An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence

JS PL (he/him)

Dedicated to Phil Scott.

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Context of Today's Story: Cartesian Differential Categories

- Cartesian differential categories (CDC) provide the categorical foundations of multivariable differential calculus over Euclidean spaces.

  R. Blute, R. Cockett, R.A.G. Seely, **Cartesian Differential Categories**

  CDC have also been successful in formalizing various important concepts in differential calculus such as: solving differential equations; exponential functions; Jacobians and gradients; linearizationl de Rham cohomolohy; etc.

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  CDC have also been successful in formalizing various important concepts in differential calculus such as: solving differential equations; exponential functions; Jacobians and gradients; linearizationl de Rham cohomolohy; etc.

- Cartesian *closed* differential categories (CCDC) provide the categorical semantics of the differential λ-calculus (and resource calculus).

  T. Ehrhard, L. Regnier **The differential λ-calculus.**

  Manzonetto, G., **What is a categorical model of the differential and the resource λ-calculi?**

  R. Cockett, J. Gallagher **Categorical models of the differential λ-calculus.**

  C(C)DC have also been quite popular in computer science, in particular since they provide categorical frameworks for differential programming languages; automatic differentiation; some machine learning algorithms; etc.

- The coKleisli category of a categorial model of Differential Linear Logic (DiLL) is a C(C)DC

Motivation of Today's Story: Taylor Series

An all-important concept in differential calculus is the notion of *Taylor series*.

- For a smooth function $f : \mathbb{R} \to \mathbb{R}$, its Taylor series at 0 (sometimes also called its Maclaurin series) is the power series:
$$\mathcal{T}(f)(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot f^{(k)}(0)x^k$$
Taylor series are very useful, and a lot of information about a function can be gained from studying its Taylor series. It is often highly desirable for Taylor series to converge (in the usual real analytical sense) and also for functions to equal their Taylor series.

Motivation of Today's Story: Taylor Series Expansion

Taylor series expansion is also important for differential λ-calculus and resource calculus.

T. Ehrhard, L. Regnier **Bohm tress, Krivine's Machine, and the Taylor Expansion of λ-terms.**

T. Ehrhard, L. Regnier **Uniformity and the Taylor Expansion of ordinary λ-terms.**

Boudes, P., He, F., Pagani, M. **A characterization of the Taylor expansion of λ-terms**

Manzonetto, G., Pagani, M. **Bohm's Theorem for resource λ-calculus through Taylor Expansion**

- In the presence of countable infinite sums, one can give full Taylor expansions of λ-terms. In particular, Taylor expansion provides a linear approximation of ordinary application, so:

$$MN = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathsf{D}^n M \cdot N^n) 0$$

  where $\mathsf{D}^n M \cdot N^n$ denotes the n-th derivative of M applied n times to N.

- Manzonetto provides the categorical interpretation this in a CCDC with infinite sums.

  Manzonetto, G., **What is a categorical model of the differential and the resource λ-calculi?**

- Taylor series expansion of coKleisli maps in a categorial model of DiLL was first considered by Ehrhard, and then further studied by Kerjean and me. This approach played a crucial role for defining *codigging* in DiLL:

  Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)

  Kerjean, M. and Lemay, J.-S. P. **Taylor Expansion as a Monad in Models of** DiLL**.**

Objective Today's Story

- While the notion of Taylor series for the differential λ-calculus is morally the same as the notion of Taylor series in classical differential calculus: there is a bit of a mismatch.
- For Taylor series in the differential λ-calculus: one assumes countable infinite sums in the algebraic sense.
- Having infinite sums clashes with models coming from the analysis side of things. There are many important examples of CDC that do not have infinite sums and yet still have a well-defined notion of Taylor series.

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- For Taylor series in the differential λ-calculus: one assumes countable infinite sums in the algebraic sense.
- Having infinite sums clashes with models coming from the analysis side of things. There are many important examples of CDC that do not have infinite sums and yet still have a well-defined notion of Taylor series.
- **OBJECTIVE:** provide a formal theory of Taylor series in an arbitrary CDC in such a way that it also gives a unified story recapturing Taylor series expansion in differential calculus via convergence and for the differential λ-calculus via algebraic infinite sums.

Today's Story & Summary of MFPS Paper

- Formalize the notion of *Taylor polynomials* of maps in a CDC by generalizing:

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- A Taylor CDC is a CDC for which this ultra*pseudo*metric is an actual metric, and so maps are completely determined by their Taylor polynomials.

Theorem

In a Taylor CDC, for every map f, its sequence of Taylor polynomials $\left(\mathcal{T}^{(n)}[f]\right)_{n=0}^{\infty}$ converges to f with respect to this ultrametric.

This ultrametric is analogues to the ultrametric of power series which makes the infinite sum converge, and this ultrametric then forces maps to be equal to their Taylor series.

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This ultrametric is analogues to the ultrametric of power series which makes the infinite sum converge, and this ultrametric then forces maps to be equal to their Taylor series.

- We also explain how for coKleisli maps in DiLL, we recapture what was done by Ehrhard and also Kerjean & Lemay.

- In a CCDC with algebraic infinite sums, we explain how being Taylor is equivalent to Manzonetto's notion of modelling Taylor series expansion.

Cartesian Differential Categories - Definition

Definition

For a commutative semiring k, a **Cartesian k-differential category** (CDC) is a category $\mathbb{X}$ with finite products such that:

- Hom-sets $\mathbb{X}(A, B)$ are k-modules such that pre-composition preserves the k-linear structure:

$$(r \cdot f + s \cdot g) \circ x = r \cdot (f \circ x) + s \cdot (g \circ x)$$

- A **differential combinator** D, which is a family of operators $\mathbb{X}(A, B) \xrightarrow{\mathrm{D}} \mathbb{X}(A \times A, B)$,

$$\frac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}$$

  where $\mathrm{D}[f]$ is called the derivative of f, and which satisfies seven axioms which capture the basics of the derivative from differential calculus (such as the chain rule, etc.)

There is a very practical (sound and complete) term logic for CDC. So we write:

$$\mathrm{D}[f](a, b) := \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

Cartesian Differential Categories - Examples

Example

Let POLY_k be the Lawvere theory of polynomials, that is, whose objects are $n \in \mathbb{N}$ and where a map $P : n \to m$ is a tuple of polynomials $P = \langle p_1(\vec{x}), \dots, p_m(\vec{x}) \rangle$ with $p_i(\vec{x}) \in k[x_1, \dots, x_n]$. Then POLY_k is a CDC where $\mathsf{D}[P] : n \times n \to m$ is:

$$\mathsf{D}[P] := \left\langle \sum_{i=1}^{n} \frac{\partial p_1(\vec{x})}{\partial x_i} y_i, \dots, \sum_{i=1}^{n} \frac{\partial p_m(\vec{x})}{\partial x_i} y_i \right\rangle$$

where $\sum\limits_{i=1}^{n} \frac{\partial p_i(\vec{x})}{\partial x_i} y_i \in k[x_1, \dots, x_n, y_1, \dots, y_n]$.

Example

Let SMOOTH be the Lawvere theory of real smooth functions, that is, whose objects are Euclidean spaces $\mathbb{R}^n$ and whose maps are real smooth functions $F : \mathbb{R}^n \to \mathbb{R}^m$. Note we have that $F = \langle f_1, \dots, f_n \rangle$ for real smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$. Then SMOOTH is a CDC, where for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is then defined as:

$$\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y} = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \dots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{x}) y_i \right\rangle$$

which is the *total* derivative of F. Note that $\mathbb{R}$-POLY is a sub-CDC of SMOOTH.

Taylor Polynomials in a CDC

We now wish to express *Taylor polynomials* of maps in a CDC by generalizing:

$$\mathcal{T}^{(n)}(f)(x) = \sum_{k=0}^{n} \frac{1}{k!} \cdot f^{(k)}(0)x^k$$

For this we need:

- Higher Order Derivatives
- Multiplying by $\frac{1}{n!}$ for $n \in \mathbb{N}$
- Evaluating at zero
- Multiplication and copying input variables for x^n
- Finite Sums

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Higher Order Derivatives

In a CDC, we can apply the differential order n-times to get:

$$\mathrm{D}^n[f] : A^{\times^{2^n}} \to B$$

called the ***n*-th total derivative** of f.

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However, by the axioms of a CDC there is a lot of redundant information in $\mathrm{D}^n[f]$.

For example, the second total derivative can be worked out to be:

$$\mathrm{D}^2[f](a, b, c, d) = \mathrm{D}^2[f](a, b, c, 0) + \mathrm{D}^2[f](a, d)$$

So we see that $\mathrm{D}^2[f]$ has a $\mathrm{D}[f]$ summand – which does not tell us any new information about f. Instead, all the new information comes from differentiating the first argument repeatedly.

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To make this precise: we need partial differentiation.

Higher Order Derivatives

In a CDC, for a map $g : C \times A \to B$, we can differentiate in C while keeping A constant by inserting zeroes in the total derivative. So its partial derivative in C is of type $C \times A \times C \to B$ and is defined as:

$$\frac{\mathrm{d}g(x,a)}{\mathrm{d}x}(c_1) \cdot c_2 = \frac{\mathrm{d}g(x,y)}{\mathrm{d}(x,y)}(c_1, a) \cdot (c_2, 0)$$

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Then for a map $f : A \to B$, its ***n*-th derivative** is defined by continuously deriving the first argument of the derivative:

$$\begin{gathered} A \to B \\ A \times A \to B \\ A \times A \times A \to B \\ \vdots \\ A \times \underbrace{A \times \ldots \times A}_{n\text{-times}} \to B \end{gathered}$$

Higher Order Derivatives

For a map $f : A \to B$, its n-**th derivative** is the map $\partial^{(n)}[f] : A \times A^{\times^n} \to B$, which we write as:

$$\partial^{(n)}[f](a_0, a_1, \ldots, a_n) := \frac{\mathrm{d}^{(n)} f(x)}{\mathrm{d}x}(a_0) \cdot a_1 \cdot \ldots \cdot a_n$$

and is defined inductively as:

$$\frac{\mathrm{d}^{(0)} f(x)}{\mathrm{d}x}(a_0) = f(a_0)$$

$$\frac{\mathrm{d}^{(n+1)} f(x)}{\mathrm{d}x}(a_0) \cdot a_1 \cdot \ldots \cdot a_n \cdot a_{n+1} = \frac{\mathrm{d}\frac{\mathrm{d}^{(n)} f(x)}{\mathrm{d}x}(y) \cdot a_1 \cdot \ldots \cdot a_n}{\mathrm{d}y}(a_0) \cdot a_{n+1}$$

$\mathrm{D}^n[f]$ can be expressed in terms of sums of $\partial^{(j)}[f]$ $(j \geq n)$, while $\partial^{(n)}[f]$ can be obtained by inserting zeroes into $\mathrm{D}^n[f]$.

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Sums, zeroes, and positive rationals

Recall that in the definition of a CDC, our homsets where k-modules.

- So we can take finite sums of parallel maps $A \to B$;
- There is a zero map[1] 0, and so we can evaluate maps at zero by $g \circ 0$
- For any map $f : A \to B$, we can scalar multiply by $r \in k$ to get $\frac{p}{q} \cdot f : A \to B$

What about scalar multiplying by $\frac{1}{n!}$?

[1] which is not a zero morphism, we only have $0 \circ f = 0$ but $f \circ 0$ does not necessarily equal 0

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What about scalar multiplying by $\frac{1}{n!}$?

For this, we now need to assume that our base semiring k is a $\mathbb{Q}_{\geq 0}$-algebra.

- So for any map $f : A \to B$, we can scalar multiply by a positive rational $\frac{p}{q} \cdot f : A \to B$

For the rest of this talk, we will assume k is a commutative $\mathbb{Q}_{\geq 0}$-algebra.

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Do we need multiplication?

The definition of a CDC does not assume we can multiply maps... even if both our mains examples (polynomials/smooth functions) we could, there are examples where we cannot! (ex. CDC of only linear maps).

Is this a problem?

Do we need multiplication?

The definition of a CDC does not assume we can multiply maps... even if both our mains examples (polynomials/smooth functions) we could, there are examples where we cannot! (ex. CDC of only linear maps).

Is this a problem? **ANSWER**: No!

The solution comes from the higher order derivatives.

Example

Consider a real smooth function $f : \mathbb{R} \to \mathbb{R}$, then we get that $\partial^{(n)}[f] : \mathbb{R}^{n+1} \to \mathbb{R}$ and is:

$$\partial^{(n)}[f](x, y_1, \dots, y_n) = f^{(n)}(x) y_1 \dots y_n$$

So we can just use higher order derivatives to avoid the need for multiplication.

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Taylor Polynomials in a CDC

Definition

In a CDC, for a map $f : A \to B$ and every $n \in \mathbb{N}$:

- Define $\mathcal{M}^{(n)}[f] : A \to B$ as the composite:

$$\mathcal{M}^{(n)}[f](x) = \frac{1}{n!} \cdot \frac{\mathrm{d}^{(n)} f(u)}{\mathrm{d}u}(0) \cdot x \cdot \ldots \cdot x$$

  which we call the ***n*-th Taylor** D-**monomial** of f.

- Define $\mathcal{T}^{(n)}[f] : A \to B$ as:

$$\mathcal{T}^{(n)}[f] = \sum_{k=0}^{n} \mathcal{M}^{(k)}[f] = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{\mathrm{d}^{(k)} f(u)}{\mathrm{d}u}(0) \cdot x \cdot \ldots \cdot x$$

  which we call the ***n*-th Taylor** D-**polymial** of f.

For polynomails and smooth functions, these recapture exactly the usual notion of Taylor polynomials.

Is polynomial the right word?

Usually, polynomials are defined using multiplication... but in a CDC we don't have a multiplication...

Is polynomial the right word?

Usually, polynomials are defined using multiplication... but in a CDC we don't have a multiplication... Another way of saying a smooth function is a polynomial is to say that its higher derivative is eventually zero.

Definition

In a CDC, a D-**polynomial** is a map $p : A \to B$ such that there is a $n \in \mathbb{N}$ such that $\partial^{(n+1)}[p] = 0$.

D-polynomials form a sub-CDC.

Lemma

In a CDC, for every map f and $n \in \mathbb{N}$, both $\mathcal{M}^{(n)}[f]$ and $\mathcal{T}^{(n)}[f]$ are D-*polynomials.*

REMARK: In some CDC, there are D-polynomials that are not Taylor D-polymials.

Convergence of Taylor Series

In a CDC we want that maps be equal to their Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{\mathsf{d}^{(n)} f(u)}{\mathsf{d}u}(0) \cdot x \cdot \ldots \cdot x$$

So we will now define a metric for which this series converges.

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In fact we will define an *ultra*metric which is defined by comparing Taylor monomials of maps.

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So we will now define a metric for which this series converges.

In fact we will define an *ultra*metric which is defined by comparing Taylor monomials of maps.

But first we define an ultra*pseudo*metric.

Ultrapseudometric

An **ultrapseudometric** on a set M is a function $\mathrm{d} : M \times M \to \mathbb{R}_{\geq 0}$ such that:

- $\mathrm{d}(x, x) = 0$
- $\mathrm{d}(x, y) = \mathrm{d}(y, x)$
- $\mathrm{d}(x, z) \leq \max\{\mathrm{d}(x, y), \mathrm{d}(y, z)\}$ (**Strong Triangle Inequality**)

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Definition

In a CDC $\mathbb{X}$, for every homset $\mathbb{X}(A, B)$, define the function $\mathrm{d}_{\mathrm{D}} : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{R}_{\geq 0}$ as:

$$\mathrm{d}_{\mathrm{D}}(f, g) = \begin{cases} 2^{-n} & \text{where } n \in \mathbb{N} \text{ is the smallest natural number such that } \mathcal{M}^{(n)}[f] \neq \mathcal{M}^{(n)}[g] \\ 0 & \text{if all } n \in \mathbb{N},\ \mathcal{M}^{(n)}[f] = \mathcal{M}^{(n)}[g] \end{cases}$$

Proposition

$\mathrm{d}_{\mathrm{D}} : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{R}_{\geq 0}$ *is an ultrapseudometric.*

Moreover, composition, pairing, scalar multiplication, and differentiation are all non-expansive. So this makes a CDC enriched over ultrapseudometric spaces.

Ultrametric

It is important to note that for an ultrapseudometric, we can have $\mathrm{d}(x, y) = 0$ for $x \neq y$.

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An **ultrametric** on a set M is a ultrapseudometric $\mathrm{d} : M \times M \to \mathbb{R}_{\geq 0}$ which also satisfies that:

- $\mathrm{d}(x, y) = 0$ implies $x = y$

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The canonical ultrapseudometric d_{D} of a CDC is not always an ultrametric! Two maps can have the same Taylor monomials but not be equal.

The canonical ultrapseudometric d_{D} is an ultrametric precisely when maps are completely determined by their Taylor monomials.

Taylor Cartesian Differential Category

Definition

A **Taylor Cartesian differential category** is a CDC such that:

- If for parallel maps $f : A \to B$ and $g : A \to B$, for all $n \in \mathbb{N}$, $\mathcal{M}^{(n)}[f] = \mathcal{M}^{(n)}[g]$. then $f = g$.

Proposition

A CDC is Taylor if and only if for every homset, the ultrapseudometric $\mathsf{d}_{\mathsf{D}} : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{R}_{\geq 0}$ *is an ultrametric.*

A Taylor CDC is enriched over ultrametric spaces.

Taylor Series Convergence in Taylor CDC

Theorem

In a Taylor CDC, for every map $f : A \to B$, the sequence $\left(\mathcal{T}^{(n)}[f]\right)_{n=0}^{\infty}$ converges to f with respect to the ultrametric d_{D}, so we may write:

$$f = \sum_{n=0}^{\infty} \mathcal{M}^{(n)}[f] \qquad\qquad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{\mathrm{d}^{(n)} f(u)}{\mathrm{d}u}(0) \cdot x \cdot \ldots \cdot x \tag{1}$$

This ultrametric forces the Taylor series to converge to f the same way that the ultrametric for power series makes the infinite sum converge.

Taylor CDC – Examples

Example

POLY_k is Taylor.

Example

SMOOTH is not Taylor since two smooth functions can have the same Taylor expansion but not be equal. On the other hand, the subcategory of real *entire* functions is a Taylor CDC.

Taylor CDC via quotienting

While not every CDC is Taylor, we can always build one!

This follows from the fact that every ultrapseudometric space can be made into an ultrametric space by quotienting.

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So let (M, d) be an ultrapseudometric space. Then we have an equivalence relation $\sim$ given by $x \sim y$ if and only if $\mathrm{d}(x, y) = 0$. Let $M_\sim$ be the set of equivalence classes of $\sim$, so $[x] = \{y \in M | x \sim y\}$. Then $(M_\sim, \mathrm{d}_\sim)$ is an ultrametric space where $\mathrm{d}_\sim([x], [y]) = \mathrm{d}(x, y)$.

Lemma

For a CDC $\mathbb{X}$, let $\mathbb{X}_\sim$ be the CDC whose objects are the same $\mathbb{X}$ and whose homsets are $\mathbb{X}_\sim(A, B) = \mathbb{X}(A, B)_\sim$. Then $\mathbb{X}_\sim$ is a Taylor CDC.

The CDC structure on $\mathbb{X}_\sim$ is defined as for $\mathbb{X}$ on the representatives, and is well-defined since $\sim$ is fully compatible with the CDC structure.

What about other ways?

What if we were in a CDC where there was another reason for maps to equal their Taylor series:

- Convergence via another metric
- Algebraic Infinite Sums

Another metric?

Suppose we are in a CDC which is metric space enriched, so homsets have another possible metric $\mathsf{b} : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{R}_{\geq 0}$.

Lemma

If for every map f, the sequence $\left(\mathcal{T}^{(n)}[f]\right)_{n=0}^{\infty}$ converges to f with respect to the metric b, *then the CDC is Taylor.*

REMARK: The converse is not true!

Infinite Sum Case

If we are in a setting with algebraic infinite sums, for every map $f : A \to B$, we can formally define the map $\mathcal{T}[f] : A \to B$ as the infinite sum of its Taylor monomials:

$$\mathcal{T}[f] = \sum_{n=0}^{\infty} \mathcal{M}^{(n)}[f]$$

Proposition

A CDC with countably infinite sums is Taylor if and only if for every map, $f = \mathcal{T}[f]$.

Modelling Taylor Expansion

Definition

A CCDC with countably infinite sums is said to **model Taylor expansion**[a] if for every map $f : C \times A \to B$ and a map $g : C \to A$,

$$f(c, g(c)) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{\mathrm{d}^{(n)} f(c,y)}{\mathrm{d}y}(0) \cdot g(c) \cdot \ldots \cdot g(c)$$

Manzonetto, G., **What is a categorical model of the differential and the resource λ-calculi?**

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[a] This is the unCurry version of Manzonetto's formula. For convenience, Manzonetto also works in a setting with addition is idempotent, so the coefficients $\frac{1}{n!}$ would dissapear.

Proposition

A CCDC with countably infinite sums is Taylor if and only if it models Taylor expansion.

CoKleisli Category of a Differential Category

Differential categories provide the categorical semantics of DiLL. In particular a differential category has a comonad ! and the coKleisli category of ! is a CDC.

In a differential category, we can define canonical natural transformations:

$$\mathsf{M}^{(n)}_A : !A \to !A \qquad \qquad \mathsf{T}^{(n)}_A : !A \to !A$$

and we can show that for any coKleisli map $f : !A \to B$, we have that:

$$\mathcal{M}^{(n)}[f] = f \circ \mathsf{M}^{(n)}_A \qquad \qquad \mathcal{T}^{(n)}[f] = f \circ \mathsf{T}^{(n)}_A$$

The natural transformations $\mathsf{T}^{(n)}$ were defined and studied by Ehrhard:

Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)

The natural transformations $\mathsf{M}^{(n)}$ were further studied in:

Kerjean, M. and Lemay, J.-S. P. **Taylor Expansion as a Monad in Models of** DiLL**.**

The notion of Taylor series in CDC from today's story recaptures what was done in the above for coKleisli maps of differential categories. These ideas played a key role in our LICS2023 paper for defining codigging.

Thanks for listening!

Oktober Fest, Kellogg College, University of Oxford – 2017

HOPE YOU ENJOYED MY TALK!

THANK YOU FOR LISTENING!

MERCI!

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`https://sites.google.com/view/jspl-personal-webpage/`