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No-go Theorems: Polynomial Comonads That do Not Distribute over Distribution Monads

Amin Karamlou, Nihil Shah

Preliminaries

Monads and comonads

A **monad** is a triple (M, η, μ) where:

1. $M\colon C \to C$ is an endofunctor.
2. The **unit** $\eta\colon id_C \to M$ is a natural transformation.
3. The **multiplication** $\mu\colon M^2 \to M$ is a natural transformation.

And the following equations hold:

$$\mu \circ M\mu = \mu \circ \mu M; \quad \mu \circ M\eta = \mu \circ \eta M = id_M$$

A **comonad** is a triple (W, ϵ, δ) where:

1. $W\colon C \to C$ is an endofunctor.
2. The **counit** $\eta\colon W \to id_C$ is a natural transformation.
3. The **compultiplication** $\delta\colon M \to M^2$ is a natural transformation.

And the following equations hold:

$$W\delta \circ \delta = \delta W \circ \delta; \quad W\epsilon \circ \delta = \epsilon W \circ \delta = id_W$$

Monads and comonads

Monads model effectful computation e.g. nondeterminism, probabilities.

An effectful computation from A to B is represented as a morphism $A \to MB$.

Comonads model contextual computation e.g. list prefixes, tree nodes.

A contextual computation from A to B is represented as a morphisms $WA \to B$.

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Natural question: When can computations that are both contextual, and effectful be modelled as morphisms in a suitable category?

Monads and comonads

- $Obj(Kl(M)) = Obj(C)$
- $Hom_{Kl(M)}(A, B) = Hom(A, MB)$
- $id_x = \eta_x$
- $g \circ f = X \xrightarrow{f} MY \xrightarrow{g^*} MZ$ where:
  - $f : X \to MY$ and $g : Y \to MZ$
  - $g^* = \mu_z \circ Mg$

- $Obj(coKl(W)) = Obj(C)$
- $Hom_{coKl(W)}(A, B) = Hom(WA, B)$
- $id_x = \epsilon_x$
- $g \circ f = WX \xrightarrow{f^*} WY \xrightarrow{g} Z$ where:
  - $f : WX \to Y$ and $g : WY \to Z$
  - $f^* = Wf \circ \delta_x$

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Distributive laws

A (mixed) **distributive law** of a comonad (W, ϵ, δ) over a monad (M, η, μ) is a natural transformation $\lambda\colon\ W \circ M \Rightarrow M \circ W$ satisfying four axioms:

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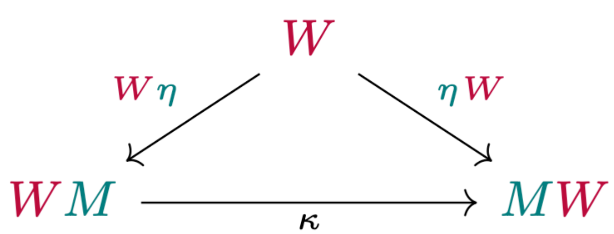
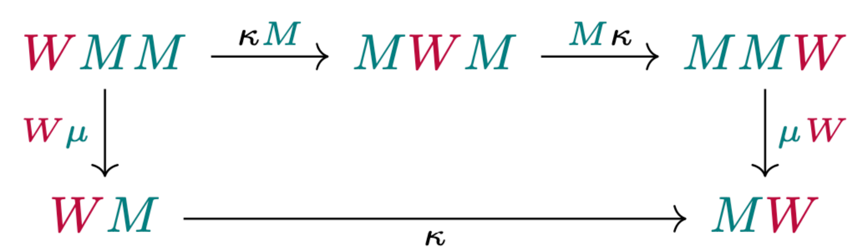
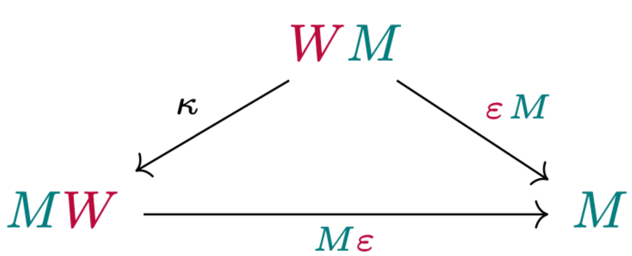
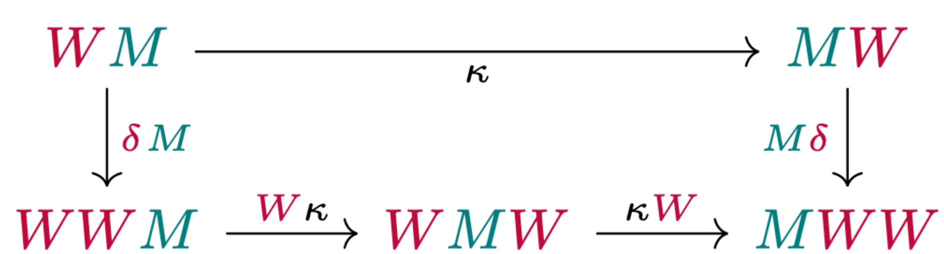


Multiplication





Counit



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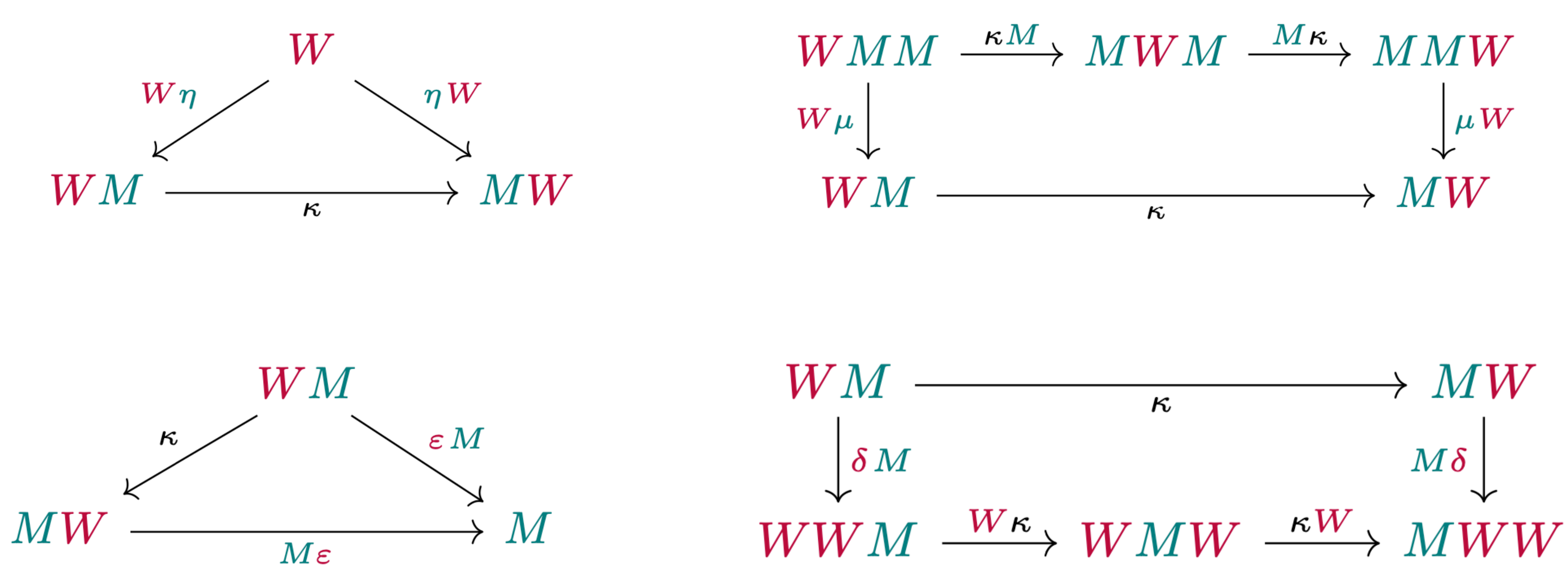

Counit **Comultiplication**

Note that each axiom can be satisfied independently. In particular, we say that there exists a **pointed law** between W and M whenever the unit axiom alone is satisfied.

BiKleisli categories

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  - $g \circ f = WX \xrightarrow{f^*} WMY \xrightarrow{\lambda_Y} MWY \xrightarrow{g^*} MZ$ where:

- $biKl(W, M)$ can be seen as the Kleisli category of M lifted to $coKl(W)$, or equivalently as the coKleisli category of W lifted to $Kl(M)$.

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No-go theorems

- Distributive laws are not guaranteed to exist, and even when they do, finding them is often difficult.

- A result attributed to Plotkin shows that the powerset monad does not distribute over the distribution monad.

- [1] Vastly generalises this result to present several families of no-go-theorems for when the existence of distributive laws between pairs of monads is impossible.

Our contribution: First examples of no-go results for comonad-monad distributive laws.

Inspired by an open question in the game comonads literature:
Do the game comonads G_k distribute over the quantum monad Q_d?

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[1] Zwart, Maaike, and Dan Marsden. "No-go theorems for distributive laws." In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1-13. IEEE, 2019.

Outline

1. Plotkin Style no-go theorem for prefix lists and powerset.

2. Generalising the comonads to almost all polynomial comonads.

3. Generalising the monads to “choice” monads.

4. Transfer theorems for generalising to (co)monads in other categories.

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1. Plotkin Style Argument

First no-go result

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- The powerset monad (P, η, μ) on $\mathbf{SET}$ is given by:
  1. $P(X)$ is the set of subsets of X.
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- The prefix list comonad (N, ϵ, δ) on $\mathbf{SET}$ is given by:
  1. $N(X)$ is the set of all non-empty lists over X.
  2. $\epsilon_X[x_1, \ldots, x_n] = x_n$.
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Theorem: There is no distributive law of the comonad (N, ϵ, δ) over the monad (P, η, μ)

Proof Sketch

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Key lemma: There is a unique pointed endofunctor law $N \circ P \Rightarrow P \circ N$ given by:

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$$[\{a,b\},\{c\}] \xrightarrow{\lambda_X} \{[a,c],[b,c]\} \xrightarrow{P\delta_X} \{[[a],[a,c]],[[b],[b,c]]\}$$

$$[\{a,b\},\{c\}] \xrightarrow{\delta_{PX}} [[\{a,b\}],[\{a,b\},\{c\}]]$$

$$\neq$$

$$[[\{a,b\}],[\{a,b\},\{c\}]] \xrightarrow{N\lambda_X} [\{[a],[b]\},\{[a,c],[b,c]\}] \xrightarrow{\lambda_{NX}} \{[[a],[a,c]],[[b],[a,c]],[[a],[b,c]],[[b],[b,c]]\},$$

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$$[[\{a,b\}],[\{a,b\},\{c\}]] \xrightarrow[N\lambda_X]{} [\{[a],[b]\},\{[a,c],[b,c]\}] \xrightarrow[\lambda_{NX}]{} \{[[a],[a,c]],[[b],[a,c]],[[a],[b,c]],[[b],[b,c]]\},$$

In fact, this shows the stronger statement that there is no natural transformation which satisfies both the unit and comultiplication axioms.

2. Generalising the Comonad

Containers

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- Containers are functors $F = [S \triangleleft P] : \mathbf{SET} \to \mathbf{SET}$ which have an associated set of shapes S and a set of positions $P(s)$ for each shape $s \in S$.

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  1. $[S \triangleleft P]X = \{(s, l) \mid s \in S, l \colon P(s) \to X\}$
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[1] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Categories of containers. In International Conference on Foundations of Software Science and Computation Structures, pages 23–38. Springer, 2003.

Examples

- N is a container where $S = Nat$ and $P(s) = \{1,2,\ldots,s\}$.

- The labelled binary suffix tree comonad $(B, \epsilon^B, \delta^B)$ on $\mathbf{SET}$ is given by:
  1. $B(X)$ is the set of all binary trees with nodes labelled by elements of X.
  2. $\epsilon_X(t)$ returns the root node of t.
  3. $\delta_X(t)$ replaces each node of t with the subtree rooted at that node.

- The pointed list comonad $(N^*, \epsilon^*, \delta^*)$ on $\mathbf{SET}$ is given by:
  1. $N^*(X)$ is the set of all pointed lists over X. A pointed list is a tuple $(\mathbf{L}, i)$ where $\mathbf{L}$ is a list and i refers to an index of $\mathbf{L}$.
  2. $\epsilon^*_X([x_1, \ldots, x_n], i) = x_i$.
  3. $\delta^*_X(\mathbf{L}, i) = ([(\mathbf{L},1), (\mathbf{L},2), \ldots, (\mathbf{L}, n)], i)$.

Examples

- Given k pebbles, the pebble list comonad $(N^k, \epsilon^k, \delta^k)$ on $\mathbf{SET}$ is given by:
  1. $N^k(X)$ is the set of non-empty list of *moves* (p, x) where $p \in [k], x \in X$.x
  2. $\epsilon^k_X[(p_1, x_1), \ldots, (p_n, x_n)] = x_n$.
  3. $\delta^k_X[(p_1, x_1), \ldots, (p_n, x_n)] = [(p_1, \mathbf{L_1}), \ldots, (p_n, \mathbf{L}_n)]$ where $\mathbf{L_i} = [(p_1, x_1), \ldots, (p_i, x_i)]$

- For a given set S, the coreader comonad $(S \times (-), \epsilon, \delta)$
  1. $S \times (-)$ is the product by S endofunctor which has a single position for each shape $s \in S$.
  2. $\epsilon_X(s, x) = x$.
  3. $\delta_X(s, x) = (s, (s, x))$.

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Theorem: Every container has a unique pointed law $[S \lhd P] \circ P \Rightarrow P \circ [S \lhd P]$ given by:

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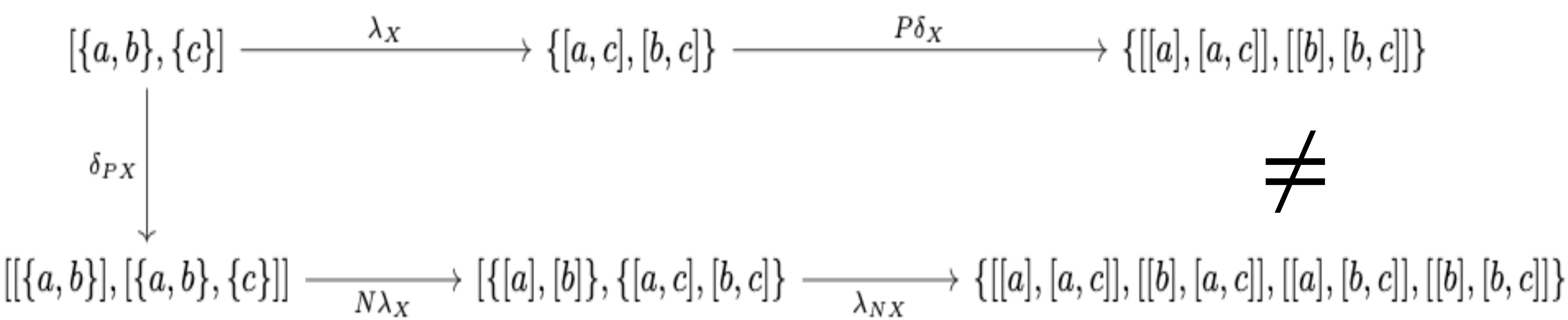

By noticing that our Plotkin style argument is essentially a cardinality argument we can generalise our no go-theorem to almost all directed containers.

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Theorem: Let (W, ϵ, δ) be a directed container with $W = [S \triangleleft P]$. W is the coreader comonad on S if and only if it has a distributive law over (P, η, μ).

3. Generalising the Monad

Uniform Choice Monads

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Definition 5.1. *Given a pointed endofunctor* (M, η) *with a natural transformation* **supp**: $M \to \mathcal{P}$, *a n-ary term* $\beta \colon \mathbf{Id}_{\mathbf{Set}} \times \cdots \times \mathbf{Id}_{\mathbf{Set}} \to M$ *for* M *is a n-*uniform choice term *if*

(1) β is idempotent: *For all* $X \in \mathbf{Set}$ *and* $x \in X$,

$$\beta(x, \ldots, x) = \eta(x);$$

(2) β is commutative: *For all* $X \in \mathbf{Set}$, $x_1, \ldots, x_n \in X$, *and permutations* $\pi \colon [\mathrm{n}] \to [\mathrm{n}]$,

$$\beta(x_1, \ldots, x_n) = \beta(x_{\pi(1)}, \ldots, x_{\pi(n)});$$

(3) **supp** preserves β: *For all* $X \in \mathbf{Set}$ *and* $x_1, \ldots, x_n \in X$,

$$\mathbf{supp}(\beta(x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}.$$

Examples

[1] Leinster, Tom. "Codensity and the ultrafilter monad." *arXiv preprint arXiv:1209.3606* (2012).

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- The distribution monad (D, η^D, μ^D) is a monad on $\mathbf{SET}$ given by:
  1. $D(X) = \{\varphi : X \to [0,1] \mid \text{supp}(\varphi) \text{ is finite}\}$ satisfying $\sum_i s_i = 1$.
  2. $\eta_X(x) = 1.x$
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- The filter monad (F, η, μ) on $\mathbf{SET}$ is another example. It is a bit cumbersome to describe so I omit the details. See e.g. [1]

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Third no-go theorem

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Theorem: If $F = [S \triangleleft P]$ is a container, (M, η, μ) is an n-uniform choice monad, and there exists a pointed endofunctor law $\lambda{:}FM{\to}MF$, then λ is “unique up to supports”:

$$supp(\lambda_X(s,l)) = \{(s, j \colon P(s) \to X) \mid \forall p \in P(s), j(p) \in supp(l(p))\}$$

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Theorem: If $(W, \epsilon, \delta) \in C_W$ and $(M, \eta, \mu) \in C_M$ then there is no distributive law.

4. Transfer Theorems

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- These (co)monads are intimately linked to longstanding techniques in computer science for studying relaxations of graph homomorphism and graph isomorphism.

- The existence of suitable distributive laws between them would allow us to consider ways of combining comonadic and monadic relaxations.

(Co)monads on graphs

[1] Abramsky, Samson, and Nihil Shah. "Relating structure and power: Comonadic semantics for computational resources." *Journal of Logic and Computation* 31.6 (2021): 1390-1428.

[2] Connolly, Adam. Game comonads and beyond: compositional constructions for logic and algorithms. Diss. 2023.

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(Co)monads on graphs

- The Ehrenfeucht–Fraïssé comonad [1] (E, ϵ, δ) on $\mathbf{GRAPH}$ is given by:
  1. E_X is the set of all non-empty lists over X.
  2. $\epsilon_X[x_1, \ldots, x_n] = x_n$.
  3. $\delta_X[x_1, \ldots, x_n] = [[x_1], [x_1, x_2], \ldots, [x_1, x_2, \ldots x_n]]$.
  4. $Edge([x_1, \ldots, x_n], [y_1, \ldots, y_m])$ iff
     a. One list is a prefix of the other.
     b. $Edge(x_n, y_m)$.

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- The BLP monad [2] (B, η, μ) on $\mathbf{GRAPH}$ is given by:
  1. $B(X)$ is the set of subsets of X.
  2. $\eta_X(x)$ is the singleton set $\{x\}$.
  3. μ_X takes a union of sets.
  4. Action on edges is a bit complicated to describe (but not terribly important).

[1] Abramsky, Samson, and Nihil Shah. "Relating structure and power: Comonadic semantics for computational resources." *Journal of Logic and Computation* 31.6 (2021): 1390-1428.

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Intuitive argument

[1] Manes, Ernie, and Philip Mulry. "Monad compositions. I: General constructions and recursive distributive laws." *Theory and Applications of Categories [electronic only]* 18 (2007): 172-208.

Intuitive argument

- Clearly, there are significant similarities between E and N. Likewise, there are similarities between B and P.

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- Clearly, there are significant similarities between E and N. Likewise, there are similarities between B and P.
- We would like to argue that if there is no distributive law "at the level of sets" then there is no hope of having a distributive law for graphs.

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Intuitive argument

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- We would like to argue that if there is no distributive law "at the level of sets" then there is no hope of having a distributive law for graphs.
- We use two transfer theorems to formalise this argument. These generalise an earlier theorems of Manes and Mulry [1].

[1] Manes, Ernie, and Philip Mulry. "Monad compositions. I: General constructions and recursive distributive laws." *Theory and Applications of Categories [electronic only]* 18 (2007): 172-208.

Transfer Theorem 1

(1) There exists categories $\mathfrak{C}, \mathfrak{D}$ with a coreflective adjunction $L : \mathfrak{C} \to \mathfrak{D} \dashv U : \mathfrak{D} \to C$ between them. We write α, β for the unit and counit of this adjunction.
(2) $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$, $(W, \varepsilon^{W}, \delta^{W})$ are comonads over $\mathfrak{D}, \mathfrak{C}$ respectively.
(3) $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$, (M, η^{M}, μ^{M}) are monads over $\mathfrak{D}, \mathfrak{C}$ respectively.

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Theorem 6.5. *Assume the following:*

(1) There exists a coKleisli law $w : WU \to U\mathbb{W}$.

(2) There exists a Kleisli law $m : U\mathbb{M} \to MU$.

(3) $\rho : \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ and $\rho' : WM \to MW$ are natural transformations satisfying the following "Yang-Baxter" condition:

$$Mw \circ \rho' U \circ Wm = m\mathbb{W} \circ U\rho \circ w\mathbb{M}$$

Then we have:

(1) If ρ is a distributive law, m is epic, and w is monic, then ρ' is a distributive law.

(2) If ρ' is a distributive law, m is monic, and w is epic, then ρ is a distributive law.

Transfer Theorem 2

Theorem 6.7. *Assume the following:*

(1) *There exists a split epic natural transformation* $w : WU \longrightarrow U\mathbb{W}$. *We write* w^- *for the section of* w.

(2) *There exists a split monic natural transformation* $m : U\mathbb{M} \longrightarrow MU$. *We write* m^- *for the retraction of* m.

(3) $\rho : \mathbb{W}\mathbb{M} \longrightarrow \mathbb{M}\mathbb{W}$ *is a natural transformation.*

Then, ρ *together with the natural transformation* $\rho' : WM \longrightarrow MW$ *defined as* $\rho' = MW\alpha^{-1} \circ Mw^- L \circ m\mathbb{W}L \circ U\rho L \circ w\mathbb{M}L \circ Wm^- L \circ WM\alpha$, *satisfy the following "Yang-Baxter" equation:*

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3. $\rho : \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ *is a natural transformation.*

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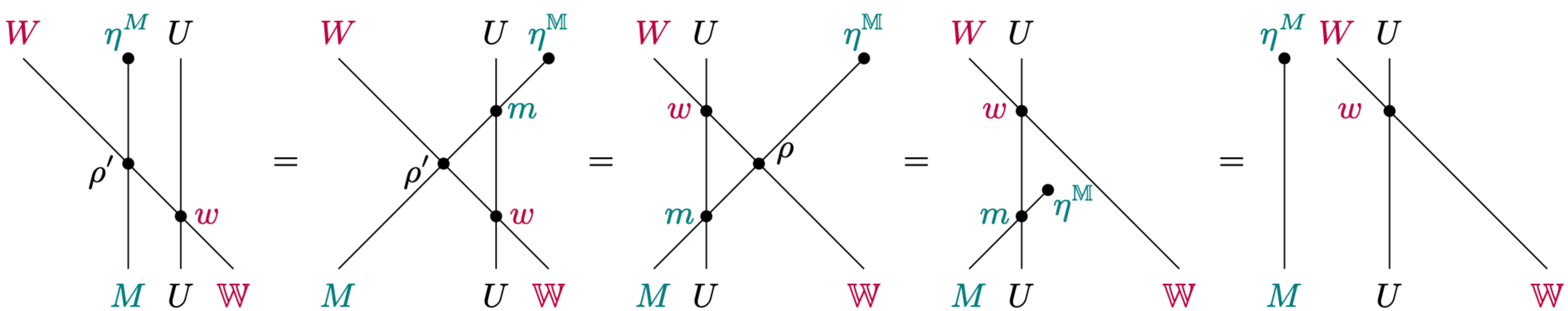
Both theorems admit elegant string diagrammatic proofs!

String diagrams



[1] Hinze, Ralf, and Dan Marsden. Introducing string diagrams: the art of category theory. Cambridge University Press, 2023.

Most general no-go theorem

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Most general no-go theorem

$C_{\mathbb{M}}(\mathfrak{D})$ denotes the class of monads $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}})$ on $\mathfrak{D}$ with an isomorphic Kleisli law $m : U\mathbb{M} \to MU$ for some $M \in C_M$.

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$\mathcal{C}_{\mathbb{W}}(\mathfrak{D})$ denotes the class of comonads $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}})$ on $\mathfrak{D}$ with an isomorphic coKleisli law $w : WU \to U\mathbb{W}$ for some $W \in \mathcal{C}_W$.

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Theorem 6.8. *If* $(\mathbb{W}, \varepsilon^{\mathbb{W}}, \delta^{\mathbb{W}}) \in \mathcal{C}_{\mathbb{W}}\mathfrak{D}$ *and* $(\mathbb{M}, \eta^{\mathbb{M}}, \mu^{\mathbb{M}}) \in \mathcal{C}_{\mathbb{M}}\mathfrak{D}$, *then there is no distributive law* $\rho : \mathbb{W}\mathbb{M} \to \mathbb{M}\mathbb{W}$ *of* (W, ε, δ) *over* (M, η, μ).

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Corollary: There is no distributive law of (E, ϵ, δ) over (B, η, μ).

5. It's not entirely hopeless!
(Work in progress)

Grading the comonads

- Many of the functors we have been considering come equipped with a natural notion of grading.
- For instance, N can be graded by the natural numbers, giving rise to a family of functors N_k each of which sends a set X to non-empty lists of length less than or equal to k filled with elements of X.
- This allows us to define a graded comonad N_k. We can then derive a graded distributive law of N_k over P.
- This seems to be a way of recovering compositionality at the cost of using more resources

[1] Gaboardi, Marco, et al. "Combining effects and coeffects via grading." *ACM SIGPLAN Notices* 51.9 (2016): 476-489.