A Convenient Topological Setting for Higher-Order Probability Theory

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1. Backstory: Higher-Order Probability Theory
2. The Protagonists: Baire, Riesz, Radon, Giry
3. The Base Category: QCB Spaces
4. Construction: a Riesz Representation Theorem
5. Finale: The Baire Probability Monad

1. Backstory: Higher-Order Probability Theory

Fundamental problem:

Theorem.[1] The category Meas of measurable spaces and measurable maps is *not* cartesian closed.

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$\rightsquigarrow$ problem in the semantics of probabilistic functional programming languages

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3. a notion of probability measure giving rise to a probability monad on C,
4. such that the Lebesgue measure on $[0, 1]$ is a probability measure of this kind.

One solution:

"A Convenient Category for Higher-Order Probability Theory"[2]

$\rightsquigarrow$ introduces *quasi-Borel spaces* (QBS), a cartesian closed extension of standard Borel spaces, together with a notion of "probability measure" adapted to this setting.

[2] Chris Heunen et al. "A convenient category for higher-order probability theory". In: *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2017, pp. 1–12.

Curious phenomenon in the setting of QBS: failure of *deterministic marginal independence.*

Theorem.[3] There is a probability measure on the QBS $\mathbb{R} \times 2^{\mathbb{R}}$, whose marginal on $2^{\mathbb{R}}$ is deterministic (given by $\delta_{\emptyset}$), which is, however, *not* the product of its marginals.

$$\text{“ } (0.1978\ldots, \emptyset),\ (0.6302\ldots, \emptyset),\ (0.4414\ldots, \emptyset),\ \ldots \text{ ”}$$

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Avoiding this phenomenon would require a *strongly affine* probability monad.

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Questions:

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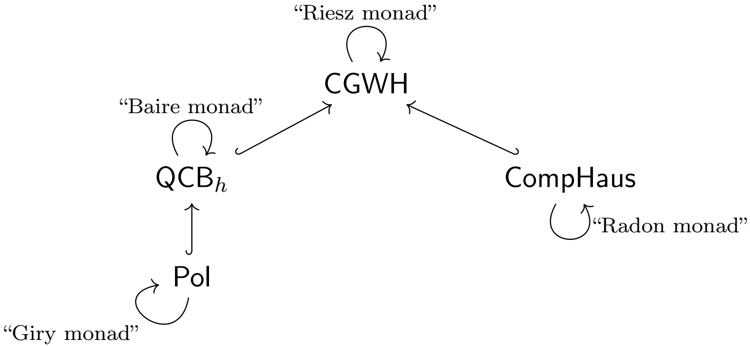
b. Can we realise higher-order probability theory using a more common notion of "sample space"?

— In particular, is there a *topological* setting for higher-order probability theory?

2. The Protagonists: Baire, Riesz, Radon, Giry

Two new monads: Riesz, Baire probability monad.



CGWH: compactly generated weakly Hausdorff (WH) spaces
QCB_h: WH quotients of countably based (QCB) spaces

Overview of various probability monads:

Name	Base Cat.	Type of Measure	Strongly Affine?	Enriched over CCC?
Giry	Meas	Any	✓	✗
–	QBS	see [3]	✗	✓
Giry	Pol	Borel	✓	✗
Radon	CompHaus	Radon	✓	✗
Riesz	CGWH	"k-regular"	?	✓
Baire	QCB_h	Baire	✓	✓

3. The Base Category: QCB Spaces

Definition.[4] A QCB space is a topological space X for which there exists a second countable space Y and a topological quotient map $Y \to X$.

[4] Matias Menni and Alex Simpson. "Topological and limit-space subcategories of countably-based equilogical spaces". In: *Mathematical Structures in Computer Science* 12.6 (2002), pp. 739–770.

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In addition, X is *weakly Hausdorff* if limits of convergent sequences are unique.

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Write QCB_h for the category of WH QCB spaces with continuous maps as morphisms.

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Example. Every separable metric space is second-countable and hence a WH QCB space.

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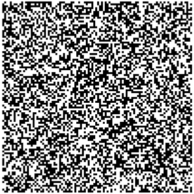
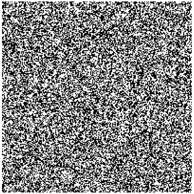
Example. The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is a WH QCB space (but *not* metrisable) with the strong topology.





Theorem. QCB_h is cartesian closed with countable limits and colimits.

All of this structure is inherited from the inclusion

$$\mathsf{QCB}_h \hookrightarrow \mathsf{CGWH}$$

from WH QCB spaces into compactly generated weakly Hausdorff (CGWH) spaces.

Remark. Any QCB space X is *sequential*, i.e. a subset $U \subseteq X$ is open *iff* for any sequence $x_n \to x \in U$, (x_n) is eventually in U.

$\rightsquigarrow$ QCB topologies are completely determined by convergent sequences.

Example. The topology of the space $C(X, Y) = Y^X$ of continuous maps between WH QCB spaces X, Y can be described as follows:

$$
\begin{aligned}
& f_n \to f \text{ in } C(X, Y) \\
\Leftrightarrow\ & f_n \to f \text{ uniformly on compact subsets of } X \\
\Leftrightarrow\ & \text{for all } x_n \to x \in X, \text{ we have that } f_n(x_n) \to f(x).
\end{aligned}
$$

Example. The space $C_b(X)$ of continuous bounded functions on a WH QCB space X carries a canonical WH QCB space topology:

$$C_b(X) := \operatorname{colim}_{n\in\mathbb{N}} C(X, B_n^{\mathbb{C}}(0))$$

A sequence (f_n) converges in $C_b(X)$ *iff* it is uniformly bounded by some constant R and it converges in $C(X, B_R^{\mathbb{C}}(0))$.

(Here, $B_R^{\mathbb{C}}(0)$ is the ball/disc of radius R centred at 0 in the complex numbers.)

4. Construction: a Riesz Representation Theorem

Theorem. The (continuous) dual of $C_b(X)$ can be identified with the space $\mathcal{M}_0(X)$ of finite complex Baire measures on X, via the bijection

$$\mathcal{M}_0(X) \to C_b(X)', \;\; \mu \mapsto \int_X (-)\, \mathrm{d}\mu.$$

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$\rightsquigarrow$ $\mathcal{M}_0(X) \subseteq C(C_b(X))$ also acquires a WH QCB topology in which $\mu_n \to \mu$ *iff* for all $f_n \to f \in C_b(X)$,

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Fact. This coincides with the weak topology when X is a Polish space and the (μ_n) are probability measures.

Definition. Let X be a WH QCB space.

$$\mathcal{M}(X) := \overline{\operatorname{span} \delta_\bullet(X)} \subseteq \mathcal{M}_0(X) \subseteq C(C_b(X)).$$

Using cartesian closedness of QCB_h, we obtain:

Theorem. The following maps are well-defined and continuous:

$$(-)_* : C(X,Y) \to C(\mathcal{M}(X), \mathcal{M}(Y)), \;\; f \mapsto f_*,$$

$$\delta_\bullet : X \to \mathcal{M}(X), \;\; x \mapsto \delta_x,$$

$$\textstyle\int : \mathcal{M}(\mathcal{M}(X)) \to \mathcal{M}(X), \;\; \pi \mapsto \left[A \mapsto \int_{\mathcal{M}(X)} \mu(A)\, \mathrm{d}\pi(\mu)\right],$$

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With this structure, $\mathcal{M}$ is an enriched monad on QCB_h!

Question: Is $\mathcal{M}$ commutative?

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Theorem. Let X, Y be WH QCB spaces, $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$, $f \in C_b(X \times Y)$. Then,

$$\int_X \int_Y f(x,y)\,\mathrm{d}\nu(y)\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

Proof. This holds for finitely supported measures and both sides are continuous in (μ, ν). □

5. Finale: The Baire Probability Monad

Definition. For X a WH QCB space,

$$\mathcal{P}(X) := \{\mu \in \mathcal{M}(X) \mid \mu \text{ probability measure}\}.$$

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Theorem. With $(-)_*, \delta, \int$ defined as for $\mathcal{M}$, $\mathcal{P}$ is a strongly affine, commutative enriched monad on QCB_h.

Takeaways:

1. In addition to the measurable-flavoured setting of quasi-Borel spaces, there are *topological* settings for higher-order probability theory.

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2. Higher-order probability theory and deterministic marginal independence are compatible.

3. In the topological setting, we have a category for higher-order probability theory whose objects are familiar kinds of spaces: we do not have to move beyond topological spaces.

Thank you!

Further details:

Benedikt Peterseim. "On Monadic Vector-Valued Integration". In: *MSc thesis, arXiv:2403.19681* (2024)

Peter Kristel and Benedikt Peterseim. "A Topologically Enriched Probability Monad on the Cartesian Closed Category of CGWH Spaces". In: *arXiv preprint arXiv:2404.08430* (2024)

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References II

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