Categorified Path Calculus

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Motivation 1:

can we have string diagrams for

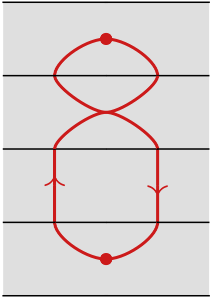
both $\otimes$ and $\oplus$

?

Motivation 1: both $\otimes$ and $\oplus$?

$$\mathrm{Tr}(A) := I \xrightarrow{\eta_A} A^* \otimes A \xrightarrow{\sigma^{\otimes}} A \otimes A^* \xrightarrow{\epsilon_A} I$$

$$\mathrm{Tr}(A) =$$



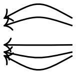
Motivation 2:
what is an amplitude
(in quantum mechanics)
?

Path counting

Feynman Lectures on Physics, Vol. III (1965), §3-1

> We wish now to put this result in terms of our new notation. First, however, we want to state our *second general principle* of quantum mechanics: When a particle can reach a given state by two possible routes, the total amplitude for the process is the *sum of the amplitudes* for the two routes considered separately. In our new notation we write that
>
> $$\langle x \mid s \rangle_{\text{both holes open}} = \langle x \mid s \rangle_{\text{through 1}} + \langle x \mid s \rangle_{\text{through 2}}. \tag{3.4}$$

Parallel paths add:

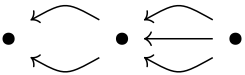
$2 + 3 = \bullet$  $\bullet = 5,$

Path counting

Feynman Lectures on Physics, Vol. III (1965), §3-1

> by way of hole 1. We can do that by using our *third general principle*: When a particle goes by some particular route the amplitude for that route can be written as the *product* of the *amplitude* to go part way with the *amplitude* to go the rest of the way. For the setup of Fig. 3–1 the amplitude to go from s to x by way of hole 1 is equal to the amplitude to go from s to 1, multiplied by the amplitude to go from 1 to x.
>
> $$\langle x \mid s \rangle_{\text{via } 1} = \langle x \mid 1 \rangle \langle 1 \mid s \rangle. \tag{3.5}$$

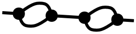
Serial paths multiply:

$2 \times 3 =$  $= 6.$

Path counting

String diagrams:

$1 + 1 =$  $= 2.$

$2 \times 2 =$  $=$  $= 4.$

Crucial use of *distributivity*: $(a + b)c = ac + bc.$

[*Composing PROPs*, Lack, 2004]
[*Categories in Control*, Baez, Erbele, 2015]
[*Interacting Hopf Algebras*, Bonchi, Sobociński, Zanasi, 2017]
[*Quantum Linear Optics via String Diagrams*, de Felice, Coecke, 2023]

Path counting

Feynman Lectures on Physics, Vol. III (1965), §8-1

> One thinks of the bracket $\langle \chi | \phi \rangle$ as being divided into two pieces. The second piece $| \phi \rangle$ is often called a *ket*, and the first piece $\langle \chi |$ is called a *bra* (put together, they make a "bra-ket"—a notation proposed by Dirac); the half-symbols $| \phi \rangle$ and $\langle \chi |$ are also called *state vectors*. In any case, they are *not* numbers, and, in general, we want the results of our calculations to come out as numbers; so such "unfinished" quantities are only part-way steps in our calculations.

$\langle \chi |$ $\qquad$ $| \phi \rangle$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Path counting

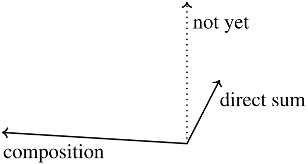
$\langle \chi | \phi \rangle$

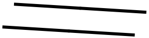
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

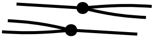
$|\phi\rangle\langle\chi|$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Path counting







$$\begin{bmatrix}1\end{bmatrix} \oplus \begin{bmatrix}1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1\\ 1\end{bmatrix} \oplus \begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$$

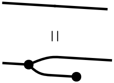
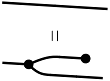
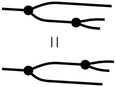
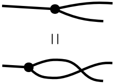
Path counting

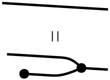
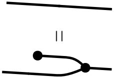
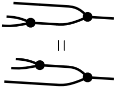
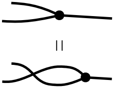
Counit, unit and empty:

$$\begin{bmatrix} \; \end{bmatrix}_{0,1} \qquad \begin{bmatrix} \; \end{bmatrix}_{1,0} \qquad \begin{bmatrix} \; \end{bmatrix}_{0,0}$$

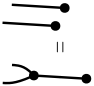
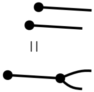
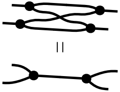
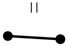
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} \; \end{bmatrix}_{1,0} \right) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

Path calculus

(l-unit)		(r-unit)	
(assoc)		(comm)	

(l-counit)		(r-counit)	
(coassoc)		(cocomm)	

Path calculus

(comul-unit) 	(counit-mul) 
(bimonoid) 	(counit-unit) 

Punchline: equivalent to the theory of matrices with natural number entries,

$$\mathbf{Mat}(\mathbb{N}, +, \times, 0, 1)$$

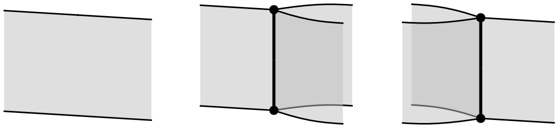
Path calculus... categorified

$$\mathsf{Mat}(\mathbb{N}, +, \times, 0, 1)$$

$$\rightsquigarrow$$

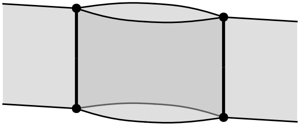
$$\mathsf{Mat}(\mathbf{FVec}, \oplus, \otimes, O, I)$$

Path calculus... categorified

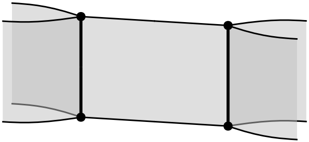


$[I]$ $\quad$ $\begin{bmatrix} I & I \end{bmatrix}$ $\quad$ $\begin{bmatrix} I \\ I \end{bmatrix}$

Path calculus... categorified

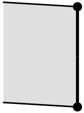


$$[I \;\; I] \begin{bmatrix} I \\ I \end{bmatrix} = [I \otimes I \oplus I \otimes I] \cong [I \oplus I]$$

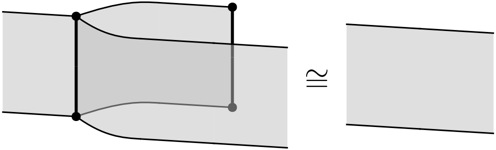


$$\begin{bmatrix} I \\ I \end{bmatrix} [I \;\; I] = \begin{bmatrix} I \otimes I & I \otimes I \\ I \otimes I & I \otimes I \end{bmatrix} \cong \begin{bmatrix} I & I \\ I & I \end{bmatrix}$$

Path calculus... categorified



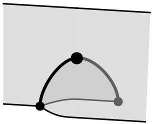
$[\;]_{1,0}$



$$[I\;\;I]\begin{bmatrix} I \\ O \end{bmatrix} = [I \otimes I \oplus I \otimes O] \cong [I]$$

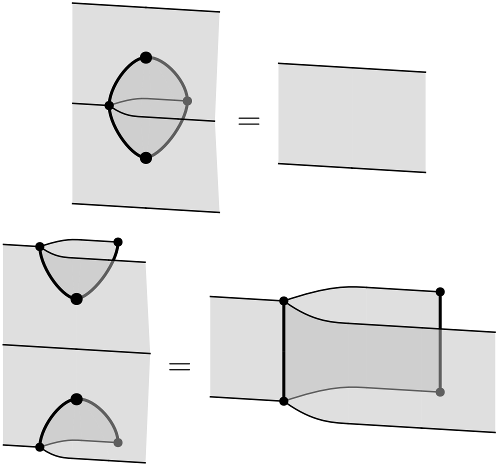
Path calculus... categorified

Coning (`r-unit`):



$$\begin{array}{c} [I] \\ \uparrow \cong \\ [I \;\; I]\begin{bmatrix} I \\ O \end{bmatrix} \end{array}$$

Path calculus... categorified

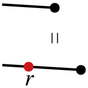
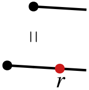
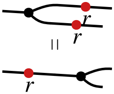
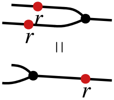


Returning to path calculus

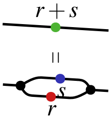
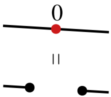
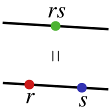
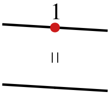
Given any semi-ring, or rig, $(R, +, \times, 0, 1)$ we introduce the diagrammatic generators:

for each $r \in R$.

Returning to path calculus

(unit-hom)		(counit-hom)	
(mul-hom)		(comul-hom)	

Returning to path calculus

`(add)` 	`(zero)` 
`(mul)` 	`(one)` 

Punchline: we get a calculus for

$$\mathsf{Mat}(R, +, \times, 0, 1)$$

Categorified path calculus

$$\mathtt{Mat}(R, +, \times, 0, 1)$$

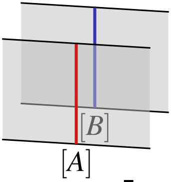
$$\rightsquigarrow$$

$$\mathtt{Mat}(\mathcal{C}, \oplus, \otimes, O, I, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \sigma^{\oplus}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, ...)$$

Crucial use of *distributivity*: $(A \oplus B) \otimes C \xrightarrow{\cong} A \otimes C \oplus B \otimes C.$

[*Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory*, Johnson, Yau, 2021]

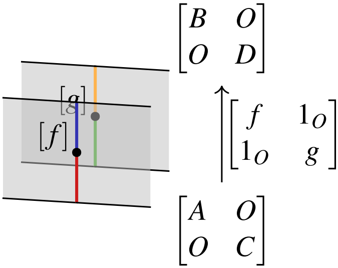
Categorified path calculus



$$[A] \boxplus [B] := \begin{bmatrix} A & O \\ O & B \end{bmatrix}$$

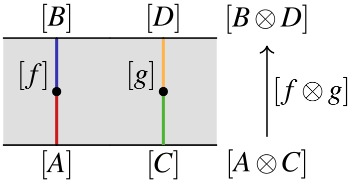
Categorified path calculus

$[f] \boxplus [g] =$



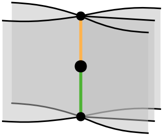
Categorified path calculus

$$[f][g] =$$



Categorified path calculus

More generally, given $f_{ij} : A_{ij} \to B_{ij}$ in $\mathcal{C}$, such as



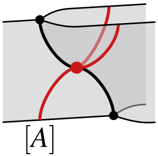
$$\begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \end{bmatrix}$$

$$\Big\uparrow \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \end{bmatrix}$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \end{bmatrix}$$

Categorified path calculus

Coning (`mul-hom`):



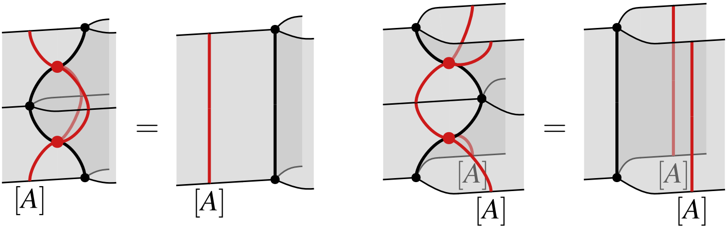
$$[I\ \ I]\begin{bmatrix}A & O\\ O & A\end{bmatrix}$$

$$\uparrow \cong$$

$$[A][I\ \ I]$$

Categorified path calculus

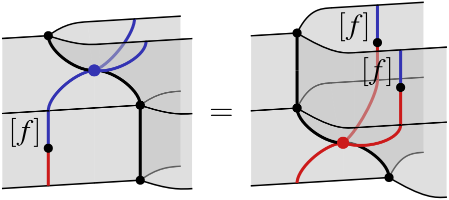
The *pulling* equations:



and *copulling*..

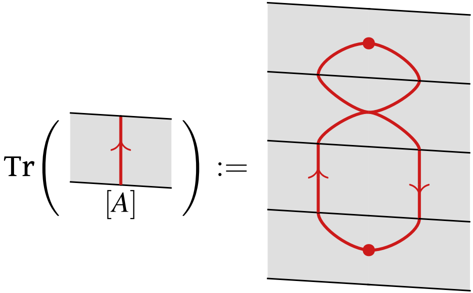
Categorified path calculus

Naturality:



Backwards compatible

$$\mathrm{Tr}(A) := I \xrightarrow{\eta_A} A^* \otimes A \xrightarrow{\sigma^{\otimes}} A \otimes A^* \xrightarrow{\epsilon_A} I$$

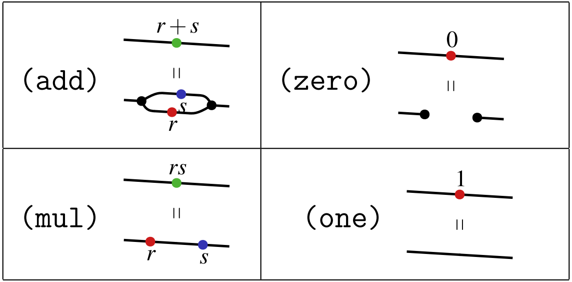


Scalars

Quantum:

$$\langle \chi | \phi \rangle \in \mathbb{C}$$

Path calculus:

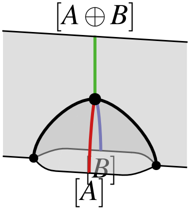


Categorified scalars

Hom categories:

$$\mathrm{Hom}(1,1) \cong \mathcal{C}$$

Categorified path calculus:



$$[A \oplus B]$$

$$\Big\uparrow \cong$$

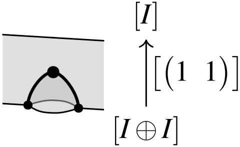
$$\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} A & O \\ O & B \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix}$$

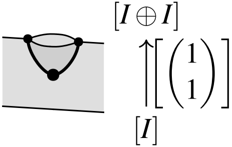
When $\oplus$ is a Biproduct in $\mathcal{C}$

We have matrix morphisms in $\mathcal{C}$, such as

$$\begin{pmatrix}1 & 1\end{pmatrix} : I \oplus I \to I, \qquad \begin{pmatrix}1 \\ 1\end{pmatrix} : I \to I \oplus I$$

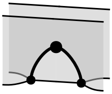
these become the cap and cup:





When $\oplus$ is a Biproduct in $\mathcal{C}$

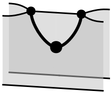
We also have another cap and cup



$$\begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$$\Uparrow \begin{bmatrix} 1 & 0_{O,I} \\ 0_{O,I} & 1 \end{bmatrix}$$

$$\begin{bmatrix} I & I \\ I & I \end{bmatrix}$$



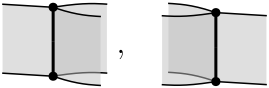
$$\begin{bmatrix} I & I \\ I & I \end{bmatrix}$$

$$\Uparrow \begin{bmatrix} 1 & 0_{I,O} \\ 0_{I,O} & 1 \end{bmatrix}$$

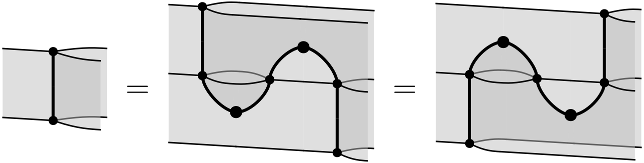
$$\begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

When $\oplus$ is a Biproduct in $\mathcal{C}$

All these caps and cups exhibit



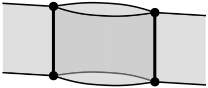
as ambidextrous adjoints:



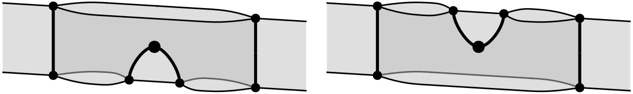
etc.

When $\oplus$ is a Biproduct in $\mathcal{C}$

Ambidextrous adjoints give a Frobenius algebra on:

$$[I \oplus I] =$$ 

Multiplication, comultiplication:

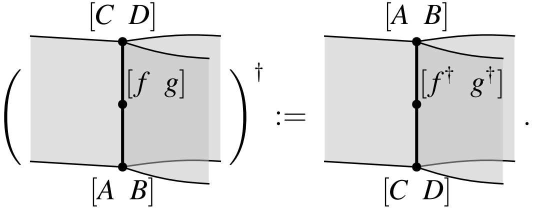


... special symmetric Frobenius algebra, or *classical bit*.

[*Frobenius algebras and ambidextrous adjunctions*, Lauda, 2006]
[*Higher quantum theory*, Vicary, 2012]

2-categorical quantum mechanics
The dagger

Given $f : A \to B$ and $g : B \to D$ in $\mathcal{C}$:

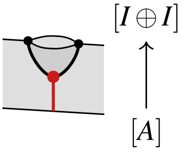


Unitary 2-cells...

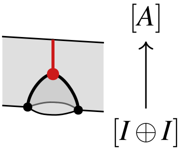
[*A categorical semantics of quantum protocols*, Abramsky, Coecke, 2004]

[*Higher quantum theory*, Vicary, 2012]

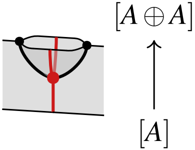
2-categorical quantum mechanics



qubit measurement



qubit preparation

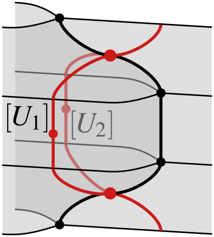


projective measurement

& these are unitary.

2-categorical quantum mechanics

Given unitaries $U_1, U_2 : A \to A$ in $\mathcal{C}$ the *controlled-U* operation is the 2-cell given by

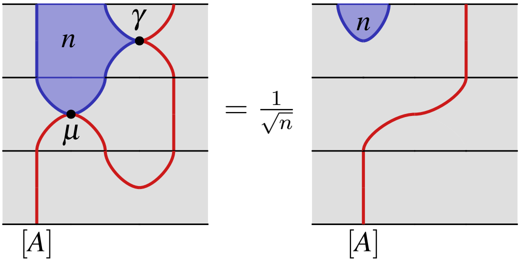


$$\begin{bmatrix} I \\ I \end{bmatrix} [A]$$

$$\uparrow$$

$$\begin{bmatrix} I \\ I \end{bmatrix} [A]$$

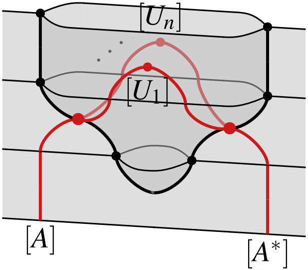
2-categorical quantum mechanics
Teleportation



2-categorical quantum mechanics
Unitary error basis

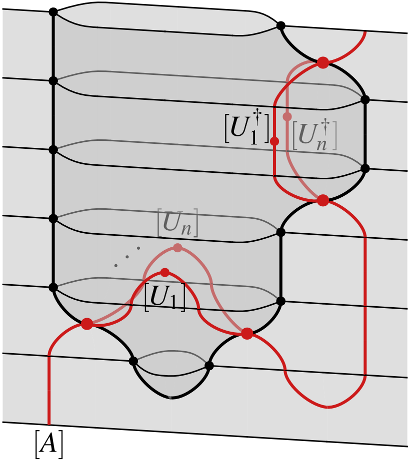
A *unitary error basis* for a system $A \in \mathcal{C}_0$ is a sequence of unitaries $U_1 : A \to A, \ldots, U_n : A \to A$ in $\mathcal{C}_1$ such that the 2-cell

$$\mu = \frac{1}{\sqrt{n}}$$



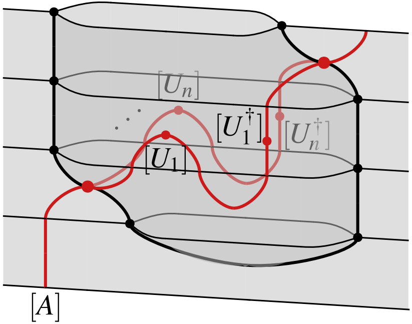
is unitary.

Teleportation

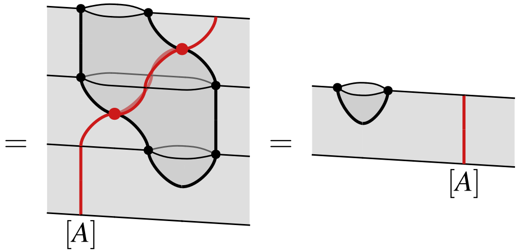


Teleportation

$=$



Teleportation



Conclusion

Motivation 1:

Can we have string diagrams for both $\otimes$ and $\oplus$? Yes.

Motivation 2:

What is an amplitude ? Um...

Todo:

Weak structure...

[*String diagrams for higher mathematics with* `wiggle.py`, Burton, 2023]

The end

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