Part I:
Graphical Symplectic Algebra

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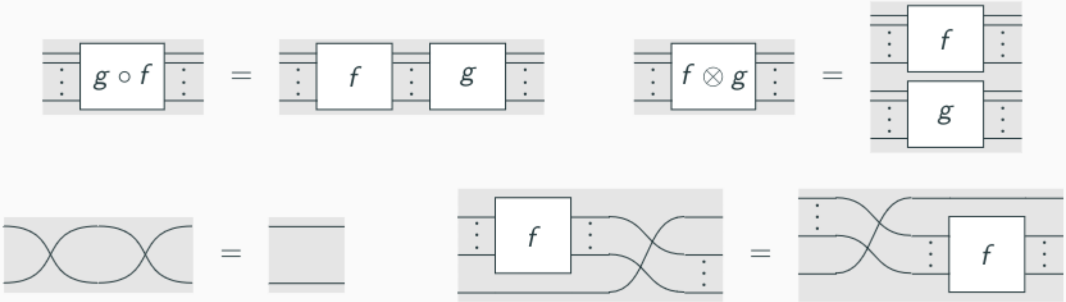
based on: arXiv:2401.07914

June 20, 2024

Motivation: I am bad at mathematics.

Props

A **prop** is a strict symmetric monoidal category generated by a single object...



A **compact prop** also allows for wires to be bent/unbent:

Graphical linear algebra

Affine matrices: generators

Given a field $\mathbb{K}$, finite dimensional affine transformations can be represented their **homogeneous coordinates matrices** (T, S are matrices, $\vec{a}, \vec{b}$ are vectors):

$$\left[\begin{array}{c|c} T & \vec{a} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} S & \vec{b} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} TS & T\vec{b}+\vec{a} \\ \hline 0 & 1 \end{array}\right]$$

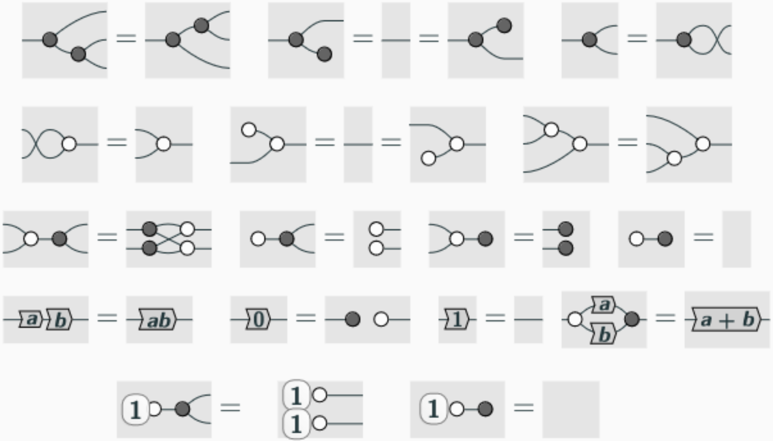
The prop of affine transformations between finite dimensional vector spaces is generated by the homogeneous coordinate matrices:

$[\![$  $]\!] = \left[\begin{array}{c|c} 1 & 0 \\ 1 & 0 \\ \hline & 1 \end{array}\right]$ $[\![$  $]\!] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline & & 1 \end{array}\right]$ $[\![$  $]\!] = \left[\begin{array}{c|c} a & 0 \\ \hline & 1 \end{array}\right]$

$[\![$  $]\!] = \left[\begin{array}{c|c} * & * \\ \hline & 0 \end{array}\right]$ $[\![$  $]\!] = \left[\begin{array}{c|c} * & 0 \\ \hline & 0 \end{array}\right]$ $[\![$  $]\!] = \left[\begin{array}{c|c} * & 0 \\ \hline & 1 \end{array}\right]$

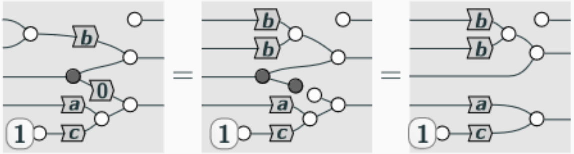
Affine matrices: axioms

Modulo the equations:

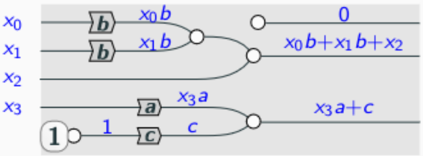


Example of matrix multiplication

The following diagram can be simplified to a normal form:



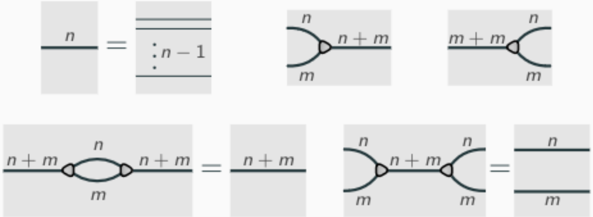
Following the paths from left to right gives us the homogeneous coordinate matrix:



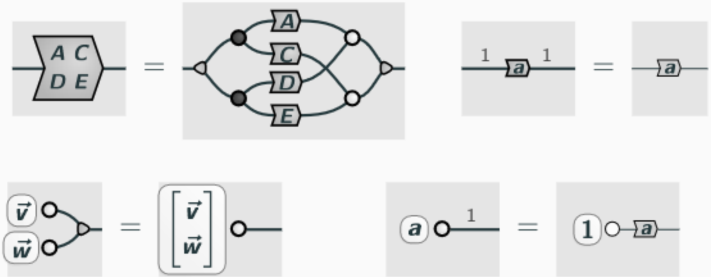
$$\rightsquigarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ b & b & 1 & 0 & 0 \\ 0 & 0 & 0 & a & c \\ \hline & & & & 1 \end{array}\right] \left[\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ x_0 b + x_1 b + x_2 \\ x_3 a + c \\ \hline 1 \end{array}\right]$$

Strictification and block matrices

Every prop can be strictified to an $\mathbb{N}$-coloured prop:



This allows us to define block matrices/vectors diagrammatically:



Affine relations (Bonchi et al. [Bon+19],Bonchi et al. [BSZ17])

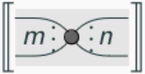
Given a field $\mathbb{K}$, the compact prop of $\mathbb{K}$-affine relations, $\mathsf{AffRel}_{\mathbb{K}}$, has:

- **Morphisms** $n \to m$ are affine subspaces $S \subseteq \mathbb{K}^n \oplus \mathbb{K}^m$.
- **Composition** relational, for $S : n \to m,\ T : m \to k$

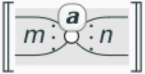
$$R \circ S := \{(\vec{x}, \vec{z}) \in \mathbb{K}^n \oplus \mathbb{K}^k \mid \exists \vec{y} \in \mathbb{K}^m : (\vec{x}, \vec{y}) \in S \quad \text{and} \quad (\vec{y}, \vec{z}) \in R\}$$

- **Symmetric monoidal structure** given by direct sum;
- **Compact structure** same as Rel.

$\mathsf{AffRel}_{\mathbb{K}}$ is generated by the following relations, for all $a \in \mathbb{K}$:



$$:= \left\{ \left(\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}, \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix} \right) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; a \in \mathbb{K} \right\}$$



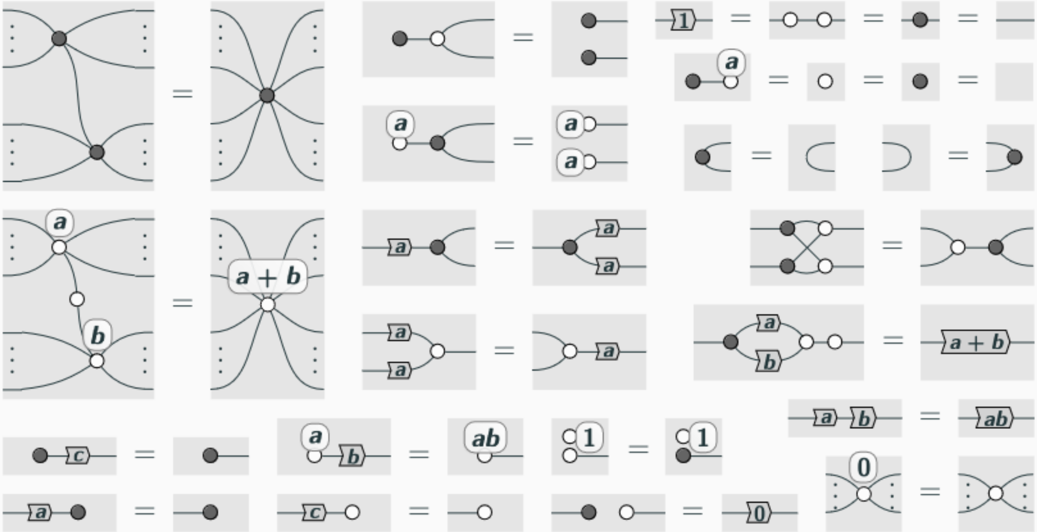
$$:= \left\{ (\vec{b}, \vec{c}) \in \mathbb{K}^n \oplus \mathbb{K}^m \;\middle|\; \sum_{j=0}^{n-1} b_j + \sum_{k=0}^{m-1} c_k = a \right\}$$

$$:= \{(b, ab) \mid b \in \mathbb{K}\}$$

Modulo, the "spiders" 

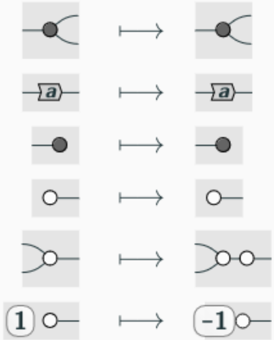
and 

being commutative, undirected and,



for all $a, b \in \mathbb{K}$, $c \in \mathbb{K}^{\times}$.

The embedding $\mathsf{AffMat}_{\mathbb{K}} \hookrightarrow \mathsf{AffRel}_{\mathbb{K}}$ taking an affine transformation $T : n \to m$ to it's graph $\{(\vec{x}, T\vec{x}) \mid \vec{x} \in \mathbb{K}^m\}$ sends:



Classical mechanics and symplectic geometry

Electrical circuits: current and voltage

The extensional behaviour of an electrical circuits is characterised by how it transforms current and voltage;

- **Ohm's law:** The voltage around the node in a circuit is equal to the current multiplied by the resistance.
- **Kirchhoff's current law:** The sum of currents flowing into a node is equal to the sum of currents flowing out of the node.

Example

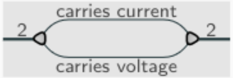
Given a linear resistor with resistance $r \in \mathbb{R}^{>0}$ on a wire with incoming current/votage (z_0, x_0) and outgoing current/voltage (z_1, x_1):

- by KCL, currents equalize: $z_0 = z_1$;
- by OL, the outgoing current becomes: $x_1 = x_0 + z_0 r$.

String diagrams for electrical circuits, take I

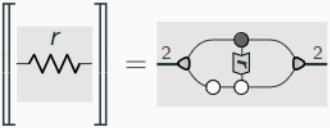
Following Baez et al. [BCR18] and Baez and Fong [BF18], we can represent electrical circuit components as real affine relations.

Using the string diagrams from Bonchi et al. [Bon+19], decompose a wire into a current and voltage



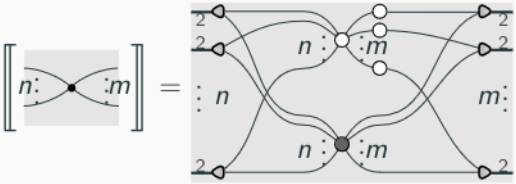
...the resistor is represented as follows:

Example



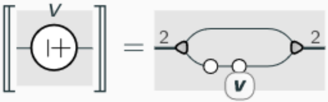
Example

Ideal wire junctions *sum currents*, and *equalize voltages*:



Example

Constant voltage source *does nothing to current* and *adds to the voltage*:



What is the more conceptual picture?

Symplectic geometry

Classical mechanical systems can be represented by the configurations of abstract **positions** Z and **momenta** X:

Classical mechanics	Z	dZ/dt	X	dX/dt
Translation	position	velocity	momentum	force
Electronic	charge	current	flux linkage	voltage
Hydraulic	volume	flow	pressure momentum	pressure
Thermal	entropy	entropy flow	temperature momentum	temperature

For n-particles in Euclidean space, *the space of possible configurations of positions/momenta* $\mathbb{R}^{2n} \cong \mathbb{R}^n_Z \oplus \mathbb{R}^n_X$ is the **phase space**.

Table adapted from Smith [Smi93, page 23, table 2.1] and Baez and Fong [BF18]

Affine Lagrangian subspaces

Definition
Two configurations $(\vec{z}, \vec{x}), (\vec{q}, \vec{p}) \in \mathbb{K}^{2n}$ of phase-space are **compatible** when:

$$\vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q} = 0$$

The bilinear map

$$\omega_n : \mathbb{K}^{2n} \oplus \mathbb{K}^{2n} \to \mathbb{K} \quad ((\vec{z}, \vec{x}), (\vec{q}, \vec{p})) \mapsto \vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q}$$

is a **symplectic form**, and the phase space $(\mathbb{K}^{2n}, \omega_n)$ is a **symplectic vector space**.

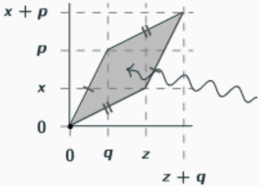
An **affine Lagrangian subspace** is a *maximally compatible* affine subspace of a symplectic vector space.

Remark (Baez and Fong [BF18], Baez et al. [BCR18])
Resistors, voltages sources and junctions of wires are affine Lagrangian subspaces.

Geometric interpretation of compatibility

Example

In the phase-space of a single particle, $(\mathbb{K}^2, \omega_1)$, the symplectic form measures area:



$$\omega_1\left(\begin{bmatrix} z \\ x \end{bmatrix}, \begin{bmatrix} q \\ p \end{bmatrix}\right)$$

Compatible points are colinear, so affine Lagrangian subspaces are lines.

An affine Lagrangian subspaces don't represent single particle; but an ensemble of particles *flowing along a trajectory*.

Affine Lagrangian relations

Definition (Guillemin and Sternberg [GS79],Weinstein [Wei82])
The **compact prop of affine Lagrangian relations** $\mathsf{AffLagRel}_{\mathbb{K}}$ has:

- **Morphisms** $n \to m$, given by (possibly empty) affine Lagrangian subspaces of $(\mathbb{K}^{2n} \oplus \mathbb{K}^{2m}, \omega_n - \omega_m : \mathbb{K}^{2(n+m)} \oplus \mathbb{K}^{2(n+m)} \to \mathbb{K})$.
- **Composition** is given by relational composition.
- **Symmetric monoidal structure** is given by the direct sum.

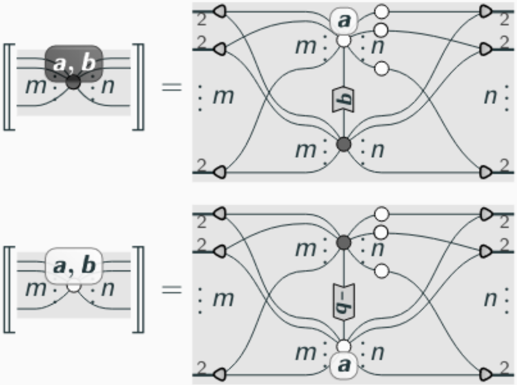
Lemma
There is an embedding $\mathsf{AffRel}_{\mathbb{K}} \to \mathsf{AffLagRel}_{\mathbb{K}}$ *given*

- **on objects by:** $n \mapsto 2n$;
- **on morphisms by:** $(S + \vec{a}) \mapsto S^{\perp} \oplus (S + \vec{a})$.

For the geometrically inclined, this is induced by the embedding of a vector space $\mathbb{R}^n \hookrightarrow T^*(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \oplus \mathbb{R}^n \cong \mathbb{R}^{2n}$ *into its cotangent bundle.*

Generators of affine Lagrangian relations (Comfort and Kissinger [CK22])

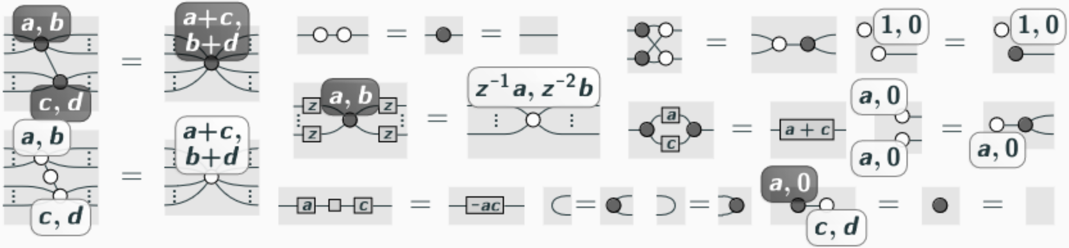
$\mathsf{AffLagRel}_{\mathbb{K}}$ is generated by two spiders decorated by $\mathbb{K}^2$; interpreted in $\mathsf{AffRel}_{\mathbb{K}}$ as:



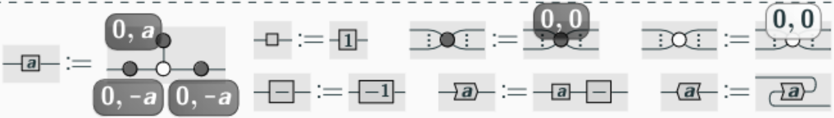
Equations of affine Lagrangian relations (Booth et al. [BCC24b])

Modulo both spiders, being commutative, undirected nodes,

as well as for all $a, b, c, d \in \mathbb{K}$ and $z \in \mathbb{K}^{\times}$:

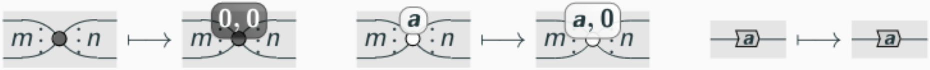


With derived generators:

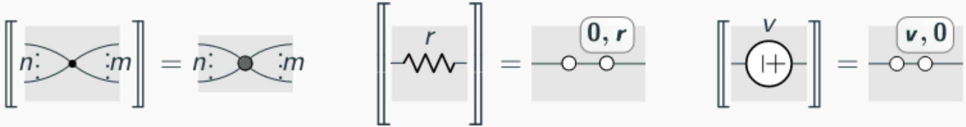


Interpreting electirical circuits

The embedding $\mathsf{AffRel}_{\mathbb{K}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{K}}$ takes:



Now that the position/momentum wires are bundled together, we have a more concise description of electrical circuit components:

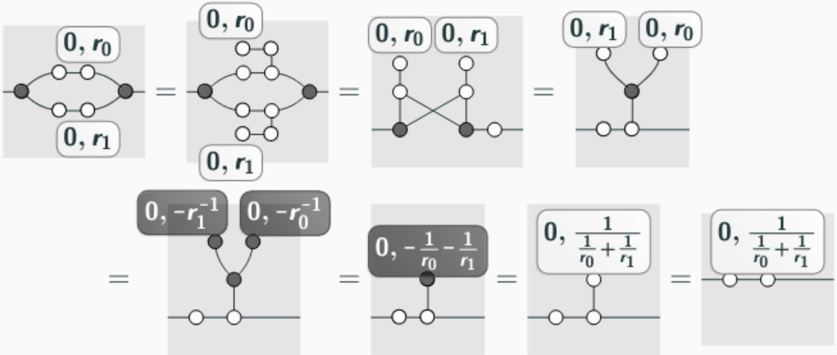


Example: composing resistors in parallel

$\mathsf{AffLagRel}_{\mathbb{R}}$ allows us to cleanly compose electrical circuits:

Example

Consider two resistors with resistances $r_0, r_1 \in \mathbb{R}^{>0}$ composed in parallel.



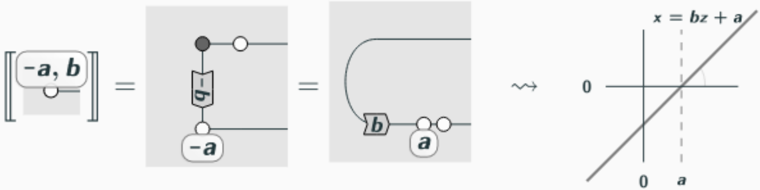
Electrons nondeterministically flow through both resistors, where they are impeded.

They extensionally behave like a resistor with resistance $1/(1/r_0 + 1/r_1)$.

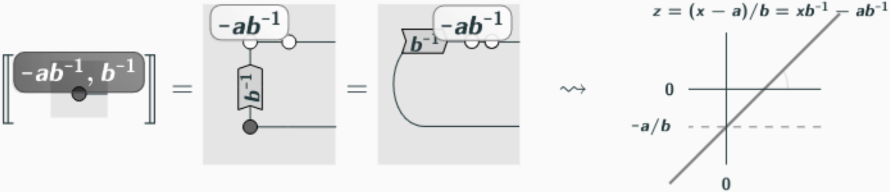
Interpreting the two spiders:

This colour-swap rule corresponds to a change of refrence frame.

Where configurations of phase space can be represented as functions of position:



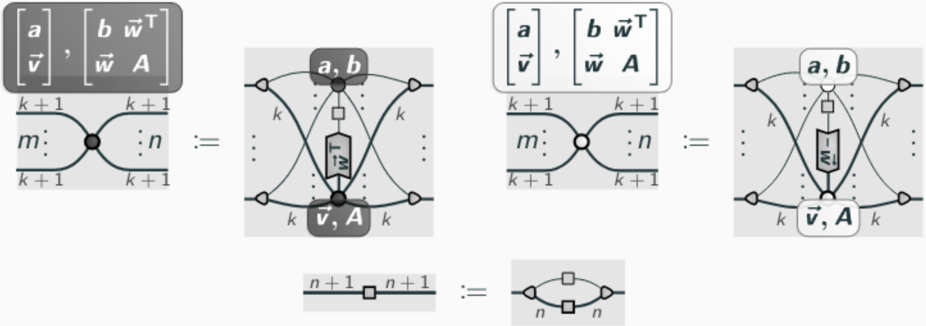
...or of momentum:



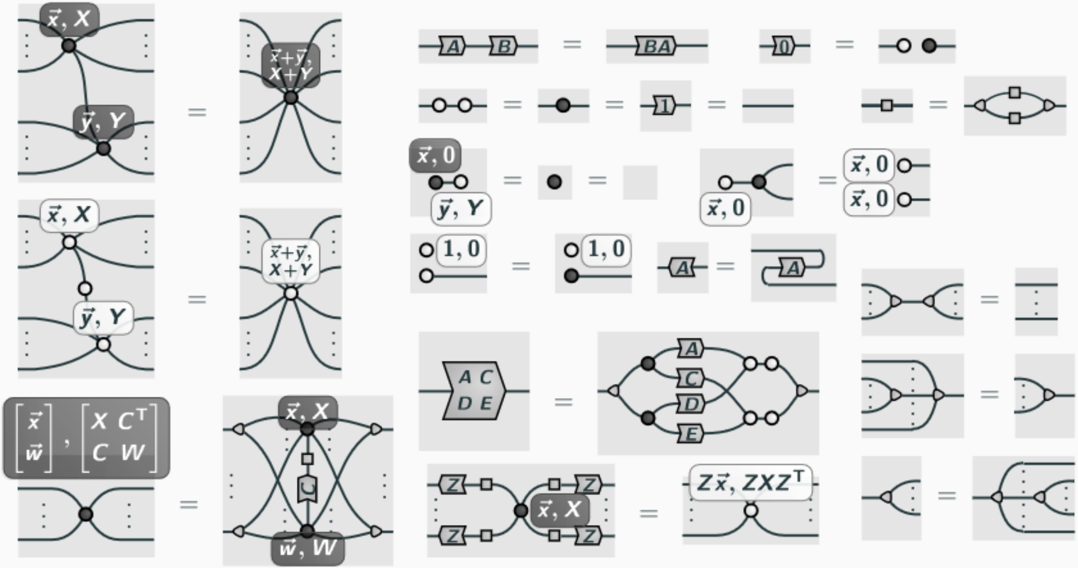
Scalable spiders

We can define higher-dimensional spiders by induction on the number of wires $k \in \mathbb{N}$.

Take $n, m \in \mathbb{N}$, $a, b \in \mathbb{K}$, $\vec{v}, \vec{w} \in \mathbb{K}^k$ and $A \in \mathrm{Sym}_k(\mathbb{K})$.



Scalable identities



Impedance matrix

Consider a network of resistors/voltage sources acting on n wires.

The extensional behaviour can be represented by a positive-definite $0 \prec R \in \mathrm{Sym}_n(\mathbb{R})$ called the **impedance matrix**, and a voltage $\vec{v} \in (\mathbb{R}^{>0})^n$

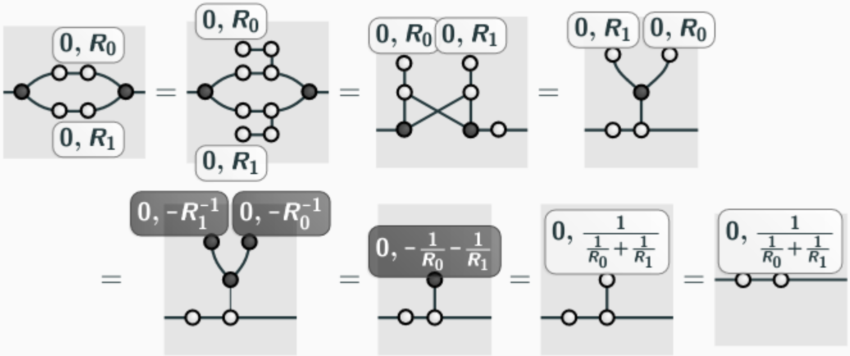
$$[\![\;\cdot\;]\!] = \left\{ \left(\begin{bmatrix} \vec{z} \\ \vec{x} \end{bmatrix}, \begin{bmatrix} \vec{z} \\ , \vec{x} + R\vec{z} + \vec{v} \end{bmatrix} \right) \mid \forall \vec{z}, \vec{x} \in \mathbb{R}^n \right\}$$

The resistance between the jth and kth wire is $r_{j,k} = r_{k,j} \in \mathbb{R}$.

The change in voltage on wire j is $v_j \in \mathbb{R}$.

Composing networks of resistors in parallel

Black-boxed networks of resistors compose in parallel in the same way as single resistors composed in parallel:



We don't know the internal structure of the two networks, but we still can compute their extensional behaviour in parallel.