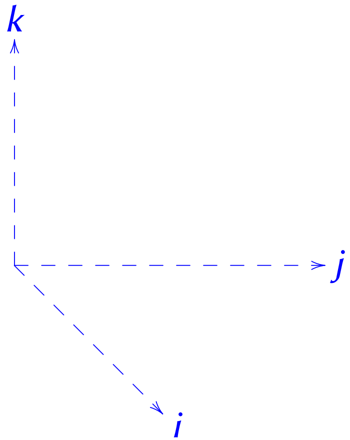
Cartesian cubical sets

Theorem. (AMP, TYPES 2014)
$\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ for $\mathbf{C} =$ (non-trivial bipointed finite sets)$^{\mathrm{op}}$ (= Grothendieck's "smallest test category") is equivalent to $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ where $\mathbf{M}$ is the monoid of all endofunctions on $\{0\} \uplus \mathbb{N} \uplus \{1\}$ that preserve 0 and 1.

From the $\mathbf{Set}^{M}_{\mathrm{fs}}$ viewpoint, cartesian cubical sets X are sets whose elements depend implicitly (via support) on finitely many named dimensions $i, j, k, \ldots \in \mathbb{N}$



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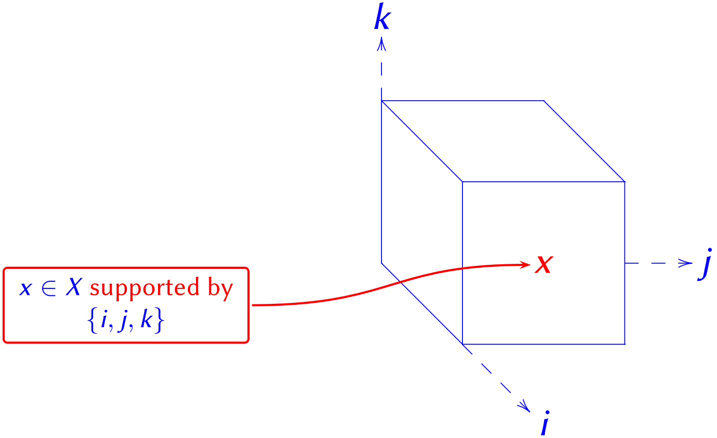

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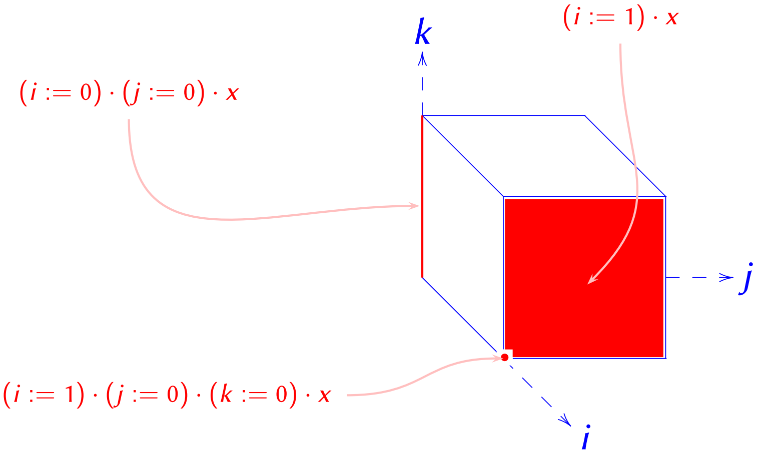

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in the version using $\mathbf{Set}^{C^{\mathrm{op}}}$
the dependency is explicit,
leading to “weakening hell”

Other flavours of cubical set

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$\mathbf{M}$ for other versions of cubical sets:

- Bezem, Coquand & Huber, *A model of type theory in cubical sets* (TYPES 2013).

  $\mathbf{M}$ = the monoid of all endofunctions m on $\{0\} \uplus \mathbb{N} \uplus \{1\}$ that preserve 0 and 1 and that are injective on $\mathbb{N} - m^{-1}\{0, 1\}$

- Cohen, Coquand, Huber & Mörtberg, *Cubical type theory: A constructive interpretation of the univalence axiom* (TYPES 2015).

  $\mathbf{M}$ = the monoid of all endomorphisms of the free de Morgan algebra on $\mathbb{N}$

Simplicial sets

Theorem. (Eric Faber, thesis, 2019) The pre-sheaf topos $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets is equivalent to $\mathbf{Set}^{M}_{\mathrm{fs}}$ where M is the monoid of order-preserving endofunctions on $\{-\infty \leq \cdots - 2 \leq -1 \leq 0 \leq 1 \leq 2 \leq \cdots \leq +\infty\}$ that preserve $-\infty$ and $+\infty$.

Reflexive globular sets

are diagrams in **Set** of shape $C_0 \; \overset{t_0}{\underset{s_0}{\leftleftarrows}} \!\!\xrightarrow{i_0}\; C_1 \; \overset{t_1}{\underset{s_1}{\leftleftarrows}} \!\!\xrightarrow{i_1}\; C_2 \; \overset{t_2}{\underset{s_2}{\leftleftarrows}} \!\!\xrightarrow{i_2}\; C_3 \; \overset{t_3}{\underset{s_3}{\leftleftarrows}} \!\!\xrightarrow{i_3}\; C_4 \quad \cdots$ satisfying

$s_n \circ i_n = \mathrm{id} = t_n \circ i_n \qquad s_n \circ s_{n+1} = s_n \circ t_{n+1} \qquad t_n \circ s_{n+1} = t_n \circ t_{n+1}$

(C_n = n-cells, s_n = source, t_n = target, i_n = identity). They are the objects of an evident pre-sheaf category.

Reflexive globular sets

are diagrams in **Set** of shape

$$C_0 \;\substack{\xleftarrow{\;t_0\;} \\ \xrightarrow{\;i_0\;} \\ \xleftarrow[\;s_0\;]{}}\; C_1 \;\substack{\xleftarrow{\;t_1\;} \\ \xrightarrow{\;i_1\;} \\ \xleftarrow[\;s_1\;]{}}\; C_2 \;\substack{\xleftarrow{\;t_2\;} \\ \xrightarrow{\;i_2\;} \\ \xleftarrow[\;s_2\;]{}}\; C_3 \;\substack{\xleftarrow{\;t_3\;} \\ \xrightarrow{\;i_3\;} \\ \xleftarrow[\;s_3\;]{}}\; C_4 \quad \cdots$$

satisfying

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Theorem. The pre-sheaf category of reflexive globular sets is equivalent to $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ for the monoid $\mathbf{M}$ whose non-identity elements

$$d^b_n \text{ (for } n \in \mathbb{N},\, b \in \{s, t\}) \quad \text{satisfy} \quad d^b_n d^{b'}_{n'} = \begin{cases} d^b_n & \text{if } n < n' \\ d^{b'}_{n'} & \text{if } n' \leq n \end{cases}$$

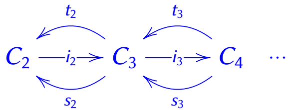
and where we regard $\mathbf{M}$ as a submonoid of $\mathbf{T}_{\mathbf{M}}$ via Cayley.

[Cf. Ross Street, *The Algebra of Oriented Simplexes*, JPAA 49(1987)283–335.]

Reflexive globular sets

$d_n^0 \cdot x$ = take the n-dimensional source of x
$d_n^1 \cdot x$ = take the n-dimensional target of x

For this monoid
x is finitely supported iff
$d_n^0 \cdot x = x$ holds for some n



satisfying

${}_{n+1} = t_n \circ t_{n+1}$

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Why bother?

We have seen that various (pre)sheaf toposes are equivalent to toposes of finitely supported **M**-sets. So what?

+ Can avoid weakening hell: possible-worlds (stages) become implicit sub-worlds of just one world (via support sets).
- Still have world-morphisms, i.e. elements of the monoid (unless the monoid is a group); and naturality conditions (but those can sometimes be avoided using (co)free functors on indexed families).
+ Some constructs look much nicer in $\mathbf{Set}_{\mathrm{fs}}^{M}$ (e.g. path types in the cubical models are given by nominal name-abstraction).

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- *Interval* $1 \underset{1}{\overset{0}{\rightrightarrows}} \mathrm{I}$ in $\mathbf{Set}^{\mathbf{M}}_{fs}$ is $\mathrm{I} = \{0\} \uplus \mathbb{N} \uplus \{1\}$ with $\mathbf{M}$-action given by function application.
- Exponential X^{I} has a simple description as an object of *paths* $[\,i\,]x$ given by a named dimension $i \in \mathbb{N}$ and an element $x \in X$, quotiented by the equivalence relation that identifies $[\,i\,]x$ with $[\,j\,]y$ iff $(i := k) \cdot x = (j := k) \cdot y$ for some/any k not in the support of (i, x, j, y) (cf. α-equivalence!).
- The interval is tiny and the right adjoint to $X \mapsto X^{\mathrm{I}}$ has a simple description (omitted)—used to construct univalent universes of Coquand-fibrations.

Some to-dos

- Fully develop the cartesian cubical set model of HoTT using finitely supported $\mathbb{M}$-sets, e.g. by translating Steve Awodey's recent extensive pre-sheaf based account.

  ["Cartesian Cubical Model Categories", arXiv:2305.00893]

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- Take the forcing notation ($m \Vdash x \in S$) seriously. Adapt Dana Scott's pre-sheaf model of IZF

OXFORD
February 1980

THE PRESHEAF MODEL FOR SET THEORY

by
Dana Scott

The whole discussion is just the putting together of two-plus-two from known facts in topos theory, but it is a useful exercise for me to get various things straight.

§1. The Construction. Let $\mathbb{C}$ be a fixed small category, called the <u>site</u>. It has <u>domains</u> (objects, types) A, B, C and <u>maps</u> $f: B \to A$, $g: C \to B$, etc. <u>Composition</u> $f \circ g: C \to A$ is written in the indicated order. The <u>identity</u> map on a domain A is written as 1_A. The usual axioms are satisfied about composition and identities. That $\mathbb{C}$ is small means the number of domains in $\mathbb{C}$ is limited and, for domains A, B, the collection $\{f \mid f: B \to A\}$ is always a set.

In making the model, we will often have need of a notation for functions (sets of ordered pairs). Thus:

$$(x_i)_{i \in I} = \{(i, x_i) \mid i \in I\},$$

where an ordered pair has $(a,b) = \{\{a\},\{a,b\}\}$. Note $(a,b) \neq \emptyset$. Therefore, if we also use the notation:

$$\langle y_j \rangle_{j \in J} = \{\emptyset\} \cup (y_j)_{j \in J},$$

then always $(x_i)_{i \in I} \neq \langle y_j \rangle_{j \in J}$. That is to say, we have functions (vectors, systems, families) in two "colours".

DEFINITION 1.1. Let A be a domain of $\mathbb{C}$. An <u>individual</u> (at stage A) is a system

$$a = (a_f)_{f: B \to A}$$

of arbitrary things indexed by $\{f \mid f: B \to A\}$. <u>Restriction</u> along a map $f: B \to A$ of $\mathbb{C}$ is given by:

$$a \restriction f = (a_{f \circ g})_{g: C \to B}.$$

If we let I_A be the <u>class</u> of all individuals at stage A, then $a \restriction f \in I_B$.

The notion of a set-valued pre-sheaf is assumed known; it is a functor $\mathcal{F}$ from $\mathbb{C}^{op}$ into Sets. If A is in $\mathbb{C}$, then $\mathcal{F}(A)$ is a set; and if $f: B \to A$ in $\mathbb{C}$, then $\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(B)$ is a function. We can define for $a \in \mathcal{F}(A)$ the family:

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