# Amortized Analysis via Coalgebra

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**In a category of algebras,**

**amortized analyses are coalgebra morphisms.**

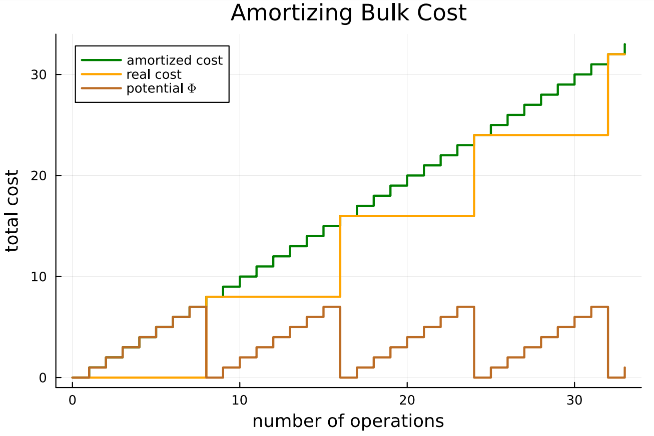
# Background

# Amortized Analysis

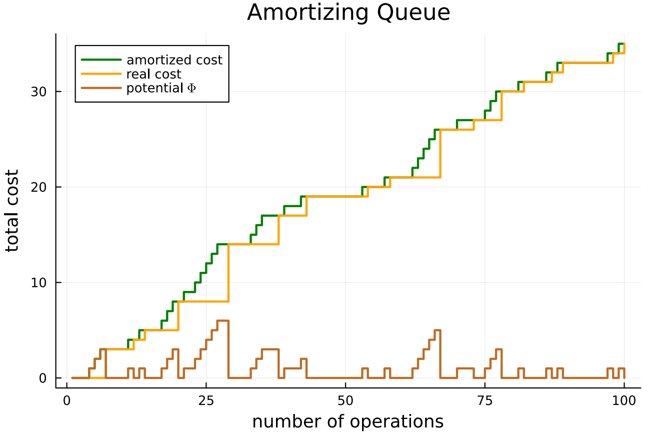
*In many uses of data structures, a* **sequence of operations**, *rather than just a single operation, is performed, and we are interested in the* **total time of the sequence**, *rather than in the times of the individual operations.*

—*Tarjan, 1985*

# Amortized Analysis (cont.)



# Amortized Analysis (cont.)



# Amortized Analysis: Potential Method

Let $d, d' \in D$ be states of a data structure. For each operation:

$$\textbf{amortized cost} = \textbf{real cost} + \Phi(d') - \Phi(d)$$

Here, $\Phi : D \to \mathbb{Z}$ is maps states to "potential", extra imagined up-front cost to offset big operations.

- Cheap operations save potential: $\Phi(d') > \Phi(d)$.
- Expensive operations spend potential: $\Phi(d') < \Phi(d)$.

# Abstract Cost Analysis via the Writer Monad

**calf** is an effectful dependent type theory for studying the cost and behavior of algorithms and data structures.

**Example**

$$
\begin{aligned}
&\mathit{isort} : \mathsf{list}(E) \rightharpoonup \mathsf{F}(\mathsf{list}(E)) \\
&\mathit{isort}\ [] = \mathsf{ret}([]) \\
&\mathit{isort}\ (x :: xs) = \\
&\quad \mathsf{bind}\, xs' \leftarrow \mathit{isort}\ xs\, \mathsf{in} \\
&\quad \mathit{insert}\ x\ xs'
\end{aligned}
$$

# Abstract Cost Analysis via the Writer Monad (cont.)

For effects, **calf** is "polarized" (à la CBPV/EEC/LNL).

**Cost Annotation**

To instrument a program with cost $c$, effect $\mathsf{charge}\langle \$c \rangle$.

**Key Idea**

Effects commute with computations: effects now are effects later.

$$\mathsf{charge}\langle \$c \rangle \,;\, (\lambda x.\ e) = \lambda x.\ (\mathsf{charge}\langle \$c \rangle \,;\, e)$$

$$\mathsf{charge}\langle \$c \rangle \,;\, (e_1, e_2) = ((\mathsf{charge}\langle \$c \rangle \,;\, e_1), (\mathsf{charge}\langle \$c \rangle \,;\, e_2))$$

# Abstract Cost Analysis via the Writer Monad (cont.)

**Semantics**

Category of cost algebras, $\mathbf{Alg}(T)$, where $T$ is the writer monad $\mathbb{C} \times (-)$, using adjunction $\mathsf{F} \dashv \mathsf{U} : \mathbf{Alg}(T) \to \mathcal{C}$.

Notation:

$$\cfrac{\cfrac{\mathsf{F}A \to X}{A \rightharpoonup X}}{A \to \mathsf{U}X}$$

$$\frac{\delta_{\$} : A \to \mathbb{C} \qquad \delta_{\circ} : A \to B}{\delta : A \rightharpoonup \mathsf{F}B}$$

# Coalgebraic Semantics of Data Structures

**Definition**

A *signature* is an endofunctor $\Sigma : \mathbf{Alg}(T) \to \mathbf{Alg}(T)$.

**Example**

The signature for queues

$$\Sigma X = (E \rightharpoonup X) \times (\mathsf{F}1 + (E \ltimes X))$$

power — copower

provides two operations

$$\begin{aligned} \text{enqueue} &: E \rightharpoonup X \\ \text{dequeue} &: \mathsf{F}1 + (E \ltimes X) \end{aligned}$$

where $X$ is the "state type".

# Coalgebraic Semantics of Data Structures (cont.)

**Definition**

A $\Sigma$-*coalgebra* $(D,\ \delta : D \to \Sigma D)$ is an implementation of $\Sigma$.

**Example**

With signature $\Sigma$ for queues as before:

- carrier $D = \mathsf{F}(\mathsf{list}(E))$, and
- transition map

$$\delta : D \to (E \rightharpoonup D) \times (\mathsf{F}1 + (E \ltimes D))$$

implements the operations.

# Coalgebraic Semantics of Data Structures (cont.)

**Definition (morphism of $\Sigma$-coalgebras)**

A morphism $(D, \delta) \to (S, \sigma)$ is a morphism $\Phi : D \to S$ that preserves the $\Sigma$-coalgebra structure:

$$\begin{array}{ccc} D & \xrightarrow{\delta} & \Sigma D \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Sigma\Phi} \\ S & \xrightarrow{\sigma} & \Sigma S \end{array}$$

**Key Idea**

The specification $S$ *simulates* the data structure $D$.

# Basic Examples

# Example: Bulk Cost

**Specification (carrier F1)**

Charges $1 every cycle:

$$\sigma : 1 \rightharpoonup \mathsf{F}1$$
$$\sigma\ * = \mathsf{charge}\langle \$1 \rangle \ ; \mathsf{ret}(*)$$

# Example: Bulk Cost

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**Implementation (carrier $\mathsf{F}(\mathsf{Fin}_8)$)**

Charges \$8 every 8 cycles:

$$\delta : \mathsf{Fin}_8 \rightharpoonup \mathsf{F}(\mathsf{Fin}_8)$$
$$\delta \; 7 = \mathsf{charge}\langle \$8 \rangle \, ; \mathsf{ret}(0)$$
$$\delta \; d = \mathsf{ret}(\mathsf{suc} \; d)$$

# Example: Bulk Cost (cont.)

Coalgebra morphism $\Phi : \mathsf{Fin}_8 \rightharpoonup \mathsf{F}1$ must satisfy:

$$\Phi \mathbin{;} \sigma = \delta \mathbin{;} \Sigma\Phi$$

# Example: Bulk Cost (cont.)

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Since $\mathsf{U}(\mathsf{F1}) \cong \mathbb{C}$, equivalently $\Phi : \mathsf{Fin}_8 \to \mathbb{C}$:

$$\Phi(d) + \sigma_{\$} = \delta_{\$}(d) + \Phi(\delta_{\circ}(d))$$

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When $\mathbb{C} = \mathbb{Z}$:

amortized cost

$$\sigma_{\$} = \delta_{\$}(d) + \Phi(\delta_{\circ}(d)) - \Phi(d)$$

real cost

new state $d'$

# Example: Bulk Cost (cont.)

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When $\mathbb{C} = \mathbb{Z}$:

$$\underbrace{\sigma_{\$}}_{\text{amortized cost}} = \underbrace{\delta_{\$}(d)}_{\text{real cost}} + \Phi(\underbrace{\delta_{\circ}(d)}_{\text{new state } d'}) - \Phi(d)$$

For example, $\Phi(d) = \$d$.

# Example: Batched Queue

$$(l_i, l_o)$$

$l_i$: inbox list; $l_o$: outbox list

- Enqueue to inbox;
- dequeue from outbox;
- move inbox to outbox when outbox empty.

$([4, 3], [1, 2])$

↓ enqueue 5

$([5, 4, 3], [1, 2])$

↓ dequeue 1

$([5, 4, 3], [2])$

↓ dequeue 2

$([5, 4, 3], [])$

↓ dequeue 3

$([], [4, 5])$

↓ enqueue 6

$([6], [4, 5])$

# Example: Batched Queue (cont.)

**Specification (carrier $\mathsf{F}(\mathsf{list}(E))$)**

Charges \$1 per enqueue, \$0 per dequeue:

$$\sigma : \mathsf{list}(E) \rightharpoonup \Sigma(\mathsf{F}(\mathsf{list}(E)))$$
$$\sigma\ .\mathsf{enqueue}\ l\ e = \mathsf{charge}\langle \$1 \rangle \mathbin{;} \mathsf{ret}(l \mathbin{+\!\!+} [e])$$
$$\sigma\ .\mathsf{dequeue} = \cdots$$

**Implementation (carrier $\mathsf{F}(\mathsf{list}(E)^2)$)**

Charges \$0 per enqueue, \$0 (usually) or $\$n$ (rarely) per dequeue.

# Example: Batched Queue (cont.)

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**Implementation (carrier $\mathsf{F}(\mathsf{list}(E)^2)$)**

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Let $\Phi(l_i, l_o) = \mathsf{charge}\langle \$(\mathsf{length}(l_i)) \rangle\ ;\ \mathsf{ret}(l_o \mathbin{+\!\!+} \mathsf{reverse}(l_i))$.

potential function (annotating $\mathsf{length}(l_i)$) — integrated behavior (annotating $l_o \mathbin{+\!\!+} \mathsf{reverse}(l_i)$)

# Amortizing Other Effects

## Non-Commutative Cost Models

Choosing $\mathbb{C} = \text{String}$, amortized string printing is buffering:

$$\Phi(\text{“hello”}) \mathbin{+\!\!+} \text{“world”} = \text{“hellowor”} \mathbin{+\!\!+} \Phi(\text{“ld”})$$

old buffer — spec print — impl print — new buffer

“Potential function” $\Phi$ is the inclusion, flushing the buffer.

# Amortizing Other Effects

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## Randomized Amortized Analysis

Using monad $\mathcal{D}(\mathbb{C} \times (-))$, amortize randomness.

# Amortizing Other Effects

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## Randomized Amortized Analysis

Using monad $\mathcal{D}(\mathbb{C} \times (-))$, amortize randomness.

## Expected Amortized Analysis

Using monad $\mathbb{C} \times \mathcal{D}(-)$, expected amortized analysis.

# Composition of Potential Functions

Potential functions are coalgebra morphisms, so they compose.

**Example**

For bulk cost amortization:

$$(D_{16}, \delta_{16}) \xrightarrow{\Phi'} (D_8, \delta_8) \xrightarrow{\Phi} (S, \sigma)$$

# Composition of Potential Functions (cont.)

To compose data structures with different signatures:

$$\int^{\Sigma} \mathbf{Coalg}(\Sigma)$$

**Example**

Amortized queues implemented via a pair of amortized stacks.

$$\mathsf{asQueue} : \Sigma_{\mathsf{Stacks}} \to \Sigma_{\mathsf{Queue}}$$

$$\Phi : \mathsf{asQueue}(D_{\mathsf{stacks}}, \delta_{\mathsf{stacks}}) \to (S_{\mathsf{queue}}, \sigma_{\mathsf{queue}})$$

# Generalizations

# Lax Amortized Analysis

Sometimes, the amortized cost is an overestimate:

$$\textbf{amortized cost} \geq \textbf{real cost} + \Phi(d') - \Phi(d)$$

In some cases, the change in potential is less than the spec.

**Key Idea**

Upgrade to bicategories: programs are ordered by inequality.

# Lax Amortized Analysis (cont.)

**Definition (colax morphism of Σ-coalgebras)**

A *colax morphism* $(D, \delta) \to (S, \sigma)$ is a morphism $\Phi : D \to S$ that "colaxly" preserves the Σ-coalgebra structure:

$$\begin{array}{ccc} D & \xrightarrow{\ \delta\ } & \Sigma D \\ {\scriptstyle \Phi}\downarrow & \overset{\varphi}{\Swarrow} & \downarrow{\scriptstyle \Sigma\Phi} \\ S & \xrightarrow[\ \sigma\ ]{} & \Sigma S \end{array}$$

Here, 2-cell $\varphi : (\Phi \mathbin{;} \sigma) \Leftarrow (\delta \mathbin{;} \Sigma\Phi)$ is a proof of inequality.

Choose $\mathcal{C} = \mathbf{Poset}$, letting all types but $\mathbb{C} = \omega$ be discrete.

$$\Phi(d) + \sigma_{\$} \geq \delta_{\$}(d) + \Phi(\delta_{\circ}(d))$$

# Lax Amortized Analysis (cont.)

**Remark**

A 2-cell $\Phi \leq \Phi'$ justifies that $\Phi$ is a tighter analysis than $\Phi'$.

**Example**

For bulk cost, both

$$\Phi(d) = \$d$$
$$\Phi'(d) = \$(d + 1)$$

are coalgebra morphisms. Now, observe that $\Phi \leq \Phi'$.

# Splitting Potential

**Example**

Some operations split data structures into parts:

$$\Sigma X = X \otimes X$$

Informally, for multiple outputs:

total output potential

$$\sigma_{\$} \geq \delta_{\$}(d) + \sum_i \Phi(\delta_{\circ}(d)_i) - \Phi(d)$$

Made formal when $T$ is commutative, using map

$$\mathsf{F}A \otimes \mathsf{F}B \xrightarrow{\sim} \mathsf{F}(A \times B)$$

to add potential.

# Splitting Potential (cont.)

## Example

Let $\Phi : \mathsf{F}A \to \mathsf{F}1$:

$$\begin{array}{ccccc} \mathsf{F}A & \xrightarrow{\delta} & \mathsf{F}A \otimes \mathsf{F}A & & \\ \downarrow{\scriptstyle \Phi} & \geq & \downarrow{\scriptstyle \Phi \otimes \Phi} & & \\ \mathsf{F}1 & \xrightarrow{\sigma} & \mathsf{F}1 \otimes \mathsf{F}1 & \overset{+}{=\!=\!=} & \mathsf{F}1 \end{array}$$

In other words:

$$\Phi(d) + \sigma_{\$} \geq \delta_{\$}(d) + \sum_{i \in \{1,2\}} \Phi(\delta_{\circ}(d)_i)$$

# Combining Potential

Some data structures, e.g. queues, support an append operation:

$$\frac{X \otimes X \to X}{X \to (X \multimap X)}$$

But, $\Sigma X = X \multimap X$ is not functorial!

Instead, use a profunctor $\Sigma : \mathbf{Alg}(T)^{\mathrm{op}} \times \mathbf{Alg}(T) \to \mathbf{Set}$. Here:

$$\Sigma(X^-, X^+) = (X^- \otimes X^-) \multimap X^+$$

As desired, this gives us:

$$\sum_{i \in \{1,2\}} \Phi(d_i) + \sigma_\$ \geq \delta_\$(d) + \Phi(\delta_\circ(d))$$

total input potential

# Conclusion

# Conclusion

## Foundation

In a category of cost algebras, a **coalgebra morphism** is a generalized **potential function** of amortized analysis.

- Integrates cost and behavior;
- Provides a theory of composition for amortized analyses;
- Elegantly supports amortization of arbitrary effects;
- Simplifies formalization.

## Extensions

- Inexact amortized analysis expressed via bicategories;
- Splitting potential expressed via monoidal products;
- Combining potential expressed via profunctors.