Semantic foundations for type-driven probabilistic modelling

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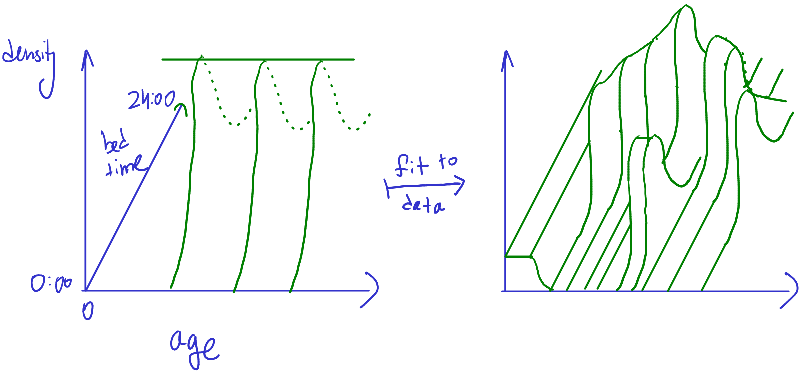
Statistical & Probabilistic Modelling



ILLUSTRATIVE PURPOSES ONLY

Prob. Modelling as Programming

distributions describe Programs

$\Downarrow$ [Saheb-Djahromi '77
Kozen '78]

Programs describe distributions

Prob. Modelling as Programming [Saheb-Djahromi '77 Kozen '78]

Programs describe distributions

Sophisticated samplers:

STLC + Sample + { datatypes, recursion, polymorphism, ... }

Sophisticated Bayesian models: Sample + observe

Prior

likelihood/density

modelling via PL features & stats components

inference (generic/customized)

systems (performance)

But Today:

Semantic foundations

Desiderata: Types

Discrete Spaces

finite spaces

infinite spaces

$\mathbb{B}\mathrm{ool}$ $[n]$ naturals $\mathbb{N}$ integers $\mathbb{Z}$ rationals $\mathbb{Q}$

success/failure (Bernoulli) coin tosses

week day

dice roll

$\rightsquigarrow$ count

$\rightsquigarrow$ Priority

Desiderata : Types

Continuous Spaces

$\mathbb{R}$ position

$[a, b]$ hour

$[0, \infty]$ scaling factors

$[0, 1]$ Probabilities

Discrete +

Desiderata : Types

Space Combinators

products

$\mathbb{R} \times \mathbb{R}$

correlated outcomes (position + velocity)

coproducts

$\mathbb{R} \amalg [n]$

coordinate or location

subspaces

$S^1 \hookrightarrow \mathbb{R}^2$

semantic invariants

quotients

$\mathrm{Bag}\,\mathbb{N}$

point processes

Discrete + Continuous

Desiderata: Probability / measure

[Kolmogorov'33 equivalent axioms]

Events $E \subseteq X$:

$\{ d \in [6] \mid$ dice roll more than 3: $d \geqslant 3 \}$

Measure $\mu \Rightarrow$ Events $\xrightarrow{\Pr_\mu} [0, \infty]$

$E \mapsto \Pr_{x \sim \mu}[x \in E]$

measure

Outcomes

where E occurs

Desiderata: probability / measure

Measure μ $\Rightarrow$ $\text{Events} \xrightarrow{\Pr_\mu} [0,\infty]$

$\Pr[\emptyset] = 0$

totality

$\Pr[E] = \Pr[E \cap F] + \Pr[E \cap F^c]$

disjoint additivity

$\Pr\left[\bigcup_n E_n\right] = \sup_n \Pr[E_n]$ $\left(E_n \subseteq E_{n+1}\right)_n$

(Scott) continuity

$\Pr\left[\begin{matrix}\text{whole}\\ \text{space}\end{matrix}\right] = 1$

Probability

No-go #1:

Thm (Vitali, 1905) Assuming Axiom of Choice:

$\nexists\, P_{r,\lambda} : P\mathbb{R} \to [0,\infty]$ measure

$P_r[[E]+a] = P_r[E]$ $\qquad$ $P_r[[a,b]] = b-a$

translation invariant $\qquad$ measures length

Borel subsets $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{P}(\mathbb{R})$

- Boolean subalgebra w.r.t. $\cap, \cup, \emptyset, (-)^c$
- ω-chain closed:

$$\forall (E_n \subseteq E_{n+1})_n \in \mathcal{B}_{\mathbb{R}}^{\omega} .\ \bigcup_n E_n \in \mathcal{B}_{\mathbb{R}}$$

- Every interval: $(a,b) \in \mathcal{B}_{\mathbb{R}}$

Thm: (Lebesgue 1902)

$\exists!\ \Pr_\lambda : B_{\mathbb{R}} \to [0,\infty]$ measure

translation invariant + measures length

Lebesgue Measure

Measurable space $A = (\underline{A}, B_A)$

set of points

Events $B_A \subseteq P_A$

σ-field

- Boolean subalgebra w.r.t. $\cap, \cup, \emptyset, (-)^c$
- ω-chain closed:

$$\forall (E_n \subseteq E_{n+1})_n \in B_{\mathbb{R}}^{\omega} \,.\, \bigcup_n E_n \in B_{\mathbb{R}}$$

measurable: $A \xrightarrow{f} B$

$$f^{-1}[E] \in B_A \Leftarrow E \in B_B$$

Classical theory [Kolmogorov 1933]

Measure space

Ω measurable space

$\mu : \mathcal{B}_\Omega \longrightarrow [0,1]$ Probability measure

Modelling Foundation: Meas

discrete $(\mathbb{N}, P(\mathbb{N}))$

+ their measures

continuous $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Combinators:

products, subspaces (limit structure)

coproducts quotients (colimit structure)

No-Go as PL Semantics:

Thm [Aumann'61] Let $\mathbb{R}^{\mathbb{R}} := \mathrm{Meas}(\mathbb{R}, \mathbb{R})$

$\nexists$ $\mathcal{B}_0 \subseteq \mathcal{P}(\mathbb{R}^{\mathbb{R}})$ σ-field

eval: $(\mathbb{R}^{\mathbb{R}}, \mathcal{B}_0) \times \mathbb{R} \xrightarrow{(f,x) \mapsto fx} \mathbb{R}$ measurable

Consequences:

$\Rightarrow$ no-go for event space $\mathcal{B}(\mathbb{B}_{\mathbb{R}})$

$\Rightarrow$ no-go for Π-types

$\Rightarrow$ no-go for:

- functional programming
- OO Programming ...

Semantic Tradition: Probabilistic Powerdomains

[Plotkin '82, Graham '87, Jones-Plotkin '89, Tix, Jung 97
Tix 98, Mislove '00+.., Keimel 03,
J. Goubault-Larrecq '07 +... + apologies]

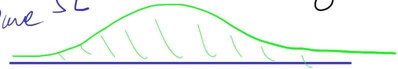
Recent developments:

- Boolean-valued models [Bacci et al.'18]
- Probabilistic coherence spaces with cones [Ehrhard et al '18, de Amorim et al '22]
- Measurable event structures [Paquet-Winskel '18]
- Banach spaces with structure [Dahlqvist, Kozen '20]
- Interval domains [Jia et al.'21a+b, Di Gianantonio, Edalat '24]
- Atomic Sheaf Toposes [Simpson '24]

Quasi-Borel Spaces [Staton et al. '17] cf. [Forré '21]

Measure Theory	Qbs Theory

sample space Ω

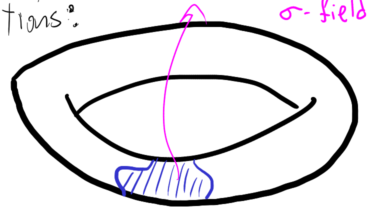


Primitive notions:

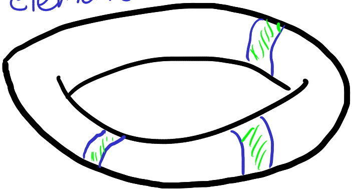
event $E \in B_A$

σ-field

random element $\alpha \in R_A$



$\underline{A}$



$\underline{A}$

Derived

random element: $(\alpha : \Omega \to A) \in R_A$

metaphorology

event $E \in B_A$

Applications

Weak de Finetti Thm

Verify Bayesian inference implementations

Design modelling languges & inference systems
+ implementations

Probabilistic Networking PL

Formalise

semantics for expected cost analysis

Also:

semantics for name generation

Applications

Weak de Finetti Thm [Staton et al '17]

Verify Bayesian inference implementations [Ścibior et al. '18]

Design modelling languages & inference systems

+ implementations: MonadBayes [Ścibior et al '18, now Tweag.Io]

LazyPPL [Staton et al. '23] (Parts of) Gen [Lew et al '20]

Domain Theory [Vákár et al. '19]

Probabilistic Networking PL [Vandenbroucke, Schrijvers '20]

Formalise [Hirata et al '19 + '23]

semantics for expected cost analysis [de Amorim '24]

Also:

semantics for name generation [Sabok et al. '21]

Rest of talk: mini-tutorial via types

1) Metaphorologies & Qbs
2) Simple-type structure
3) Standard spaces

Course

PhD Programs

CDT: Dependable AI

THE NATIONAL ROBOTARIUM
PEOPLE CENTRED :: INTELLIGENCE DRIVEN

lfcs Laboratory for Foundations of Computer Science

Engineering and Physical Sciences Research Council

HERIOT WATT UNIVERSITY

THE UNIVERSITY of EDINBURGH

Theme: Working with quasi Borel spaces:

1) types for abstracting over measurability

2) types for organising probabilistic concepts

Not today:

I) Colimit structure & Probability spaces

II) dependently-typed structure

III) random variable spaces & stochastic processes

See course

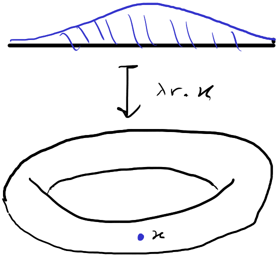
Def: metaphorology over X set

"random elements" $\mathcal{R}$ "points"

$\mathcal{R} \subseteq X^{\mathbb{R}}$ $\models$ 3 closure axioms:

- Constants:

$$\frac{x \in \lfloor X \rfloor}{\underline{x} := (\lambda r. x) \in \mathcal{R}}$$



- Precomposition:

- recombination

Def: Metaphorology over X Set

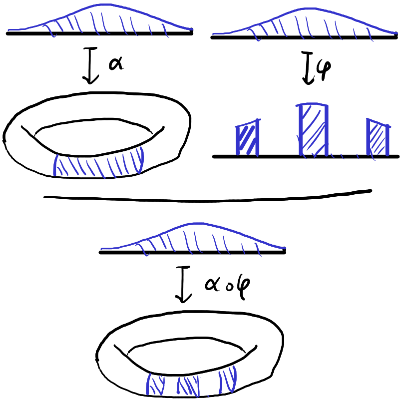
"random elements" "points"

$\mathcal{R} \subseteq X^{\mathbb{R}} \models$ 3 Closure axioms:

- Precomposition:

$\alpha \in \mathcal{R}_X \quad \varphi : \mathbb{R} \to \mathbb{R}$ Borel measurable

$\varphi \circ \alpha : \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} X, \in \mathcal{R}$



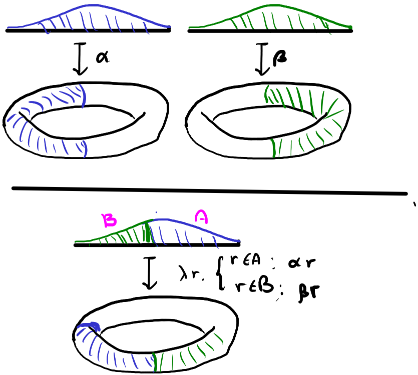
Def: Metaphorology over X Set

"random elements" $\qquad R$ "points"

$R \subseteq X^{\mathbb{R}} \models$ 3 Closure axioms:

- recombination

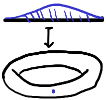
$$\frac{\vec{\alpha} \in R_X^{\mathbb{N}} \qquad \vec{E} \in B_{\mathbb{R}}^{\mathbb{N}} \qquad \mathbb{R} = \biguplus_{n=0}^{\infty} E_n}{[E_n . \alpha_n]_n := \lambda r. \begin{cases} \vdots \\ r \in A_n : \alpha_n r \end{cases} \in R}$$

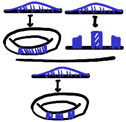


Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

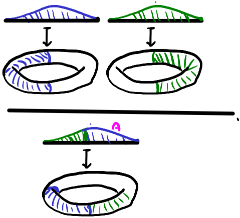
Set + metaphorology

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- Constants: 

- Precomposition: 

- recombination



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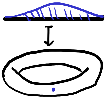
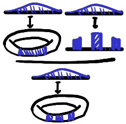
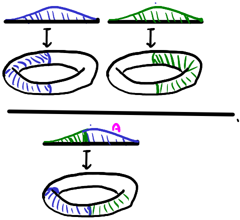
- Constants: 

- precomposition: 

- recombination 

Topos Theorists:

Concrete sheaves over

$(\mathbf{Sbs}, \text{Countable covers})$

Examples

- $\underline{\mathbb{R}} = (\mathbb{R}, \mathrm{Meas}(\mathbb{R}, \mathbb{R}))$ qbs underlying $\mathbb{R}$

- Generalise:

$$\frac{A \text{ meas space}}{\underset{\lfloor qbs \rfloor}{A} := (\underline{A}, \mathrm{Meas}(\mathbb{R}, A)) \text{ qbs}}$$

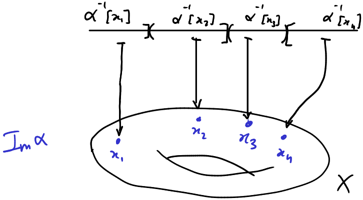
Examples

Def: $\alpha : \mathbb{R} \to X$ σ-Simple: $\alpha = [E_n . x_n]_{n \in I}$

recombination of constants

$I \hookrightarrow \mathbb{N}$ $\quad \vec{E} \in \mathcal{B}_{\mathbb{R}}^{I}$

$\vec{x} \in X^{I}$ $\quad \mathbb{R} = \biguplus_{n} E_n$



discrete qbs on X

$$\frac{X \text{ set}}{\ulcorner X^{\mathrm{Qbs}} \urcorner := \left(X,\ \sigma\text{-Simple}(\mathbb{R}, X)\right) \text{ qbs}}$$

Examples

Indiscrete qbs on X

$$\frac{X \text{ set}}{\lfloor X \rfloor_{\mathrm{Qbs}} := \left(X, \underbrace{X^{\lfloor \mathbb{R} \rfloor}}_{\text{all functions}}\right)}$$

Def: Qbs morphism $f : X \to Y$

- function $f : \lfloor X \rfloor \to \lfloor Y \rfloor$

- $\forall \ \alpha : \mathbb{R} \to \lfloor X \rfloor \in R_X \ . \quad f \circ \alpha : \mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \xrightarrow{f} \lfloor Y \rfloor \in R_Y$

Example

- Constant functions:

since:

$$f \text{ constant} \implies f \circ \alpha \text{ constant}$$

Example extending Meas

$$\frac{A, B \text{ Meas spaces} \quad f : A \to B \text{ in Meas}}{\lfloor A \rfloor_{Qbs} \xrightarrow{f} \lfloor B \rfloor_{Qbs}}$$

Example Category Qbs

Identities:

$$\mathrm{id}_A := (\lambda x.x) : A \to A$$

Composition:

$$\begin{tikzcd} A \arrow[rr, "\lambda x.\, g(f x)"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Meas} \arrow[d, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \\ \mathbf{Qbs} \arrow[d, "\lfloor - \rfloor"] \\ \mathbf{Set} \end{tikzcd}$$

generates limits & colimits

(lifts + preserves)

$$
\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor \bar{-} \rfloor_{\mathbf{Qbs}}"] \\
\mathbf{Qbs} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Meas}}"] \arrow[u, phantom, "\dashv"] \arrow[d, "\lfloor - \rfloor"] \\
\mathbf{Set} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Qbs}}"] \arrow[u, bend right=60, "\lfloor \bar{-} \rfloor_{\mathbf{Qbs}}"'] \arrow[u, phantom, "\dashv \quad \dashv"]
\end{tikzcd}
$$

N.B. for C.B.: not cohesive

Rest of talk: mini-tutorial via types

1) Metaphorologies & Qbs

2) Simple-type structure

3) Standard spaces

Course

Product $(X \times Y, \pi_1, \pi_2)$:

correlated random elements

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$
- $R_{X \times Y} = \{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \}$

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\lambda r. \langle f r, g r \rangle"] & \\
X & X \times Y \arrow[l, "\pi_1 := \lambda \langle x, y \rangle . x"] \arrow[r, "\pi_2 := \lambda \langle x, y \rangle . y"'] & Y
\end{tikzcd}$$

Ex

$\mathbb{R}^n$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$

$(+), (\cdot) : \mathbb{R}^2 \to \mathbb{R}$

$\inf, \sup$

$\liminf, \limsup : \overline{\mathbb{R}}^{\mathbb{N}} \to \overline{\mathbb{R}}$

Qbs $\models$ 1^{st} order simple-type theory (like meas)

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\,\text{var} \qquad \frac{\Gamma \vdash M : A \qquad f : A \to B \text{ in Qbs}}{\Gamma \vdash \dashv\!\!| f |\!\!\vdash M : B}\,\text{reflect}$$

$$\frac{\Gamma \vdash M_i : A_i}{\Gamma \vdash \langle M_1, \ldots, M_n \rangle : A_1 \times \cdots \times A_n}\,\text{tuple} \qquad \frac{\Gamma \vdash M : A_1 \times \cdots \times A_n \quad \Gamma, (x_i : A_i) \vdash K : B}{\Gamma \vdash \text{let}\,(x_1, \ldots, x_n) = M \text{ in } K : B}\,\text{let}$$

Smooth internalisation/externalisation

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathbf{Qbs}(X, Y)$
- $R_{Y^X} := \mathrm{uncurry}[\mathbf{Qbs}(\mathbb{R} \times X, Y)]$

  $= \{ \alpha : \mathbb{R} \to \lfloor Y^X \rfloor \mid \lambda(r, x).\, \alpha\, r\, x : \mathbb{R} \times X \to Y \}$
- $\mathrm{eval} : Y^X \times X \to Y$

  $\mathrm{eval}(f, x) := f\, x$

Simple type theory

$\mathbf{Qbs} \models \text{STLC}$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.\, M : B^A} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash P : A}{\Gamma \vdash M\, P : B}$$

Ex.

$$\limsup : \overline{\mathbb{R}}^{\mathbb{N} \times \mathbb{R}} \longrightarrow \overline{\mathbb{R}}^{\mathbb{R}}$$

$$\limsup \vec{f} := \lambda r.\ \limsup_{n \to \infty} f_n\, r$$

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X^{\mathbb{R}} \rfloor \Rightarrow \alpha : \mathbb{R} \to X$ in Qbs.

$\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow \mathrm{id} \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ \mathrm{id} \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \Rightarrow \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \Rightarrow \alpha : \mathbb{R} \to X$

$\Rightarrow \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Random element Space

Ex: Riemann-Stieltjes integrals & Lebesgue-Stieltjes measures

$$\underline{R}X := R_X$$

$$\check{\delta} : X \to \underline{R}X$$

$$\check{\delta}_x := \underline{x}$$

$$\check{\oint} : \underline{R}X \times (\underline{R}Y)^X \to \underline{R}Y$$

$$\int : \underline{R}X \times [0,\infty]^X \to [0,\infty]$$

$$\int d\alpha\, \varphi := \int d\mathcal{N}\, (\underbrace{\varphi \circ \alpha}_{\mathbb{R} \to [0,\infty]})$$

$$\check{\oint} d\alpha\, f \quad \begin{tikzcd} \mathbb{R} \arrow[r, "\cong"] \arrow[d] \arrow[d, phantom, shift left=4, ":="] & \mathbb{R}\times\mathbb{R} \arrow[r, "\alpha\times id"] & X\times\mathbb{R} \arrow[d, "f\times id"] \\ Y & & \underline{R}Y\times\mathbb{R} \arrow[ll, "eval"] \end{tikzcd}$$

Subspaces

$$\frac{A \text{ qbs} \qquad X \subseteq \lfloor A \rfloor \text{ set}}{\mathrm{Cost} : \{x \mid x \in X\} \hookrightarrow A} \text{ subspace}$$

$$R_{\{X\}} := \{ \alpha : \mathbb{R} \to X \mid \alpha \in R_A \} \qquad \mathrm{Cost}\ x := x$$

Ex: 1) $S^1 := \{ \vec{x} \in \mathbb{R}^2 \mid \|x\| = 1 \} \hookrightarrow \mathbb{R}^2$

2) Skorokhod Representation:

$$\mathrm{cdf} : R\overline{\mathbb{R}} \xrightarrow{\cong} \{ f : \overline{\mathbb{R}} \to \mathbb{R} \mid f(-\infty) = 0, f(\infty) = 1, f \text{ càdlàg, monotone} \} \hookrightarrow \mathbb{R}^{\overline{\mathbb{R}}}$$

Events

$$\mathcal{B}_{\lfloor A \rfloor} := \{ A \subseteq \lfloor A \rfloor \mid \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \} \qquad \sigma\text{-field}$$

$$\mathcal{B}_{\lfloor A \rfloor} \cong \mathrm{Qbs}(A, \mathbb{B}\mathrm{ool}) \implies \mathcal{B}_A \cong \mathbb{B}\mathrm{ool}^A$$

($\lfloor \mathcal{B}_{(\mathcal{B}_{\mathbb{R}})} \rfloor$ are the Borel-on-Borel sets from descriptive set theory.
Cf. [Sabok et al. '21])

$$\frac{A \ \text{qbs}}{\ulcorner A \urcorner^{\mathrm{Meas}} := (A, \mathcal{B}_A) \ \text{Meas space}}$$

$$\begin{tikzcd} \mathbf{Meas} \arrow[d, shift left=2, "\lfloor - \rfloor_{\mathbf{Qbs}}"] \\ \mathbf{Qbs} \arrow[u, bend left=60, "\ulcorner - \urcorner^{\mathbf{Meas}}"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

Meas vs Qbs

By generalities:

σ-field on $\mathbf{Meas}(\mathbb{R}, \mathbb{R})$

$$\begin{tikzcd}
\ulcorner \mathbb{R}^{\mathbb{R}}_{\mathbf{Meas}} \urcorner \times \mathbb{R} \arrow[r] \arrow[rr, bend right, "\ulcorner \mathrm{eval}^{\mathbf{Meas}} \urcorner"'] & \ulcorner \mathbb{R}^{\mathbb{R}}_{\mathbf{Meas}} \urcorner \times \mathbb{R} \arrow[r, dashed, "\times"] & \ulcorner \mathbb{R} \urcorner = \mathbb{R}
\end{tikzcd}$$

No factorisation by Aumann's Theorem.

So: $\left(\ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner \right)$

Rest of talk: mini-tutorial via types

1) Metaphorologies & Qbs

2) Simple-type Structure

3) Standard spaces

Course

Borel Embedding:

$$e : A \hookrightarrow B$$

when:

$$\lambda x. ex : A \xrightarrow{\cong} e[A] \hookrightarrow B \quad \& \quad e[A] \in \mathcal{B}_B$$

Ex: $\mathrm{Null} = \{ E \text{ Lebesgue null-set} \} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

characterised by: $\lambda E. \left(\int \mathrm{d}\,\mathrm{id}\, [- \in E] \stackrel{?}{=} 0 \right)$

Non-examples $\rightsquigarrow$ [Sabok et al. '21]

- $\{A \in \mathcal{B}_{\mathbb{R}} \mid A \neq \emptyset\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

- $\{(A_1, A_2) \in \mathcal{B}_{\mathbb{R}}^2 \mid A \subseteq B\} \hookrightarrow \mathcal{B}_{\mathbb{R}}^2$

- $\{A \in \mathcal{B}_{\mathbb{R}} \mid A \text{ open}\} \hookrightarrow \mathcal{B}_{\mathbb{R}}$

Standard Borel spaces

Def: A qbs S is standard Borel

$$S \hookrightarrow \mathbb{R}$$

Slogan:

Standard $\equiv$ concrete $\equiv$ well behaved

Standard Borel spaces

Def: A qbs S is standard Borel

$$S \hookrightarrow \mathbb{R}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Qbs} \arrow[l, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathbf{Sbs} \arrow[u, hook] \arrow[l, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Classical measure theory

1) Define σ-field

2) Prove it standard

3) show relevant operations measurable

With Qbs

1) use type formers
⇝ support operations by construction

2) Prove it standard

3) characterise events

Uniform convergence space

Space of continuous functions:

1) $C_0 := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

$\mathrm{eval} : C_0 \times \mathbb{R} \to \mathbb{R}$

$\mathrm{eval}\langle f, x \rangle := (\mathrm{cast}\ f)\, x$

$$\frac{\Gamma \vdash f : \mathbb{R} \to \mathbb{R} \qquad \forall r.\ [\![f]\!]\, r \text{ cts}}{\Gamma \vdash (f, \mathrm{cts}) : C_0}$$

Uniform convergence space

2) Prove it standard:

$$\mathrm{Code} \hookrightarrow \mathbb{R}^{\mathbb{Q}}$$

$$:= \left\{ \vec{y} \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a,b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^{+} \\ \exists \delta \in \mathbb{Q}^{+}\ \forall p,q \in \mathbb{Q}^{+} \cap [a,b], \\ |p-q| < \delta \Rightarrow |y_p - y_q| < \varepsilon \end{array} \right\}$$

idea:

f cts on $\mathbb{R} \iff$

$f|_{[a,b]}$ uniformly cts

Uniform convergence space

2) Prove it standard:

$$\begin{tikzcd}[column sep=huge]
C_0 \arrow[r, bend left, "-|_{\mathbb{Q}} := \lambda f.\, \lambda q.\, f\, q"] \arrow[r, phantom, "\cong"] & \mathrm{Code} \arrow[l, bend left]
\end{tikzcd}$$

$\lambda \vec{y}.\ \lambda x.$
let $\vec{q} : \mathbb{Q}^{\mathbb{N}}$
$= \mathrm{rationalApprox}\ x$
in $\lim_{n \to \infty} y_{q_n}$

idea:
f cts on $\mathbb{R}$ $\Leftrightarrow$
$f|_{[a,b]}$ uniformly cts

Characterising $\mathcal{B}_A$

$d : \underline{A} \times \underline{A} \to [0,\infty]$ metric $\qquad$ A qbs

d compatible: $\quad d : A \times A \to [0,\infty]$ measurable

$\mathcal{O}_d$ has measurable limits:

$\mathrm{Converges}_d := \{\vec{a} \text{ converges}\} \hookrightarrow A^{\mathbb{N}}$

and $\quad \lim : \mathrm{Converges}_d \to A \quad$ measurable.

Thm: A-compatible w/ measurable limits d.

d separable $\Rightarrow$ $\sigma[\mathcal{O}_d] = \mathcal{B}_A$

Thm: A-compatible w/ measurable limits d:

d separable $\Rightarrow$ $\sigma[\mathcal{O}_d] = \mathcal{B}_A$

Application:

3) $\mathcal{B}_{C_0} = \sigma(\text{Uniform topology}) = \sigma[\mathcal{O}_{d_U}]$:

Proof:

$d_U(f,g) := \sup_{r \in \mathbb{R}} |fr - gr| = \sup_{r \in \mathbb{Q}} |fr - gr|$ meas.

$\text{Converge}_{d_U} = \text{Cauchy}_{d_U} \ni \vec{f} \iff \forall \varepsilon \in_{\mathbb{Q}} (0,\infty)\, \exists n.\, \forall i,j \geq n\; d_U(f_i, f_j) < \varepsilon$

$\lim \vec{f} := \lambda x.\, \lim_{n\to\infty} f_n x$

d seperable via Weierstrass's Approx. Thm + Rational Berenstein Polynomials

Ditto for the Skorokhod space:

$$\mathbb{D}[a,b) := \{ f : [a,b) \to \mathbb{R} \mid f \text{ càdlàg} \} \hookrightarrow \mathbb{R}^{[a,b)}$$

Wip: Effros quasi-Borel space

$$\frac{A \text{ Polish}}{\mathcal{E}_A := \{ F \in \mathcal{B}_A \text{ closed} \} \not\hookrightarrow \mathcal{B}_A}$$

$\rightsquigarrow \quad \mathcal{E}_{\ell_2} =: \mathrm{Pol}$ a Tarski universe for Polish spaces

Rest of talk: mini-tutorial via types

1) Metaphorologies & Qbs
2) Simple-type structure
3) Standard spaces

Talk to me:

I) Colimit structure & Probability spaces
II) dependently-typed structure
III) random variable spaces & stochastic processes

PhD

Course