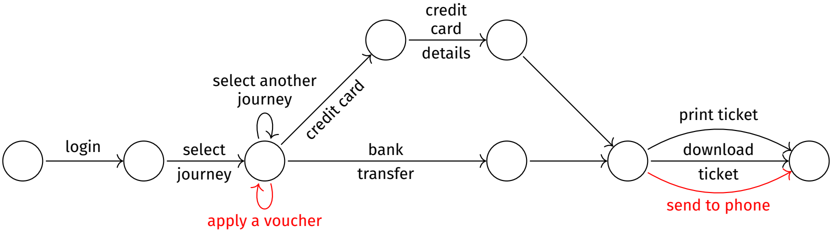
On Kleisli liftings and decorated trace semantics

Joint work with Harsh Beohar and Sebastian Küpper

Daniel Luckhardt University of Sheffield June 19, 2024

Motivation

Featured Transition Systems (FTS): Conditions that come from configurations of features



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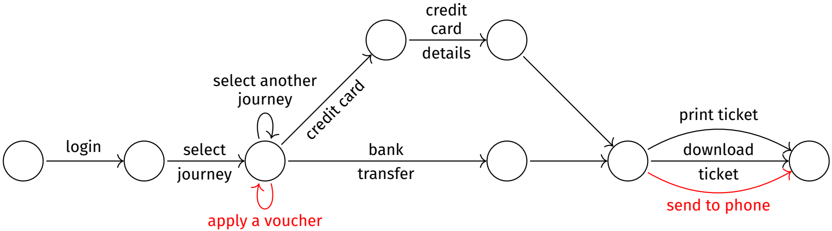

Adding upgrades

Assume that at any moment the configurations may be upgraded. Upgrades (i.e. relations $k' \leq k$ of conditions k, k') come with an order and the property that they add possible transitions, i.e. if $k' \leq k$, then $x \xrightarrow{a,k} y \implies x \xrightarrow{a,k'} y$.

Conditional transition systems

Fix: set X state space; set $\mathbb{A}$ of action; finite poset $\mathbb{K}$ of conditions (or configurations in the language of FTS).

Definition

A *conditional transition system* (CTS) is $(X, \mathbb{A}, \mathbb{K}, \rightarrow)$, where $\rightarrow \subseteq X \times \mathbb{A} \times \mathbb{K} \times X$ is the transition relation (also written $x \xrightarrow{a,k} y$) satisfying :

$$\forall_{x,y \in X, a \in \mathbb{A}, k,k' \in \mathbb{K}} \left(x \xrightarrow{a,k} y \wedge k' \leq k\right) \implies x \xrightarrow{a,k'} y.$$

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Special cases

- For $\mathbb{K}$ trivial (singleton): Labelled transition system (LTS).
- For $\mathbb{K}$ unordered: featured transition systems.

Understanding LTSs

For labelled transition system, there are several notions of equivalences, e.g.

Language equivalence

Fix a set $\downarrow \subseteq X$ of terminating states. Two CTSs on X are language equivalent, if each state admits the same *traces*, i.e. admissible actions $x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \ldots \to \bullet \in \downarrow$.

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Other equivalences: ready equivalence, failure equivalence

In terms of category theory

These equivalences can be formulated in terms of final coalgebras in Kleisli categories.

For
language equivalence ($1 = \{\bullet\}$):

$$\frac{X \to \mathbb{A}^* + 1 \in \mathbf{K}\ell(\mathcal{P})}{X \to \mathcal{P}(\mathbb{A}^* + 1) \in \mathbf{Set}},$$

i.e. with for $B(X) = \mathbb{A} \times X + 1$:

$$\begin{array}{ccc} B(X) & \longrightarrow & B(\mathbb{A}^* + 1) \\ \uparrow & & \uparrow \\ X & \longrightarrow & \mathbb{A}^* + 1 \end{array}$$

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Consider again—for simplicity—the case of language equivalence: We need to expand

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$$\mathbb{K} \times X \to \mathcal{P}(\mathbb{A}^* + 1) \qquad \left(\text{equiv. } X \to (\mathcal{P}(\mathbb{A}^* + 1))^{\mathbb{K}}\right)$$

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Solution

Let $G(X) = \mathbb{K} \times X$. So we have type $GX \to \mathcal{P}GY$. This is the type of the **relative monad induced by the monad $\mathcal{P}$ and the endofunctor G!**

What is a relative monad?

Let $G \colon \mathbf{C} \to \mathbf{C}$ be an endofunctor.

Our presentation of relative monad is an instance of a more general formulation in which G is not necessarily an endofunctor.

Definition

A *G-relative monad* comprises the following data:

1. an object mapping $T \colon \mathbf{C} \to \mathbf{C}$;
2. for every object $X \in \mathbf{C}$, there is a *unit* map $\eta_X \in \mathbf{C}(GX, TX)$;
3. for every arrow $f \in \mathbf{C}(GX, TY)$ there is its *Kleisli lifting* $f^\sharp \in \mathbf{C}(TX, TY)$ satisfying

left unit law	right unit law	associativity
$\begin{array}{ccc} GX & \xrightarrow{\eta_X} & TX \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f^\sharp} & \\ TY & & \end{array}$	$\begin{array}{c} TX \\ {\scriptstyle \mathrm{id}_{TX}}\downarrow \; \downarrow{\scriptstyle \eta_X^\sharp} \\ TX \end{array}$	$\begin{array}{ccc} TX & \xrightarrow{(g^\sharp \circ f)^\sharp} & TZ \\ {\scriptstyle f^\sharp}\downarrow & \nearrow {\scriptstyle g^\sharp} & \\ TY & & \end{array}$.

Functoriality and Kleisli category

Is T actually a functor? Yes, set $Tf = (\eta_Y \circ G(f))^\sharp$ for $f\colon X \to Y$ in $\mathbf{C}$:

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Kleisli category $\mathbf{K}\ell_G(T)$

on objects	on morphism	identities	composition
$\dfrac{X \in \mathbf{C}}{X \in \mathbf{K}\ell_G T}$	$\dfrac{f\colon GX \to TY \in \mathbf{C}}{f\colon X \to Y \in \mathbf{K}\ell_G T}$	$\dfrac{\eta_X \in \mathbf{C}}{\mathrm{id}_X \in \mathbf{K}\ell_G T}$	$\dfrac{GX \xrightarrow{f} TY \xrightarrow{g^\sharp} TX}{g \bullet f \in \mathbf{K}\ell_G T}$

The solution

Given an ordinary monad $T = (T, \eta, (\_)^\sharp)$ and an endofunctor $G \colon \mathbf{C} \to \mathbf{C}$, we can always construct a relative monad $T^G = (T^G, \eta^{T^G}, (\_)^\sharp)$ given by

1. $T^G(X) = TGX$ for $X \in \mathbf{C}$
2. $\eta_X^{T^G} = \eta_{GX}$ for $X \in \mathbf{C}$
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$\mathbf{K}\ell_G(T)$ is what we sought!

Our Problem

Putting our discussions of language (or other kinds of) equivalences together, we need lifting of $B\colon \mathbf{C} \to \mathbf{C}$ to a functor $\widetilde{B}\colon \mathbf{K}\ell(T^G) \to \mathbf{K}\ell(T^G)$, i.e. for $L\colon X \mapsto X, f \mapsto \eta_{\operatorname{cod} f} \circ G(f)$ the diagram on the right commutes.

$$\begin{array}{ccc} \mathbf{K}\ell(T^G) & \xrightarrow{\widetilde{B}} & \mathbf{K}\ell(T^G) \\ {\scriptstyle L}\big\uparrow & & \big\uparrow{\scriptstyle L} \\ \mathbf{C} & \xrightarrow{B} & \mathbf{C} \end{array}$$

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Preliminary result

If G preserves B, i.e. there is a natural isomorphism $\rho\colon GB \cong BG$, the existence of a Kleisli lifting $\mathbf{K}\ell(T) \xrightarrow{\overline{B}} \mathbf{K}\ell(T)$ of B implies the existence of a Kleisli lifting $\mathbf{K}\ell(T^G) \xrightarrow{\widetilde{B}} \mathbf{K}\ell(T^G)$ of B as defined on the right.

$$\begin{array}{ccc} GBX & \overset{\widetilde{B}f}{\dashrightarrow} & TGBY \\ \cong \downarrow \rho_X & & T\rho_Y^{-1} \uparrow \cong \\ BGX & \xrightarrow{\bar{B}f} & TBGY \end{array}$$

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Unfortunately, too strong conditions on G—take $\mathbb{K}$ nontrivial:
$GB(X) = \mathbb{K} \times (\mathbb{A} \times X + 1) \neq \mathbb{A} \times \mathbb{K} \times X + 1 = BG(X)$.

Lifting the machine endofunctor

Our aim is now to construct a Kleisli lifting of the machine endofunctor

$$B(X) = \mathbb{A} \times X + O \qquad \text{(shorthand } A = \mathbb{A} \times \_\text{).}$$

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Provided these axioms we can build a relative Kleisli lifting $\widehat{B}$ by

$$\begin{array}{ccc}
G(AX+O) & \xrightarrow{\quad \widehat{B}f \quad} & TG(AY+O) \\
{\scriptstyle G\iota_{AX} \nabla G\iota_O} \uparrow \cong & & \uparrow {\scriptstyle TG\iota_{AY} \nabla TG\iota_O} \\
GAX + GO & \xrightarrow[\widetilde{A}f + \eta_{GO}]{\qquad\qquad} & TGAY + TGO
\end{array}$$

where $\nabla\colon (X \xrightarrow{f} Y, X' \xrightarrow{f'} Y) \mapsto (X + X' \xrightarrow{f \nabla f'} Y)$ given by coproduct.

Preservation properties

For this relative Kleisli lifting $\widehat{B}$ we have:

- If G preserves colimits and the initial algebra $h\colon B(\mu_B) \xrightarrow{\cong} \mu_B$ of B exists in $\mathbf{C}$, then $Lh\colon \widehat{B}(\mu_B) \xrightarrow{\cong} \mu_B$ is the initial algebra of $\widehat{B}$ in $\mathbf{K}\ell(T^G)$.

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- By a result of Freyd 1992:
  μ_B is the final coalgebra of $(Lh)^{-1}\colon \mu_B \to \widehat{B}(\mu_B)$ of $\widehat{B}$ in $\mathbf{K}\ell(T^G)$.

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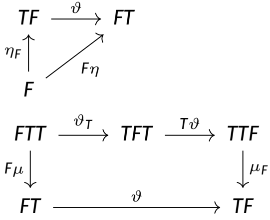
- The final coalgebra characterises the behaviour of the systems of interest.

Constructing Kleisli laws ϑ

The crucial point was

$$\frac{\textbf{A3}\ \overline{A}\colon \mathbf{K}\ell(T) \to \mathbf{K}\ell(T) \text{ is a Kleisli lifting of } A.}{\mathbf{K}\ell\text{-law } \vartheta\colon FT \Rightarrow TF}$$

Recall, that a $\mathbf{K}\ell$-law is a natural transformation such that the diagram on the right commute.



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In applications this will be based on

$$\begin{aligned} \mathbf{Set}(X, \mathcal{P}Y) &\overset{\theta_{X,Y}}{\cong} \mathcal{P}(X \times Y) \cong \mathbf{Set}(X \times Y, 2), \\ \mathbf{Pos}(X, \mathcal{P}_{\downarrow}Y) &\overset{\theta_{X,Y}}{\cong} \left\{ R \subseteq X \times Y \;\middle|\; \begin{array}{c} R \text{ is up closed in } X, \\ R \text{ is down closed in } Y \end{array} \right\} \cong \mathbf{Pos}(X \times Y^o, \underbrace{\{0 \le 1\}}_{2}), \end{aligned}$$

where $\mathcal{P}_{\downarrow}$ = downset monad and Y^o = dual poset of Y.

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where $\mathcal{P}_{\downarrow}$ = downset monad and Y^{o} = dual poset of Y.
We can view a Kleisli law ϑ_X as a relation on $\mathcal{P}F(X) \times F(X)$ ($T = \mathcal{P}, \mathcal{P}_{\downarrow}$).

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For relations

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Set IX to be X or the dual poset, resp.; and further:

$$\in_X \;=\; \theta_{\mathcal{P}X,X}(\mathrm{id}_{\mathcal{P}X}) \in \Phi(\mathcal{P}X \times IX), \quad \text{the \emph{element relation}}$$

$$\sigma_X\colon \Phi X \to \Phi(\mathbb{A} \times X), \quad R \mapsto \{(a,r) \mid a \in \mathbb{A}\}, \quad \text{the \emph{predicate lifting}}$$

Applying predicate lifting we get $\sigma_{\mathcal{P}X\times IX}(\in_X) \in \Phi F(\mathcal{P}X \times IX)$; further let

$$\lambda_{X,Y} = F(\mathrm{pr}_X) \vartriangle F(\mathrm{pr}_{IY})\colon F(X \times IY) \to FX \times FIY = FX \times IFY$$

where $f \vartriangle f'$ is the “diagonal operation” provided by the universal property of the product on the codomain;

$$\exists_f(V) = \{x \mid f(x) \in V\}, \quad \text{\emph{direct image} of } V \subseteq \operatorname{cod} f$$

$$\vartheta_X \stackrel{\text{def}}{=} \theta^{-1}_{F\mathcal{P}X,FX} \circ \exists_{\lambda_{\mathcal{P}X,X}} \circ \sigma_{\mathcal{P}X\times IX}\,(\in_X)\,.$$

Let $\mathbf{C} \in \{\mathbf{Set}, \mathbf{Pos}\}$ and $\Omega = 2$. If F preserves weak pullbacks, then ϑ is a $\mathbf{K}\ell$-law.

Axiomatisation

A3.1 There is an indexed category $\Phi\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}$ (also written $\Phi f = f^*$) with a bifibration structure, i.e. for each $f\colon X \to Y \in \mathbf{C}$ there is an adjoint situation $\exists_f \dashv f^*\colon \Phi X \to \Phi Y$.

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A3.3 There is a monad (T, η, μ) on $\mathbf{C}$ with the following correspondence

$$\theta_{X,Y}\colon \mathbf{C}(X, TY) \cong \Phi(X \times IY) \qquad \text{(for each } X, Y \in \mathbf{C}\text{)}$$

such that the following diagrams commute for each $f\colon X \to X', g\colon Y \to Y' \in \mathbf{C}$.

$$\begin{array}{ccc} \mathbf{C}(X, TY) & \xrightarrow{\theta_{X,Y}} & \Phi(X \times IY) \\ {\scriptstyle \_\circ f}\uparrow & & \uparrow{\scriptstyle (f\times IY)^*} \\ \mathbf{C}(X', TY) & \xrightarrow{\theta_{X',Y}} & \Phi(X' \times IY) \end{array} \qquad \begin{array}{ccc} \mathbf{C}(X, TY) & \xrightarrow{\theta_{X,Y}} & \Phi(X \times IY) \\ \downarrow{\scriptstyle Tg\circ\_} & & \downarrow{\scriptstyle \exists_{(X\times Ig)}} \\ \mathbf{C}(X, TY') & \xrightarrow{\theta_{X,Y'}} & \Phi(X \times IY') \end{array} .$$

Axiomatisation

A3.1 There is an indexed category $\Phi\colon \mathbf{C}^{op} \to \mathbf{Pos}$ (also written $\Phi f = f^*$) with a bifibration structure, i.e. for each $f\colon X \to Y \in \mathbf{C}$ there is an adjoint situation $\exists_f \dashv f^*\colon \Phi X \to \Phi Y$.

A3.2 There is an endofunctor $I\colon \mathbf{C} \to \mathbf{C}$ such that $F \circ I = I \circ F$.

A3.3 There is a monad (T, η, μ) on $\mathbf{C}$ with the following correspondence

$$\theta_{X,Y}\colon \mathbf{C}(X, TY) \cong \Phi(X \times IY) \qquad \text{(for each } X, Y \in \mathbf{C}\text{)}$$

such that the following diagrams commute for each $f\colon X \to X', g\colon Y \to Y' \in \mathbf{C}$.

$$\begin{array}{ccc} \mathbf{C}(X, TY) & \xrightarrow{\theta_{X,Y}} & \Phi(X \times IY) \\ {\scriptstyle \_\circ f}\uparrow & & \uparrow{\scriptstyle (f\times IY)^*} \\ \mathbf{C}(X', TY) & \xrightarrow{\theta_{X',Y}} & \Phi(X' \times IY) \end{array} \qquad \begin{array}{ccc} \mathbf{C}(X, TY) & \xrightarrow{\theta_{X,Y}} & \Phi(X \times IY) \\ \downarrow{\scriptstyle Tg\circ\_} & & \downarrow{\scriptstyle \exists_{(X\times Ig)}} \\ \mathbf{C}(X, TY') & \xrightarrow{\theta_{X,Y'}} & \Phi(X \times IY') \end{array} .$$

A3.4 There is an indexed morphism (aka predicate liftings) $\sigma\colon \Phi \Rightarrow \Phi F$.

Kleisli-law ϑ in a general setting

Using θ and axioms **A3.1-4** we can repeat the definitions of ϑ_X from above.

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Kleisli-law ϑ in a general setting

Using θ and axioms **A3.1-4** we can repeat the definitions of ϑ_X from above. Further we can abstractly define

$$\Delta_X = \theta_{X,X}(\eta_X), \quad \text{the diagonal}$$

$$S \odot R = \theta_{X,Z}\left(\theta^{-1}_{Y,Z}(S) \bullet \theta^{-1}_{X,Y}(R)\right), \quad \textit{relation composition}$$

$$\odot \colon \Phi(Y \times IZ) \times \Phi(X \times IY) \to \Phi(X \times IZ)$$

$$\tilde{\sigma}_{X,Y} = \exists_{\lambda_{X,IY}} \circ \sigma_{X \times IY}, \quad \textit{relational predicate lifting}$$

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Crucial instance of Beck-Chevalley condition (*)

If
$$\begin{array}{ccc} F(X \times IY) & \xrightarrow{\lambda_{X,Y}} & FX \times IFY \\ {\scriptstyle F(f \times IY)}\downarrow & & \downarrow{\scriptstyle Ff \times IFY} \\ F(X' \times IY) & \xrightarrow[\lambda_{X',Y}]{} & FX' \times IFY \end{array}$$
commutes, so
$$\begin{array}{ccc} \Phi(F(X \times IY)) & \xrightarrow{\exists_{\lambda_{X,Y}}} & \Phi(FX \times IFY) \\ {\scriptstyle (F(f \times IY))^*}\uparrow & & \downarrow{\scriptstyle (Ff \times IFY)^*} \\ \Phi(F(X' \times IY)) & \xrightarrow[\exists_{\lambda_{X',Y}}]{} & \Phi(FX' \times IFY) \end{array}.$$

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Results in the general setting

Assuming this Beck-Chevalley condition (*):

$$\exists_{\lambda_{X,IY}} \circ \sigma_{X \times IY}(\Delta_X) = \Delta_{FX} \quad \Longleftrightarrow \quad \begin{array}{ccc} TF & \xrightarrow{\vartheta} & FT \\ {\scriptstyle \eta_F} \big\uparrow & \nearrow {\scriptstyle F\eta} & \\ F & & \end{array}$$

$$\tilde{\sigma}_{X,Y} \text{ preserves } \odot \quad \Longleftrightarrow \quad \begin{array}{ccccc} FTT & \xrightarrow{\vartheta_T} & TFT & \xrightarrow{T\vartheta} & TTF \\ {\scriptstyle F\mu} \big\downarrow & & & & \big\downarrow {\scriptstyle \mu_F} \\ FT & & \xrightarrow{\qquad \vartheta \qquad} & & TF \end{array}$$

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Assuming this Beck-Chevalley condition (*):

$$\exists_{\lambda_{X,IY}} \circ \sigma_{X \times IY}(\Delta_X) = \Delta_{FX} \iff \begin{array}{ccc} TF & \xrightarrow{\vartheta} & FT \\ {\scriptstyle \eta_F}\uparrow & \nearrow{\scriptstyle F\eta} & \\ F & & \end{array}$$

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Advantages of the abstract approach

- easily verifiable criterion
- further applications like quantitative extensions, e.g. with probabilities or weights from a semiring.

Concluding remarks

Application to language equivalence

Let $I = \begin{cases} \text{identity} \\ \text{dual poset} \end{cases}$ and

$c\colon X \to \mathbb{A} \times X + O \in \mathbf{K}\ell(T^G)$ with $T \in \{\mathcal{P}, \mathcal{P}_\downarrow\}$ be a coalgebra.

- Assumption **A1**, **A2** and **A3.1-4** are easily checked.
- Case $O = 1$ gives language equivalence. Thus we can calculate language equivalence by a final coalgebra.

Concluding remarks

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Finally,

- Same works for ready equivalence.
- Our results extend to **Cppo**-enrichment.
- further research: quantitative enrichment of transitions

Relationship among $\mathbf{K}\ell(T)$, $\mathbf{K}\ell(T^G)$, and $\mathbf{K}\ell_{(}T)$

There is the following relationship between the classical $\mathbf{K}\ell(T)$, $\mathbf{K}\ell_G(T)$ and $\mathbf{K}\ell(T^G)$:

$$
\begin{array}{c} X \\ \uparrow \\ X \end{array}
\qquad
\begin{array}{c} \eta_{GY} \circ GF \\ \uparrow \\ f \end{array}
\qquad
\begin{array}{ccc}
\mathbf{K}\ell(T^G) & \xrightarrow{D} & \mathbf{K}\ell(T) \\
{\scriptstyle L}\uparrow & \searrow^{R} & {\scriptstyle L'}\uparrow \dashv \downarrow{\scriptstyle R'} \\
\mathbf{C} & \xrightarrow[G]{} & \mathbf{C}
\end{array}
\qquad
L' \dashv R' \text{ classical adj.}
$$

Cppo-enrichement

Recall: A category **C** is a **Cppo**-enriched category whenever its hom-set forms a ω-cpo with a bottom and the composition of arrows is a continuous function.

Cppo-enrichement

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For the preliminary result

Moreover, if $\overline{B}$ is locally continuous (when $\mathbf{K}\ell(T)$ is **Cppo**-enriched), then so is $\widetilde{B}$ defined above.

Assuming **A1-3**

If $\mathbf{K}\ell(T)$ is **Cppo**-enriched, $\overline{A}$ is locally continuous, and the operation $\_ + g$ commutes with the ω-directed joins, i.e. for any increasing families of arrows $(f_i \in \mathbf{K}\ell(T)(X, Z))_{i \in \mathbb{N}}$ we have

$$\bigvee_{i \in \mathbb{N}} (f_i + g) = (\bigvee_{i \in \mathbb{N}} f_i) + g,$$

then $\widehat{B}$ is locally continuous.

Quantitative example

Working in **Set** take

- Ω the Lawvere quantale $([0,1], \geq)$ (with $r \oplus r' = \min(r + r', 1)$);
- $T = \mathcal{P}_\Omega$, the Ω-valued powerset monad,
  $\mathcal{P}_\Omega(X) = \Omega^X$ on objects,
  $Tf(g)(y) = \inf_{f(x)=y} g(x)$ (for $f : X \to Y$) on arrows,
  $\mu_X(G)(x) = \inf_{g \in \mathcal{P}_\Omega X} G(g) \oplus g(x)$ multiplication;
- $F = \mathcal{D}$, the distribution monad;
- $\sigma_X(p)(\mu) = \mathbb{E}_\mu(p) = \sum_{x \in X} p(x) \cdot \mu(x) \qquad$ (for each $\mu \in \mathcal{D}X$).

The left adjoint $\exists_{\lambda_X}(M)(\mu, \nu) = \inf_{\lambda_X(\omega)=(\mu,\nu)} M(\omega)$ computes the optimal transport between two distributions μ, ν in M.
So using our theorem we obtain a **Kℓ**-law

$$\vartheta_X(M)(\mu) = \inf_{\lambda_{TX,X}(\omega)=(M,\mu)} \mathbb{E}_\omega(\in_X) = \inf_{\lambda_{\mathcal{D}X,X}(\omega)=(M,\mu)} \sum_{(p,x) \in \mathcal{P}_\Omega X \times X} p(x) \cdot \omega(p, x),$$

where $M \in \mathcal{D}(\mathcal{P}_\Omega X)$ and $\mu \in \mathcal{D}(X)$.