Part II:
Complete equational theories for classical and quantum Gaussian relations

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Recap

Recall that the phase-space on n particles in Euclidean space is the symplectic vector spaces $(\mathbb{R}^{2n} \cong (\mathbb{R}^n)_Z \oplus (\mathbb{R}^n)_X, \omega_n)$:

Classical mechanics	Z	dZ/dt	X	dX/dt
Translation	position	velocity	momentum	force
Electronic	charge	current	flux linkage	voltage
Hydraulic	volume	flow	pressure mom'um	pressure
Thermal	entropy	entropy flow	temperature mom'um	temperature

Where (maximally compatible, affinely constrained) mechanical circuits can be represented by string diagrams for $\mathsf{AffLagRel}_{\mathbb{R}}$.

Quantized fragments of quantum mechanics

"Quantized fragments" of quantum mechanics admit nondeterministic phase-space semantics:

Classical mechanics	Z	dZ/dt	X	dX/dt
Translation	position	velocity	momentum	force
Electronic	charge	current	flux linkage	voltage
Hydraulic	volume	flow	pressure mom'um	pressure
Thermal	entropy	entropy flow	temperature mom'um	temperature

Quantized mechanics	Z	dZ/dt	X	dX/dt
Stabiliser QM *(finite dimensional)*	Pauli Z	Pauli Z flow	Pauli X	Pauli X flow
Gaussian QM *(infinite dimensional)*	$\hat{q}$	$\hat{q}$ flow	$\hat{p}$	$\hat{p}$ flow

Stabiliser quantum mechanics

The Pauli group

Finite dimensional quantum mechanics "lives in" $(\mathrm{FVect}_{\mathbb{C}}, \otimes, \mathbb{C})$...

Definition
Fix some odd prime p. The state space of a **quopit** is the p-dimensional vector space:

$$\mathcal{H}_d := \ell^2(\mathbb{Z}/d\mathbb{Z}) = \mathrm{span}_{\mathbb{C}}\{|0\rangle, \cdots, |d-1\rangle\}$$

Definition
The n-quopit **Pauli group** $\mathcal{P}_p^{\otimes n} \subset \mathrm{U}(p^n)$ is generated under tensor product and composition by:

$$\mathcal{X}|k\rangle := |k+1\rangle \quad \text{and} \quad \mathcal{Z}|k\rangle := e^{i\frac{2\pi}{p}k}|k\rangle$$

Lemma
Because $\mathcal{X}\mathcal{Z} = e^{-i\frac{2\pi}{p}}\mathcal{Z}\mathcal{X}$ *every element of* $\mathcal{P}_p^{\otimes n}$ *has the following form,*

$$\chi(a)\mathcal{W}(\vec{z}, \vec{x}) := e^{i\frac{2\pi}{p}a}\bigotimes_{j=0}^{n-1}\mathcal{Z}^{z_j}\mathcal{X}^{x_j}$$

for some $a \in \mathbb{F}_p$, $\vec{z}, \vec{x} \in \mathbb{F}_p^n$.

Lemma
Up to scalars, a maximal Abelian subgroups $S \subseteq \mathcal{P}_p^{\otimes n}$ uniquely determines a normalised state $|S\rangle : \mathcal{H}_p^{\otimes n}$ such that for all $P \in S$, $P|S\rangle = |S\rangle$.

Such states are called **stabiliser states**.

Remark

Two n-quopit Pauli operators $\chi(a)\mathcal{W}(\vec{z},\vec{x})$ and $\chi(b)\mathcal{W}(\vec{q},\vec{p})$ commute if and only if $\omega_n((\vec{z},\vec{x}),\ (\vec{q},\vec{p})) = 0$.

Corollary (Gross [Gro06])
There is a bijection:

$$\begin{aligned}\{\text{Maximal Abelian subgroups } S \subseteq \mathcal{P}_p^{\otimes n}\} &\cong \{\text{affine Lagrangian subspaces of } \hat{S} \subseteq (\mathbb{F}_p^{2n}, \omega_n)\} \\ &\cong \{\text{stabiliser states } |S\rangle : \mathcal{H}_p^{\otimes n}\}\end{aligned}$$

Given a Pauli $\chi(a)\mathcal{W}(\vec{z},\vec{x}) \in S$:

- *$\vec{z}$ are the positions;*
- *$\vec{x}$ are the momenta;*
- *a is determined by the affine shift.*

Phase-space representation of stabiliser states

Using the compact-closed structure of $(\mathsf{FVect}_{\mathbb{C}}, \otimes, \mathbb{C})$:

Definition
The compact prop of quopit **stabiliser circuits** is generated under tensor and composition of the linear operators:

- All quopit stabiliser states $0 \to n$;
- Caps $|j\rangle \otimes |k\rangle \mapsto \delta_{i,j}$ of type $2 \to 0$;
- The cup $\sum_{j=0}^{p-1} |j\rangle \otimes |j\rangle$ is already a stabiliser state of type $0 \to 2$.

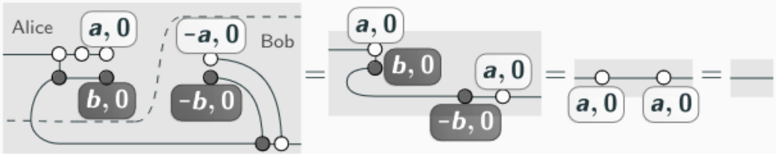
The composition of $\mathsf{AffLagRel}_{\mathbb{F}_p}$ agrees with that of in $\mathsf{FVect}_{\mathbb{C}}$:

Theorem (Comfort and Kissinger [CK22])
$\mathsf{AffLagRel}_{\mathbb{R}_p}$ *isomorphic to quopit stabiliser circuits, modulo scalars.*

Remark
The presentation of $\mathsf{AffLagRel}_{\mathbb{F}_p}$ *is the stabiliser ZX-calculus of Poór et al. [Poó+23], modulo scalars.*

Picturing quantum teleportation

This is powerful enough to do quantum teleportation à la Abramsky and Coecke [AC04] and Coecke and Kissinger [CK18]:



Gaussian quantum mechanics

Continuous-variable quantum mechanics

Definition
The continuous-variable 1-D quantum state space is the Hilbert space:

$$L^2(\mathbb{R}) := \left\{\varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \,\mathrm{d}x < \infty\right\}$$

The morphisms are bounded linear maps $(L^2(\mathbb{R}))^{\otimes n} \to (L^2(\mathbb{R}))^{\otimes m}$.

Definition
The **displacement** operators $\hat{Z}, \hat{X} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the CV-version of Paulis:

$$\hat{Z}(s) \circ \varphi(r) := e^{i2\pi rs}\varphi(r) \quad \text{and} \quad \hat{X}(s) \circ \varphi(r) := \varphi(r-s) \quad \text{for all} \quad r, s \in \mathbb{R},\ \varphi \in L^2(\mathbb{R})$$

The n-qumode **Heisenberg-Weyl group** $\mathcal{HW}^{\otimes n}$ is generated by displacement operators by tensor product and composition, where every Heisenberg-Weyl operator has the form:

$$\chi(a)\mathcal{W}(\vec{z}, \vec{x}) := e^{i2\pi a} \bigotimes_{j=0}^{n-1} \hat{Z}(z_j)\hat{X}(x_j)$$

The failure of CV stabilizer states

Lemma
Affine Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_n)$ *are in bijection with maximally Abelian subgroups of* $\mathcal{HW}^{\otimes n}$*, modulo scalars.*

Problem: *Given an affine Lagrangian subspace* $S \subseteq (\mathbb{R}, \omega_n)$*, there is no non-zero state* $|S\rangle : (L^2(\mathbb{R}))^{\otimes n}$ *such that* $\mathcal{W}(\vec{z}, \vec{x}) \, |S\rangle$ *for all* $(\vec{z}, \vec{x}) \in \mathbb{R}^n$*!*

None of the states in $\mathrm{AffLagRel}_{\mathbb{R}}$ can be represented in Hilbert spaces!!!

$$\{\text{Maximal Abelian subgroups } S \subseteq \mathcal{HW}^{\otimes n}\} \cong \{\text{affine Lagrangian subspaces of } \hat{S} \subseteq (\mathbb{R}^{2n}, \omega_n)\}$$
$$\ncong \{\text{stabiliser states } |S\rangle : (L^2(\mathbb{R}))^{\otimes n}\}$$

Definition

An n-variate **Gaussian distribution** $\mathcal{N}(\Sigma, \vec{\mu})$ consists of a positive semidefinite covariance matrix $\Sigma \in \mathrm{Sym}_n(\mathbb{R})$ and a **mean** vector $\vec{\mu} \in \mathbb{R}^n$.

When Σ is positive-definite, $\mathcal{N}(\Sigma, \vec{\mu})$ admits a probability density function.

Proposition

A $2n$-variate Gaussian probability distribution $\mathcal{N}(\Sigma, \vec{\mu})$ on phase-space $(\mathbb{R}^{2n}, \omega_n)$ corresponds to a bounded state on $(L^2(\mathbb{R}))^{\otimes n}$ if and only if:

- *Σ is positive definite;* } *so that $\mathcal{N}(\Sigma, \vec{\mu})$ has a density function*
- $\det(\Sigma) = 1$*;*
- $\Sigma + i \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ *is positive semidefinite.*

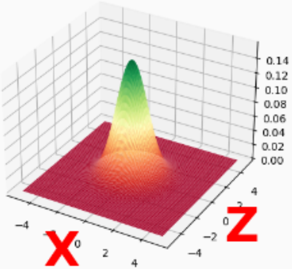
} *respects Heisenberg's uncertainty principle for pure states*

Call this a **quantum Gaussian distribution**.

Gaussian distributions in phase space: the vacuum state

Example

The **quantum vacuum state** $|0\rangle : L^2(\mathbb{R})$ is represented by the Gaussian distribution Φ_1 on $(\mathbb{R}^2, \omega_1)$:



$$\leftrightsquigarrow \mathcal{N}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right) =: \Phi_1$$

Φ_1 is the unique quantum Gaussian distribution on $(\mathbb{R}^2, \omega_1)$ invariant under rotation.

The Quantum Gaussian distribution Φ_n for $|0\rangle^{\otimes n}$ has the same universal property of being invariant under rotations (symplectic rotations $SO(\mathbb{R}, 2n) \cap Sp(\mathbb{R}, 2n)$).

Phase-space diagrams generated by Strawberry Fields/matplotlib

The isomorphisms in $\mathsf{AffLagRel}_{\mathbb{K}}$ have the form:

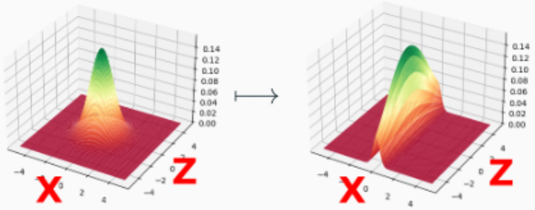
Definition
An affine automorphism on $(\mathbb{K}^{2n}, \omega_n)$ is a **symplectomorphism** when it preserves the symplectic form.

Lemma
Quantum Gaussian states are vacuum states acted on by affine symplectomorphisms.

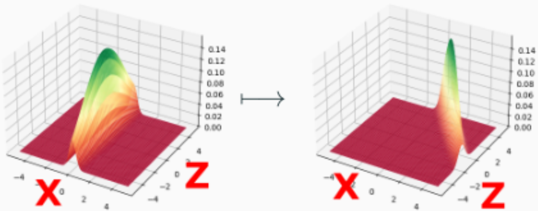
Example
For $n = 1$, recall that $\omega_1 : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ measures area in $\mathbb{R}^2$.

Therefore, quantum Gaussian states on $(\mathbb{R}^2, \omega_1)$ are generated by acting on the vacuum state with area-preserving affine isomorphisms.

For example, we can squeeze the Gaussian distribution for the vacuum state state:

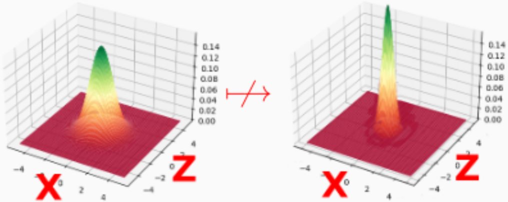


Changing the mean and rotating still is allowed.



Picturing area-preservation

But we can not make Φ_1 more concentrated:



This violates Heisenberg's uncertainty principle.

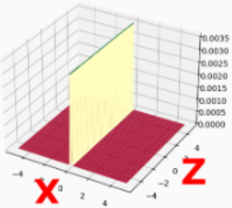
Approximating stabiliser states with Gaussian convolution

In phase-space CV stabiliser states do not have strictly positive definite covariance.

So they are not quantum Gaussian states.

However, they can be approximated with quantum Gaussian states:

Dirac delta distribution

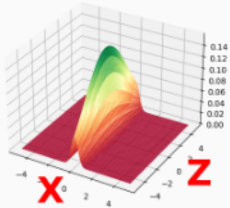


convolution by

$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)$$

$\longmapsto$

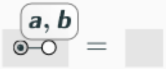
Gaussian density function

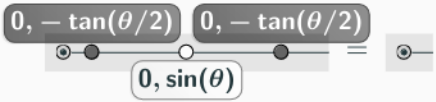


The Gaussian ZX-calculus

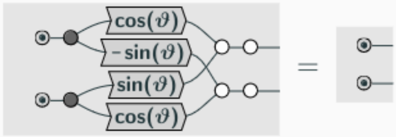
Because the vacuum state is the unique permissible Gaussian distribution in phase-space distribution invariant under rotation:

Theorem (Booth et al. [BCC24a])
The Gaussian state can be freely added to $\mathsf{AffLagRel}_{\mathbb{R}}$ *as a generator* ⊙–, *such that for all* $\vartheta \in [0, 2\pi)$ *and* $\theta \in (-\pi, \pi)$*:*





rotation of $\mathbb{R}^2$ *by* θ
decomposed into three shears



rotation of $\mathbb{R}^2$ *by* ϑ*, embedded along* $\mathsf{AffRel}_{\mathbb{R}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{R}}$

This contains both quantum Gaussian states and formal CV stabilisers.

Quantum Gaussian states and complexification

There is an equivalent formulation using the complex numbers

Proposition

Quantum Gaussian states/CV stabilisers can be represented by affine Lagrangian subspaces $S + \vec{a} \subseteq (\mathbb{C}^{2n}, \omega_n)$*, where:*

- $\vec{a}$ *is real;*
- *for all* $\vec{x} \in S$, $i\omega_n(\overline{\vec{x}}, \vec{x}) \geqslant 0$.

In other, words, we can represent the vacuum state as follows:

Theorem (Booth et al. [BCC24a])

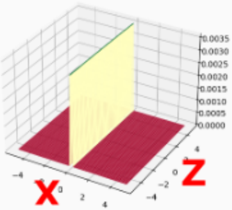
The Gaussian ZX-calculus is equivalent to adding the state 

to the image of the embedding $\mathsf{AffLagRel}_{\mathbb{R}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{C}}$.

Picturing Gaussian convolution

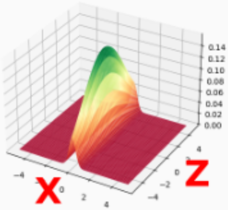
Dirac delta distribution

Gaussian density function



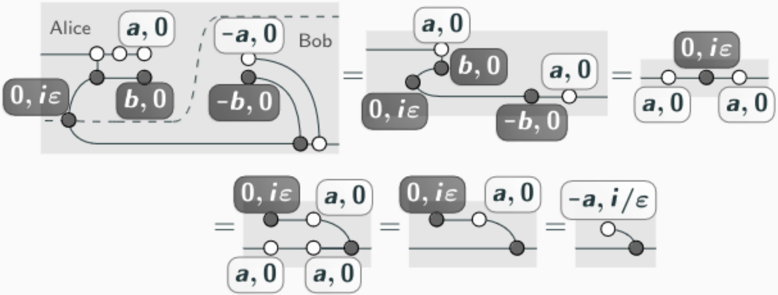
convolution by

$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\ 0\end{bmatrix}\right)$$



Picturing continuous-variable quantum teleportation

We can interpret the continuous-variable quantum teleportation algorithm of Braunstein and Kimble [BK98]:



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