Parametricity via Cohesion

C.B. Aberlé (she/her)

June 20, 2024

On Parametricity

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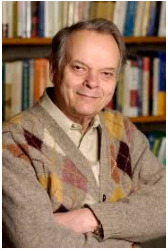

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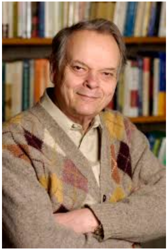

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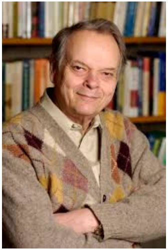
- Using Reynolds' technique, can show e.g. that all closed terms of type $\forall X.X \to X$ in System F are equivalent to the polymorphic identity function.



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- As yet no unifying axiomatic framework for such approaches to internal parametricity.
- This work constitutes a first step toward such a unifying framework, based on the category-theoretic concept of *cohesion*.

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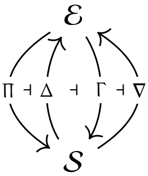

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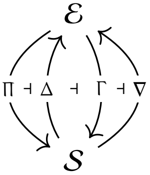

  where Δ, ∇ are fully faithful and Π preserves finite products.

- Induces a string of adjoint endofunctors on $\mathcal{E}$:

$$\int \dashv \flat \dashv \sharp$$

  with $\int, \sharp$ idempotent monads, and $\flat$ an idempotent comonad.

Example: Reflexive Graphs

The category of reflexive graphs **RGph** is cohesive over the category of sets **Set**.

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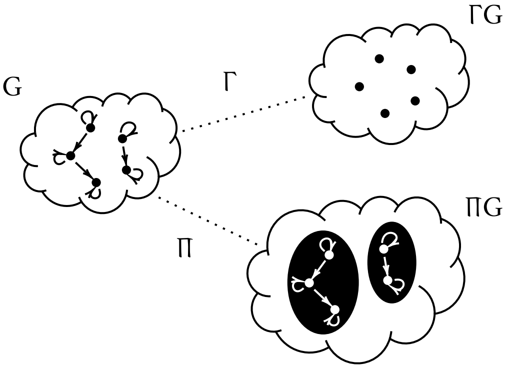
Γ maps a reflexive graph G to its set of vertices, Π maps G to its set of *weakly connected components*.



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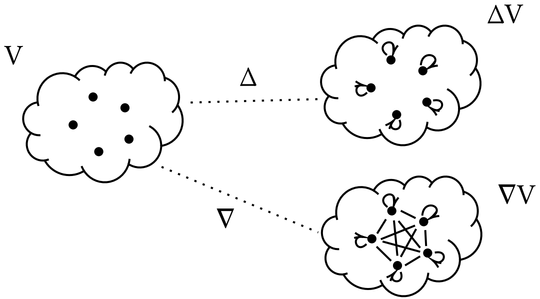
Δ maps a set V to the *discrete* graph with vertex set V, and ∇ maps V to the *codiscrete* (i.e. complete) graph on V.



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- More generally, for any base topos $\mathcal{S}$, the corresponding topos $\mathbf{RGph}(\mathcal{S})$ of internal reflexive graphs in $\mathcal{S}$ is cohesive over $\mathcal{S}$. Hence the internal language of reflexive graphs can be used to derive parametricity results for any topos.

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- In fact, this same setup of cohesion is interpretable, *mutatis mutandis*, in the case where $\mathcal{E}, \mathcal{S}$ are not (1-)topoi, but rather ∞-topoi, i.e. models of homotopy type theory (HoTT).

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- In fact, this same setup of cohesion is interpretable, *mutatis mutandis*, in the case where $\mathcal{E}, \mathcal{S}$ are not (1-)topoi, but rather ∞-topoi, i.e. models of homotopy type theory (HoTT).
- We can thus use the language of HoTT – suitably extended with cohesive modalities – to work *synthetically* with the structure of such a cohesive ∞-topos.

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Problem: the $\flat$ modality is not well-defined in arbitrary contexts, but only in those consisting entirely of *discrete* variables.

Solution: modify the structure of contexts to keep track of which variables are *discrete*.

Type-Theoretic Cohesion

Contexts now of the form $\Delta \mid \Xi$ where Δ consists of *discrete* variables, while Ξ consists of ordinary variables. The type of an ordinary variable may depend on both ordinary and discrete variables, but the type of a discrete variable can only depend upon other discrete variables.

$$\frac{\Delta \mid \Xi \ \mathsf{Ctx} \qquad \Delta \mid \Xi \vdash \mathsf{S}\ \mathsf{Type}}{\Delta \mid \Xi, x : \mathsf{S}\ \mathsf{Ctx}} \qquad \frac{\Delta \mid \Xi \ \mathsf{Ctx} \qquad \Delta \mid - \vdash \mathsf{S}\ \mathsf{Type}}{\Delta, x : \mathsf{S} \mid \Xi \ \mathsf{Ctx}}$$

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Rules for $\flat$ are then essentially those of a Pfenning-Davies-style modal necessity operator:

$$\frac{\Delta \mid - \vdash \mathsf{S}\ \mathsf{Type}}{\Delta \mid \Xi \vdash \flat\mathsf{S}\ \mathsf{Type}} \qquad \frac{\Delta \mid - \vdash s : \mathsf{S}}{\Delta \mid \Xi \vdash s^{\flat} : \flat\mathsf{S}}$$

$$\frac{\Delta \mid \Xi \vdash s : \flat\mathsf{S} \qquad \Delta \mid \Xi, z : \flat\mathsf{S} \vdash \mathrm{R}\ \mathsf{Type} \qquad \Delta, x : \mathsf{S} \mid \Xi \vdash r : \mathrm{R}[x^{\flat}/z]}{\Delta \mid \Xi \vdash \mathsf{let}\ x^{\flat} = s\ \mathsf{in}\ r : \mathrm{R}[s/z]}$$

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For any type S, we have $\epsilon_{\mathsf{S}} : \flat\mathsf{S} \to \mathsf{S}$ given by

$$\epsilon(s) :\equiv \mathsf{let}\ x^{\flat} = s\ \mathsf{in}\ x$$

S is *discrete* if ϵ_{S} is an equivalence.

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- The existence of an object with these two properties is equivalent to what Lawvere called *sufficient cohesion*.

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Proof: let S be a discrete type. A path in S is a function $f : I \longrightarrow S$. Since S is discrete, f factors as

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for some $f_\flat : \mathrm{I} \longrightarrow \flat\mathrm{S}$. But then since $\int \dashv \flat$, it follows that there is $f_\int : \int \mathrm{I} \longrightarrow \mathrm{S}$ such that

$$\begin{array}{ccc} \mathrm{I} & \xrightarrow{\eta_\mathrm{I}} & \int \mathrm{I} \\ {\scriptstyle f_\flat}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f_\int} \\ \flat\mathrm{S} & \xrightarrow[\epsilon_\mathrm{S}]{} & \mathrm{S} \end{array}$$

where η is the unit for the monad $\int$. Then since I is connected, $\int \mathrm{I} \simeq 1$ and so f factors through 1, i.e. f is constant.

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- Given a family of types $i : \mathrm{I} \vdash \mathrm{S}(i)\ \mathsf{Type}$, a path from $s_0 : \mathrm{S}(\mathbf{0})$ to $s_1 : \mathrm{S}(\mathbf{1})$ is a dependent function

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- A type S is *path-discrete* if for all $s_0, s_1 : \mathrm{S}$, the canonical map $s_0 =_{\mathrm{S}} s_1 \rightarrow \mathsf{Path}_{i.\mathrm{S}}(s_0, s_1)$ is an equivalence.

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- The above lemma says that, if a type is discrete, then it is path-discrete.

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- In the case where $i \equiv \mathbf{0}$, we therefore have $\mathsf{Gph1}^{i}_{x:\mathrm{S}}\mathrm{T}(x) \simeq \mathrm{S}$, and we strengthen this equivalence into the following judgmental equalities:

$$\mathsf{Gph1}^{\mathbf{0}}_{x:\mathrm{S}}\mathrm{T}(x) \equiv \mathrm{S} \qquad \frac{p : \mathsf{Gph1}^{\mathbf{0}}_{x:\mathrm{S}}\mathrm{T}(x)}{\pi_1(p) \equiv p} \qquad \frac{(s,t) : \mathsf{Gph1}^{\mathbf{0}}_{x:\mathrm{S}}\mathrm{T}(x)}{(s,t) \equiv s}$$

The Polymorphic Identity

Lemma: given $\alpha : \prod_{X:\mathsf{Type}} X \to X$, for any *path-discrete* type A together with $x : A \vdash B(x)$ Type and $a : A$ with $b : B(a)$, the type $B(\alpha\ A\ a)$ is inhabited.

Three steps to prove parametricity:

1. Define a function $\mathsf{step1} : \prod_{i:I} \mathsf{Gph1}^{i}_{x:A} B(x)$ such that $\mathsf{step1}(\mathbf{0}) \equiv \alpha\ A\ a$

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3. Taking the first projection of $\mathsf{step1}(i)$ for $i : I$ gives a path $\mathsf{step3} : \mathsf{Path}_{i.A}(\alpha\ A\ a,\ \pi_1(\mathsf{step1}(\mathbf{1})))$, and since A is path-discrete, this yields an identity $\alpha\ A\ a =_A \pi_1(\mathsf{step1}(\mathbf{1}))$, along which we can transport $\mathsf{step2}$ to obtain an inhabitant of $B(\alpha\ A\ a)$. □

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- The approach to this problem via internal parametricity in cohesive HoTT suffers none of these defects, and easily handles examples such as the circle, for which the analogous Awodey-Frey-Speight encoding is already quite complex.

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- In this talk we have mainly considered *unary* parametricity, but this approach handles binary and n-ary parametricity just as well.
  - Jason Reed has a nice formalization of K-ary parametricity for any type K with decidable equality: https://github.com/jcreedcmu/aberle-parametricity-exercise/blob/main/ExerciseN.agda

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  - Further work: internal parametricity for linear programs (in some form of linear dependent type theory?)

Thank you!