Continuous Domains for Function Spaces Using Spectral Compactification

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Introduction

- Close link between topology and the theory of computation
- Clearly manifested in domain theory

Domain theory and mathematical analysis:

- (Non-algebraic) domains provide a natural computational framework for mathematical analysis
- Initiated by Edalat's work on dynamical systems (Edalat 1995)
- Further developments:
  - differential equation solving (Edalat and Pattinson 2007a)
  - stochastic processes (Bilokon and Edalat 2017)
  - reachability analysis of hybrid systems (Edalat and Pattinson 2007b; Moggi et al. 2018)
  - robustness analysis of neural networks (Zhou et al. 2023).

Motivation

- Local compactness: a desirable topological property.
- What to do in the absence of local compactness?
    - **substitute** constructions.

- Examples:
    - robustness analysis of systems with state spaces that are not (locally) compact (Farjudian and Moggi 2023)
    - solution of initial value problems (IVPs) with temporal discretization (Edalat, Farjudian, and Li 2023).

IVPs

$$\begin{cases} y'(t) = f(y(t)), \\ y(t_0) = y_0, \end{cases} \tag{1}$$

- $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous vector field.

Assume that a solution exists over a lifetime of $[t_0, T]$

- We search for a solution of (1) in the space of functions from $[t_0, T]$ to the interval domain:

$$\mathbb{IR} := \{\mathbb{R}\} \cup \{[a, b] \mid a, b \in \mathbb{R} \text{ and } a \leq b\},$$

ordered by: $\forall X, Y \in \mathbb{IR} : X \sqsubseteq Y \overset{\Delta}{\iff} X \supseteq Y$.

Temporal Discretization

- By integrating both sides of (1): $\forall t \in [t_0, T], h \in [0, T - t]$:

$$y(t+h) = y(t) + \int_{t}^{t+h} f(y(\tau))\,\mathrm{d}\tau.$$

A general schema:

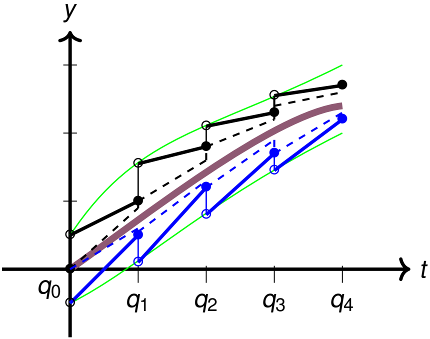
1. For some $k \geq 1$, consider a partition $Q = (q_0, \ldots, q_k)$ of the interval $[t_0, T]$.
2. Let $Y(t_0) := y_0$.
3. For each $j \in \{0, \ldots, k-1\}$ and $h \in (0, q_{j+1} - q_j]$:

$$Y(q_j + h) := Y(q_j) + I(q_j, h),$$

where $I(q_j, h)$ is an interval enclosure of $\int_{t}^{t+h} f(y(\tau))\,\mathrm{d}\tau$.

Fixpoint Formulation (Flawed)

$$\Phi(Y)(x) := \begin{cases} y_0, & \text{if } x = t_0, \\ Y(q_j) + I(q_j, x - q_j), & \text{if } q_j < x \leq q_{j+1}. \end{cases}$$



- True solution (dark red), Initial enclosure (green)
- Upper bound (black): Not upper semi-continuous
- Lower bound (blue): Not lower semi-continuous

Upper Limit Topology

- In the function space $[[t_0, T] \rightarrow \mathbb{IR}]$:
  - $\mathbb{IR}$ should be equipped with the Scott topology.
  - What about $[t_0, T]$?
- The Euclidean topology is not appropriate, since the upper (resp. lower) bounds in the previous figure were not upper (resp. lower) semi-continuous.
- The coarsest topology with respect to which the enclosures are continuous is the **upper limit topology** on $[t_0, T]$ with the collection:

  $$\{(a, b] \mid a, b \in \mathbb{R}\}$$

  of half-open intervals as its base.
- But, this topology is not locally compact (Edalat, Farjudian, and Li 2023, Proposition 4.5).

Local Compactness

Theorem 1 (Erker, Escardó, and Keimel 1998)

For any topological space $\mathbb{X}$ and non-singleton bc-domain $\mathbb{D}$, the function space $([\mathbb{X} \to \mathbb{D}], \sqsubseteq)$ *is a bc-domain* $\iff$ $\mathbb{X}$ *is core-compact.*

- Also, for sober spaces, core compactness and local compactness are equivalent.
    - We work primarily with Sober spaces.

- As such, **with the upper limit topology, we cannot obtain a continuous domain of functions**.

Substitute Construction

Basic idea: When $\mathbb{X}$ is not core-compact, construct a topological space $\hat{\mathbb{X}}$ with the following properties:

1. $\hat{\mathbb{X}}$ is core-compact.
2. $\mathbb{X}$ can be embedded into $\hat{\mathbb{X}}$ as a dense subspace.
3. The function spaces $[\mathbb{X} \to \mathbb{D}]$ and $[\hat{\mathbb{X}} \to \mathbb{D}]$ are related via a Galois connection.

Then:

- The (non-continuous) dcpo $[\mathbb{X} \to \mathbb{D}]$ can be used for implementation of algorithms,
- analysis of computability can be carried out over the continuous domain $[\hat{\mathbb{X}} \to \mathbb{D}]$,
  - subject to the existence of a suitable effective structure over $[\hat{\mathbb{X}} \to \mathbb{D}]$.

Basic Galois Connection

Assume that $\iota : \mathbb{X} \to \mathbb{Y}$ is a dense embedding of T_0 spaces, and $\mathbb{D}$ is a bc-domain. Define:

- $\forall g \in [\mathbb{Y} \to \mathbb{D}] : \ g^* := g \circ \iota$,
- $\forall f \in [\mathbb{X} \to \mathbb{D}] : \forall y \in Y : \ f_*(y) := \bigvee \left\{ \bigwedge f(\iota^{-1}(U)) \,\middle|\, y \in U \in \tau_{\mathbb{Y}} \right\}$.

Theorem 2 (Galois connection)

The maps $(\cdot)^*$ *and* $(\cdot)_*$ *form a Galois connection:*

$$\begin{tikzcd}
[\mathbb{X} \to \mathbb{D}] \arrow[r, shift left, "(\cdot)_*"] \arrow[r, phantom, "\top"] & {[\mathbb{Y} \to \mathbb{D}],} \arrow[l, shift left, "(\cdot)^*"]
\end{tikzcd}$$

in the category $\mathcal{P}o$ *of posets and monotonic maps. Furthermore:*

1. *The map* $(\cdot)^*$ *is surjective, and* $(\cdot)_*$ *is injective.*
2. $(\cdot)^* \circ (\cdot)_* = \mathrm{id}_{[\mathbb{X} \to \mathbb{D}]}$, *i. e.,* $\forall f \in [\mathbb{X} \to \mathbb{D}] : (f_*)^* = f$.
3. *The left adjoint* $(\cdot)^*$ *is Scott continuous.*

Core-Compactification

Definition 3 (Core-compactification)

A core-compact space $\mathbb{X}'$ is a *core-compactification* of $\mathbb{X}$ $\overset{\Delta}{\Longleftrightarrow}$ $\mathbb{X}$ can be embedded as a dense sub-space of $\mathbb{X}'$.

Examples: Classical compactification methods, e. g.

1. The one-point (Alexandroff) compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$.
   - Applicable only to locally compact spaces.
2. The Stone-Čech compactification $\beta\mathbb{X}$ of a Tychonoff space $\mathbb{X}$.
   - Lack of an explicit description even for simple topological spaces $\mathbb{X}$.

Viable Base

Definition 4 (Viable base)

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a topological space. We say that $\Omega_0 \subseteq \tau_{\mathbb{X}}$ is a *viable base* for $\mathbb{X}$ if:

1. it is closed under finite unions and finite intersections, and
2. it forms a base for the topology $\tau_{\mathbb{X}}$.

Remarks:

1. Ω_0 must contain $\emptyset = \cup\emptyset$ and $X = \cap\emptyset$.
2. Ω_0 is a bounded distributive lattice with $\bigwedge := \bigcap$ and $\bigvee := \bigcup$.
3. There is always at least one viable base, i. e., $\Omega_0 = \tau_{\mathbb{X}}$.

Bounded Distributive Lattice and Spectral Spaces

Since every viable base is a bounded distributive lattice, we may refer to the following equivalence of categories to construct a spectral compactification (Abramsky and Jung 1994, Section 7).:

$$\begin{tikzcd}
\mathcal{BDLat}^{op} \arrow[r, shift left=1.5ex, "\mathrm{Idl}^{op}"] \arrow[r, phantom, "\top"] & \mathcal{Afal}^{op} \arrow[l, shift left=1.5ex, "\mathcal{K}^{op}"] \arrow[r, shift left=1.5ex, "\mathrm{pt}"] \arrow[r, phantom, "\top"] & \mathcal{Spec} \arrow[l, shift left=1.5ex, "\Omega"]
\end{tikzcd}$$

Definition 5 (Spectral compactification: $\hat{\mathbb{X}}_{\Omega_0}$)

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a T_0 topological space and $\Omega_0 \subseteq \tau_{\mathbb{X}}$ is a viable base of $\mathbb{X}$. By the **spectral compactification of $\mathbb{X}$ generated by** Ω_0 we mean the topological space:

$$\hat{\mathbb{X}}_{\Omega_0} \equiv (\hat{X}_{\Omega_0}, \hat{\tau}) := \mathrm{pt}(\mathrm{Idl}(\Omega_0)),$$

in which $\hat{\tau}$ is the hull-kernel topology.

Clarification

$$\begin{tikzcd}
\mathit{BDLat}^{op} \arrow[r, shift left, "\mathrm{Idl}^{op}"] \arrow[r, phantom, "\top"] & \mathit{Afal}^{op} \arrow[l, shift left, "\mathcal{K}^{op}"] \arrow[r, shift left, "\mathrm{pt}"] \arrow[r, phantom, "\top"] & \mathit{Spec} \arrow[l, shift left, "\Omega"]
\end{tikzcd}$$

- *BDLat*

  Objects: bounded distributive lattices

  Arrows: bounded lattice homomorphisms

- *Afal*

  Objects: algebraic fully (i. e., $\top \ll \top$) arithmetic lattices

  Arrows: frame homomorphisms

- *Spec*

  Objects: spectral (i. e., compact, sober, coherent, and strongly locally compact) spaces

  Arrows: spectral maps (i. e., $f^{-1}(K)$ is compact-open for all compact-open K)

Strongly Locally Compact Spaces

Among the various features of spectral spaces, the following stands out:

Definition 6 (Strongly Locally Compact)

We say that $\mathbb{Y}$ is a *strongly locally compact* space if its topology has a base of compact-open subsets.

Proposition 1

Let $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$ *be a topological space. A set* $Q \subseteq Y$ *is compact-open if and only if* Q *is a finite element of the complete lattice* $(\tau_{\mathbb{Y}}, \subseteq)$*, i. e.,* $Q \ll Q$*.*

Lemma 7

A topological space $\mathbb{Y} = (Y, \tau_{\mathbb{Y}})$ *is strongly locally compact if and only if* $(\tau_{\mathbb{Y}}, \subseteq)$ *is an algebraic lattice.*

The pt Functor

$$\begin{tikzcd}
\mathcal{BDLat}^{op} \arrow[r, shift left, "\mathrm{Idl}^{op}"] \arrow[r, phantom, "\top"] & \mathcal{Afal}^{op} \arrow[l, shift left, "\mathcal{K}^{op}"] \arrow[r, shift left, "\mathrm{pt}"] \arrow[r, phantom, "\top"] & \mathcal{Spec} \arrow[l, shift left, "\Omega"]
\end{tikzcd}$$

- For any complete lattice $\mathbb{L} \equiv (L, \sqsubseteq)$, by a point of $\mathbb{L}$ we mean a completely prime filter $F \subseteq L$.
- We let $\mathrm{pt}(\mathbb{L})$ denote the set of points of $\mathbb{L}$ with the so-called hull-kernel topology:
  - Open sets $\mathcal{O}_u := \{x \in \mathrm{pt}(\mathbb{L}) \mid u \in x\}$, where u ranges over all the elements of L
- For any morphism $g : \mathbb{L} \to \mathbb{K}$ in $\mathcal{Afal}^{op}$ (i. e., a frame homomorphism $g : \mathbb{K} \to \mathbb{L}$) the function $\mathrm{pt}(g) : \mathrm{pt}(\mathbb{L}) \to \mathrm{pt}(\mathbb{K})$ maps every completely prime filter x of $\mathbb{L}$ to $g^{-1}(x)$.

Hull-Kernel Topology $\hat{\tau}$ on $\hat{\mathbb{X}}_{\Omega_0}$

In the hull-kernel topology $\hat{\tau}$ on $\hat{\mathbb{X}}_{\Omega_0}$, every open set is of the form:

$$\mathcal{O}_I := \{y \in \hat{X}_{\Omega_0} \mid I \in y\},$$

in which I ranges over $\mathrm{Idl}(\Omega_0)$.

Theorem 8

The set $\{\mathcal{O}_{\downarrow W} \mid W \in \Omega_0\}$ forms a base for the hull-kernel topology $\hat{\tau}$ on $\hat{\mathbb{X}}_{\Omega_0}$.

Corollary 9

When Ω_0 is countable, $\hat{\mathbb{X}}_{\Omega_0}$ is second-countable.

Rational Upper Limit Topology

Example 10 (rational upper limit topology)

- Let $\mathbb{R}_{(\mathbb{Q}]} \equiv (\mathbb{R}, \tau_{(\mathbb{Q}]})$ denote the topological space with $\mathbb{R}$ as the carrier set endowed with the rational upper limit topology $\tau_{(\mathbb{Q}]}$,
  - with $B_{(\mathbb{Q}]} := \{(a, b] \mid a, b \in \mathbb{Q}\}$ as a base.
- As for a viable Ω_0, an immediate option is $\tau_{(\mathbb{Q}]}$.
  - By Theorem 8, it does not lead to a second-countable $\hat{\mathbb{X}}_{\Omega_0}$.
- Instead, we take Ω_0 to consist of all the finite unions of elements of $B_{(\mathbb{Q}]}$.
  - This is a countable set which can be effectively enumerated. By Corollary 9, the space $\hat{\mathbb{X}}_{\Omega_0}$ must also be second-countable.

Core-Compactification

- The rational upper limit topology was used in (Edalat, Farjudian, and Li 2023) for solution of IVPs with temporal discretization.
- In (Edalat, Farjudian, and Li 2023), the domain $[\mathbb{Y} \to \mathbb{D}]$ is constructed by rounded ideal completion of a suitable abstract basis of step functions.
- Here, we work directly on $\mathbb{X}$:

Theorem 11 (Core-compactification)

Assume that $\mathbb{X}$ is a T_0 topological space. If Ω_0 is a viable base of $\mathbb{X}$, then the spectral space $\hat{\mathbb{X}}_{\Omega_0}$ is a core-compactification of $\mathbb{X}$.

- As we will see (Theorem 15) the two approaches lead to equivalent outcomes.

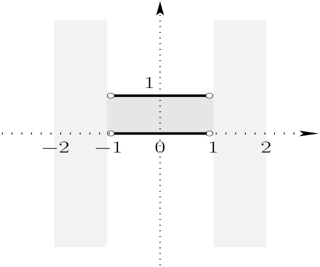
Step Functions (Reminder)

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ is a topological space, and $\mathbb{D} \equiv (D, \sqsubseteq)$ is a pointed directed-complete partial order (pointed dcpo), with bottom element $\perp$.

- For every open set $O \in \tau_{\mathbb{X}}$, and every element $b \in D$, we define the single-step function $b\chi_O : X \to D$ as follows:

$$b\chi_O(x) := \begin{cases} b, & \text{if } x \in O, \\ \perp, & \text{if } x \in X \setminus O. \end{cases}$$

Example:
$[0, 1]\chi_{(-1,1)} : [-2, 2] \to \mathbb{IR}$



- By a step-function we mean the join of a (consistent) finite set of single-step functions.

Continuous Domain of Functions

Theorem 12

Assume that $\mathbb{X} \equiv (X, \tau_{\mathbb{X}})$ *is a topological space and* $\Omega_0 \subseteq \tau_{\mathbb{X}}$ *is a viable base of* $\mathbb{X}$*. Let* $\mathbb{D} \equiv (D, \sqsubseteq)$ *be a bc-domain and assume that* $D_0 \subseteq D$ *is a basis for* $\mathbb{D}$*. Then,* $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$ *is a bc-domain with a basis* $\hat{\mathbb{B}}$ *of step-functions of the form:*

$$\hat{\mathbb{B}} = \left\{ \bigvee_{i \in I} b_i \chi_{O_{\downarrow W_i}} \mid I \text{ is finite, } \left\{ b_i \chi_{O_{\downarrow W_i}} \,\middle|\, i \in I \right\} \text{ is consistent, } \forall i \in I : W_i \in \Omega_0, b_i \in D_0 \right\}. \quad (2)$$

Corollary 13

If Ω_0 *is countable and* $\mathbb{D}$ *is* ω*-continuous, then* $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$ *is also* ω*-continuous.*

Way-Below Relation over $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$

On single-step functions, assuming that $b \neq \bot$, we have:

$$b\chi_{\mathcal{O}_{\downarrow W}} \ll b'\chi_{\mathcal{O}_{\downarrow W'}} \iff W \subseteq W' \text{ and } b \ll b'.$$

Lemma 14

The way-below relation on step functions of $\hat{\mathbb{B}}$ in (2) *can be expressed as:*

$$\bigvee_{i \in I} b_i\chi_{\mathcal{O}_{\downarrow W_i}} \ll \bigvee_{j \in J} b'_j\chi_{\mathcal{O}_{\downarrow W'_j}} \iff \forall i \in I : W_i \subseteq U_i,$$

in which $U_i \in \Omega_0$ satisfies $\mathcal{O}_{\downarrow U_i} = \left(\bigvee_{j \in J} b'_j\chi_{\mathcal{O}_{\downarrow W'_j}}\right)^{-1}(\Uparrow b_i)$.

Lemma 14 suggests an alternative approach to obtaining a domain of functions based on *abstract bases* without referring to Stone duality.

Equivalent Construction via Abstract Bases

We define the abstract basis $(\mathbb{B}_{\text{abs}}, \lhd)$ as follows:

$$\mathbb{B}_{\text{abs}} := \Big\{ f : X \to D \mid f = \bigvee_{i \in I} b_i \chi_{O_i}, \quad I \text{ is finite}, \forall i \in I : O_i \in \Omega_0 \text{ and } b_i \in D_0 \Big\}.$$

As for the binary relation $\lhd$, considering Lemma 14, we define:

$$\bigvee_{i \in I} b_i \chi_{O_i} \lhd \bigvee_{j \in J} b'_j \chi_{O'_j} \iff \forall i \in I : O_i \subseteq \Big(\bigvee_{j \in J} b'_j \chi_{O'_j}\Big)^{-1}(\Uparrow b_i).$$

Theorem 15

Assume that the domain $\mathcal{W}$ *is the rounded ideal completion of* $(\mathbb{B}_{\text{abs}}, \lhd)$. *Then* $\mathcal{W} \cong [\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$.

Galois Connection Revisited

Recall from Theorem 2 the following Galois connection in the category $\mathcal{Po}$ of posets and monotonic maps:

$$\begin{tikzcd}
{[\mathbb{X} \to \mathbb{D}]} \arrow[r, shift left=2, "(\cdot)_*"] \arrow[r, phantom, "\top"] & {[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}] \cong \mathcal{W},} \arrow[l, shift left=2, "(\cdot)^*"]
\end{tikzcd}$$

in which:

- the map $(\cdot)^*$ is surjective, and $(\cdot)_*$ is injective.
- $(\cdot)^* \circ (\cdot)_* = \mathrm{id}_{[\mathbb{X} \to \mathbb{D}]}$, i. e., $\forall f \in [\mathbb{X} \to \mathbb{D}] : (f_*)^* = f$.

Hence:

- computations take place in the dcpo $[\mathbb{X} \to \mathbb{D}]$.
- computable analysis is done in the (effectively given) domain $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}] \cong \mathcal{W}$.
- the left and right adjoints are used for moving between the two function spaces.

Comparison with Type-II Theory of Effectivity (TTE)

- We investigated a computational framework for function spaces over topological spaces that are not core-compact, e. g.
    - Upper limit topology (IVP solving)
    - Infinite-dimensional Banach spaces (PDE solving, functional analysis, etc.)
- In our framework, computability is analyzed in the continuous domain $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$.
- In Type-II Theory of Effectivity (TTE) (Weihrauch 2000), computability is analyzed via admissible representations of the function space $\mathbb{D}^{\mathbb{X}}$.

Question 1

In what ways are our framework and TTE related?

- *In particular, what is the relationship between the topology on $\mathbb{D}^{\mathbb{X}}$ induced by an admissible representation, and the Scott topology on $[\hat{\mathbb{X}}_{\Omega_0} \to \mathbb{D}]$?*

Applications

Ordinary Differential Equations (ODEs):

- In (Edalat, Farjudian, and Li 2023), we constructed a domain using abstract bases for solution of IVPs with temporal discretization.
- In Theorem 15, we showed that the same domain (up to isomorphism) can be obtained using spectral compactification.

Partial Differential Equations (PDEs):

- We expect spectral compactification to be useful in domain theoretic solution of partial differential equations (PDEs) as well.

Stochastic Processes with Right-Continuous Jumps:

- *lower limit topology* (not core-compact).

Stable Compactification

- Spectral compactification provides another angle on the construction obtained via abstract bases in (Edalat, Farjudian, and Li 2023).
- We believe that the construction based on compactification has some theoretical advantages.
  - Compactification is a central topic in topology.
  - Our construction can be obtained as a special case of Smyth's stable compactification (Smyth 1992) by considering fine quasi-proximities.

Question 2

Are there any concrete applications for non-spectral stable compactification (obtained via non-fine quasi-proximities) in the way that spectral compactification has been useful in IVP solving?

References: I

- [AJ94] Samson Abramsky and Achim Jung. “Domain Theory”. In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum. Vol. 3. Clarendon Press, Oxford, 1994, pp. 1–168.
- [BE17] Paul Bilokon and Abbas Edalat. “A Domain-Theoretic Approach to Brownian Motion and General Continuous Stochastic Processes”. In: *Theoretical Computer Science* 691 (2017), pp. 10–26. DOI: `10.1016/j.tcs.2017.07.016`.
- [Eda95] Abbas Edalat. “Dynamical Systems, Measures and Fractals via Domain Theory”. In: *Information and Computation* 120.1 (1995), pp. 32–48.

References: II

- [EEK98] Thomas Erker, Martín Hötzel Escardó, and Klaus Keimel. “The Way-Below Relation of Function Spaces Over Semantic Domains”. In: *Topology and its Applications* 89.1 (1998), pp. 61–74. DOI: `10.1016/S0166-8641(97)00226-5`.
- [EFL23] Abbas Edalat, Amin Farjudian, and Yiran Li. “Recursive Solution of Initial Value Problems with Temporal Discretization”. In: *Theoretical Computer Science* (2023), p. 114221. DOI: `10.1016/j.tcs.2023.114221`.
- [EP07a] Abbas Edalat and Dirk Pattinson. “A domain-theoretic account of Picard’s theorem”. In: *LMS Journal of Computation and Mathematics* 10 (2007), pp. 83–118.
- [EP07b] Abbas Edalat and Dirk Pattinson. “Denotational semantics of hybrid automata”. In: *The Journal of Logic and Algebraic Programming* 73.1 (2007), pp. 3–21.

References: III

- [FM23] Amin Farjudian and Eugenio Moggi. “Robustness, Scott continuity, and Computability”. In: *Mathematical Structures in Computer Science* 33.6 (2023), pp. 536–572. DOI: `10.1017/S0960129523000233`.
- [Mog18] Eugenio Moggi, Amin Farjudian, Adam Duracz, and Walid Taha. “Safe & Robust Reachability Analysis of Hybrid Systems”. In: *Theoretical Computer Science* 747 (2018), pp. 75–99.
- [Smy92] Michael B. Smyth. “Stable Compactification I”. In: *Journal of the London Mathematical Society* s2-45.2 (1992), pp. 321–340. DOI: `10.1112/jlms/s2-45.2.321`.
- [Wei00] Klaus Weihrauch. *Computable Analysis, An Introduction*. Springer, 2000.

References: IV

- [Zho23] Can Zhou, Razin A. Shaikh, Yiran Li, and Amin Farjudian. “A domain-theoretic framework for robustness analysis of neural networks”. In: *Mathematical Structures in Computer Science* 33.2 (2023), pp. 68–105. DOI: `10.1017/S0960129523000142`.