# Algebraic Reasoning over Relational Structures

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# Outline

- In modern algebraic approaches to the semantics of programming languages, data types and computational effects, models often involve an intricate interplay between algebraic features and relational features.
- Essential result in such algebraic approaches: Birkhoff's Variety Theorem (also known as the HSP Theorem).
- Examples of relational features:
  - order (ordered algebras due to Bloom),
  - distance (quantitative algebras due to Mardare, Panangaden, and Plotkin),
  - generalized distance (generalized quantitative algebras due to Mio, Sarkis, and Vignudelli).
- We provide an intuitive common roof of these scenarios.

# Birkhoff's Variety Theorem

Let $\Sigma$ be a *finitary signature*, i.e. a set whose elements are called *operations* and each of them has an associated finite *arity* (a natural number).
An *algebra* is a set $A$ together with operations $\sigma_A \colon A^n \to A$ for each $n$-ary $\sigma \in \Sigma$.

Given a set of equations $t_1 = t_2$, we can restrict ourselves to algebras satisfying these equations, e.g. for semigroups one can use the equation $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
In general, a class of algebras that can be determined by a set of equations is called a *variety of algebras*.
Examples of varieties: semigroups, groups, vector spaces.
Non-example: fields.

A *homomorphism* is a function between algebras that preserves operations. Given a set of algebras one can form their *product* (cartesian product with operations defined component-wise). Given an algebra, its subset is a *subalgebra* if it is closed under operations. Finally, a *quotient* is simply a surjective homomorphism.

**Birkhoff's Variety Theorem.**

A class of algebras is a variety iff it is closed under products, subalgebras, and quotients.

Reminder: A class being closed under quotients means that if $A$ is in the class and $A \twoheadrightarrow B$ is a quotient, then $B$ is in the class.

# Relational Features

Often, it is useful to have relational features on the algebra and to have the operations compatible with these features. On the side of equations we will work with the *c-clustered equations*:

*Ordered Algebras*:

$$x_i \leq y_i \ \ (i \in I) \ \ \vdash \ \ t_1 \leq t_2 \qquad \text{or} \qquad x_i \leq y_i \ \ (i \in I) \ \ \vdash \ \ t_1 = t_2,$$

*Quantitative Algebras*:

$$x_i =_{\varepsilon_i} y_i \ \ (i \in I) \ \ \vdash \ \ t_1 =_{\varepsilon} t_2,$$

*Generalized Quantitative Algebras*:

$$x_i =_{\varepsilon_i} y_i \ \ (i \in I) \ \ \vdash \ \ t_1 =_{\varepsilon} t_2 \qquad \text{or} \qquad x_i =_{\varepsilon_i} y_i \ \ (i \in I) \ \ \vdash \ \ t_1 = t_2,$$

where $x_i, y_i$ are variables, $t_1, t_2$ are terms, $\varepsilon_i, \varepsilon \in [0, \infty]$, and a certain connectedness condition is imposed on the variables $x_i, y_i$.

# Common Roof

**Definition.**

A *(finitary) relational signature* $\mathscr{S}$ is a set of *relation symbols* with associated positive *arity* $\mathsf{ar}(R) \in \mathbb{N}_+$ for each $R \in \mathscr{S}$. An $\mathscr{S}$*-structure* $(A, (R_A)_{R\in\mathscr{S}})$ is given by a set $A$ equipped with an $n$-ary relation $R_A \subseteq A^n$ for every $n$-ary relation symbol $R \in \mathscr{S}$.

## Definition.

A *morphism* $h\colon A \to B$ of $\mathscr{S}$-structures is a relation-preserving map: for each $n$-ary $R \in \mathscr{S}$ and $a_1, \dots, a_n \in A$,

$$R_A(a_1, \dots, a_n) \implies R_B(h(a_1), \dots, h(a_n)).$$

Conversely, a map $h\colon A \to B$ is said to *reflect relations* if for each $n$-ary $R \in \mathscr{S}$ and $a_1, \dots, a_n \in A$,

$$R_A(a_1, \dots, a_n) \impliedby R_B(h(a_1), \dots, h(a_n)).$$

An *embedding* is an injective map $m\colon A \rightarrowtail B$ that both preserves and reflects relations.

## Notation.

We denote the category of $\mathscr{S}$-structures and their morphisms by $\mathsf{Str}(\mathscr{S})$.

## Definition.

An *infinitary Horn clause* over a set $X$ of variables is an expression of either of the types

$$R_i(x_{i,1},\dots,x_{i,n_i}) \ \ (i\in I) \ \ \vdash \ \ R(x_1,\dots,x_n), \tag{4.1}$$

$$R_i(x_{i,1},\dots,x_{i,n_i}) \ \ (i\in I) \ \ \vdash \ \ x_1 = x_2, \tag{4.2}$$

where (a) $I$ is a set, (b) $x_k, x_{i,k} \in X$ for all indices $i, k$, and (c) $R_i$ ($i \in I$) and $R$ are relation symbols in $\mathscr{S}$ with arities $n_i$ and $n$, respectively.

## Definition.

Let $A$ be an $\mathscr{S}$-structure.

1. The structure $A$ *satisfies* the clause (4.1) if for every map $h\colon X \to A$,

$$(R_i)_A(h(x_{i,1}),\dots,h(x_{i,n_i})) \text{ for all } i\in I \qquad \text{implies} \qquad R_A(h(x_1),\dots,h(x_n)).$$

2. Similarly, $A$ *satisfies* the clause (4.2) if for every map $h\colon X \to A$,

$$(R_i)_A(h(x_{i,1}),\dots,h(x_{i,n_i})) \text{ for all } i\in I \qquad \text{implies} \qquad h(x_1) = h(x_2).$$

**Notation.**

From now on, we fix a relational signature $\mathscr{S}$ and a set Ax of infinitary Horn clauses over $\mathscr{S}$. We denote the full subcategory of structures satisfying all clauses in Ax by

$$\mathscr{C} \hookrightarrow \mathsf{Str}(\mathscr{S}).$$

## Example.

Our leading example is that of generalized metric spaces [MSV22]. A *fuzzy relation* on a set $A$ is a map $d\colon A \times A \to [0,1]$. Let $\mathrm{Ax}_{\mathrm{GM}}$ be a fixed subset of the following axioms:

$$\forall a \in A : d(a,a) = 0 \tag{Refl}$$

$$\forall a,b \in A : d(a,b) = 0 \implies a = b \tag{Pos}$$

$$\forall a,b \in A : d(a,b) = d(b,a) \tag{Sym}$$

$$\forall a,b,c \in A : d(a,c) \leq d(a,b) + d(b,c) \tag{Tri}$$

$$\forall a,b,c \in A : d(a,c) \leq \max\{d(a,b), d(b,c)\} \tag{Max}$$

A *generalized metric space* is a set $A$ with a fuzzy relation $d_A\colon A \times A \to [0,1]$, subject to the axioms in $\mathrm{Ax}_{\mathrm{GM}}$. A map $h\colon A \to B$ between generalized metric spaces is *nonexpansive* if $d_B(h(a), h(a')) \leq d_A(a,a')$ for $a, a' \in A$. We let GMet denote the category of generalized metric spaces and nonexpansive maps.

## Example (continued).

We can regard generalized metric spaces as relational structures as follows. Consider the relational signature $\mathscr{S} = \{ =_\varepsilon : \varepsilon \in [0,1] \}$ where $\mathrm{ar}(=_\varepsilon) = 2$ for each $\varepsilon \in [0,1]$. Let Ax be the corresponding subset of the following Horn clauses, where $\varepsilon, \varepsilon' \in [0,1]$:

$$\vdash x =_0 x \qquad \text{(Refl}'\text{)}$$

$$x =_0 y \vdash x = y \qquad \text{(Pos}'\text{)}$$

$$x =_\varepsilon y \vdash y =_\varepsilon x \qquad \text{(Sym}'\text{)}$$

$$x =_\varepsilon y,\ y =_{\varepsilon'} z \vdash x =_{\varepsilon+\varepsilon'} z \qquad (\varepsilon + \varepsilon' \leq 1) \qquad \text{(Tri}'\text{)}$$

$$x =_\varepsilon y,\ y =_{\varepsilon'} z \vdash x =_{\max\{\varepsilon,\varepsilon'\}} z \qquad \text{(Max}'\text{)}$$

$$x =_\varepsilon y \vdash x =_{\varepsilon'} y \qquad (\varepsilon < \varepsilon') \qquad \text{(Up)}$$

$$x =_{\varepsilon'} y \ \ (\varepsilon' > \varepsilon) \vdash x =_\varepsilon y \qquad \text{(Arch)}$$

Then $\mathscr{C} = \mathsf{GMet}$.

## Examples.

- The category Set of sets and functions is specified by the empty relational signature and the empty set of axioms.
- The category Pos of partially ordered sets (posets) and monotone maps is specified by the relational signature $\mathscr{S}$ consisting of a single binary relation symbol $\leq$ and the axioms

$$\begin{aligned} &\vdash\ x \leq x, \\ x \leq y,\, y \leq z &\vdash\ x \leq z, \\ x \leq y,\, y \leq x &\vdash\ x = y. \end{aligned}$$

## Definition.

A functor $G\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Str}(\mathscr{S})$ is a *lifting* of $F\colon \mathsf{Set} \to \mathsf{Set}$ if the square below commutes:

$$\begin{array}{ccc} \mathsf{Str}(\mathscr{S}) & \xrightarrow{G} & \mathsf{Str}(\mathscr{S}) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathsf{Set} & \xrightarrow{F} & \mathsf{Set} \end{array}$$

## Definition.

An *(infinitary) algebraic signature* is a set $\Sigma$ of operation symbols $\sigma$ with prescribed arity $\mathsf{ar}(\sigma)$, a cardinal number. A *lifted algebraic signature* $\widehat{\Sigma}$ is given by a signature $\Sigma$ together with a lifting $L_\sigma\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Str}(\mathscr{S})$ of the functor $(-)^n\colon \mathsf{Set} \to \mathsf{Set}$ for every $n$-ary operation symbol $\sigma \in \Sigma$. Given $A \in \mathsf{Str}(\mathscr{S})$ we write $L_\sigma(R_A)$ for the interpretation of the relation symbol $R \in \mathscr{S}$ in the structure $L_\sigma(A)$:

$$L_\sigma(A) = (A^n, (L_\sigma(R_A))_{R \in \mathscr{S}}).$$

## Notation.

From now on, we fix a lifted algebraic signature $\widehat{\Sigma}$ with associated lifted functors $L_\sigma$ ($\sigma \in \Sigma$). We assume that each $L_\sigma$ preserves embeddings. Moreover, we choose a regular cardinal $\kappa$ such that every operation symbol in $\Sigma$ has arity $< \kappa$; hence $\Sigma$ is a *$\kappa$-ary signature*.

**Definition.**

A $\widehat{\Sigma}$-*algebra* is given by an $\mathscr{S}$-structure $A$ equipped with $n$-ary operations

$$\sigma_A\colon (A^n, (L_\sigma(R_A))_{R\in\mathscr{S}}) \to (A, (R_A)_{R\in\mathscr{S}}) \qquad \text{in} \quad \mathsf{Str}(\mathscr{S})$$

for every $n$-ary operation symbol $\sigma \in \Sigma$.

**Definition.**

A *morphism* $h\colon A \to B$ of $\widehat{\Sigma}$-algebras is a map from $A$ to $B$ that is both a $\mathsf{Str}(\mathscr{S})$-morphism and a $\Sigma$-algebra morphism; the latter means that $h(\sigma_A(a_1, \ldots, a_n)) = \sigma_B(h(a_1), \ldots, h(a_n))$ for each $n$-ary operation symbol $\sigma \in \Sigma$ and $a_1, \ldots, a_n$.

**Definition.**

We let $\mathbf{Alg}(\widehat{\Sigma})$ denote the category of $\widehat{\Sigma}$-algebras and their morphisms, and $\mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ the full subcategory of $\widehat{\Sigma}$-*algebras over* $\mathscr{C}$, that is, $\widehat{\Sigma}$-algebras whose underlying $\mathscr{S}$-structure lies in the full subcategory $\mathscr{C} \hookrightarrow \mathrm{Str}(\mathscr{S})$ given by Ax.

## Examples.

For every relational signature $\mathscr{S}$, there are two simple choices of a lifting $L_\sigma\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Str}(\mathscr{S})$ for an $n$-ary operation symbol $\sigma \in \Sigma$:

1. The *discrete lifting* $L_\sigma^{\mathrm{disc}}$ maps $A \in \mathsf{Str}(\mathscr{S})$ to $A^n$ equipped with empty relations. Then the operation $\sigma_A\colon A^n \to A$ of a $\widehat{\Sigma}$-algebra $A$ is just an arbitrary map that is not subject to any conditions.
2. The *product lifting* $L_\sigma^{\mathrm{prod}}$ maps $A \in \mathsf{Str}(\mathscr{S})$ to the product structure $A^n$ in $\mathsf{Str}(\mathscr{S})$. Then the operation $\sigma_A\colon A^n \to A$ is relation-preserving w.r.t. the product structure:

$$R_{A^n}((a_{i,1})_{i<n}, \ldots, (a_{i,m})_{i<n}) \iff \forall i < n : R_A(a_{i,1}, \ldots, a_{i,m}).$$

## Examples.

For the signature $\mathscr{S} = \{=_\varepsilon : \varepsilon \in [0,1]\}$ and $\mathscr{C} = \mathsf{GMet}$ we obtain the quantitative $\widehat{\Sigma}$-algebras by Mio et al. [MSV22]. In *op. cit.* and in [MSV23] the authors consider two non-trivial liftings which are motivated by applications in quantitative term rewriting and machine learning:

1. The *Lipschitz lifting* for a fixed parameter $\alpha \in [1, \infty)$ for which the operation $\sigma_A \colon A^n \to A$ of a quantitative $\widehat{\Sigma}$-algebra $A$ is an $\alpha$-Lipschitz map w.r.t. the product metric $d$ on $A^n$.
2. The *Łukaszyk–Karmowski* lifting such that given a quantitative $\widehat{\Sigma}$-algebra $A$ the binary operation $\sigma_A \colon A^2 \to A$ is nonexpansive w.r.t. the Łukaszyk–Karmowski distance.

## Examples.

For the signature $\mathscr{S} = \{ \leq \}$ and $\mathscr{C} = \mathrm{Pos}$ we obtain various notions of *ordered algebras*, i.e. algebras carried by a poset.

1. The discrete lifting and the product lifting correspond to ordered algebras with arbitrary or monotone operations, respectively. The latter are standard ordered algebras due to Bloom.
2. These two liftings admit a common generalization: for a fixed subset $S \subseteq \{1, \ldots, n\}$ and $\sigma \in \Sigma$, let $L_\sigma^S$ be the lifting that sends $A \in \mathrm{Str}(\mathscr{S})$ to $A^n$ with the relation $(a_i)_{i<n} \leq (a'_i)_{i<n}$ iff $a_i \leq a'_i$ for every $i \in S$. An operation $\sigma_A \colon A^n \to A$ is then monotone in precisely the coordinates from $S$.

## Definition.

- A morphism $e \colon A \to B$ in $\mathrm{Str}(\mathscr{S})$ is *c-reflexive* if for every substructure $B_0 \subseteq B$ of cardinality $|B_0| < c$, there exists a substructure $A_0 \subseteq A$ such that $e$ restricts to an isomorphism in $\mathrm{Str}(\mathscr{S})$ (i.e. a bijective embedding) $e_0 \colon A_0 \xrightarrow{\cong} B_0$.
- If additionally $e$ is surjective, then $e$ is a *c-reflexive quotient*.
- By extension, a quotient in $\mathbf{Alg}(\widehat{\Sigma})$ is *c-reflexive* if its underlying quotient in $\mathrm{Str}(\mathscr{S})$ is $c$-reflexive.

## Definition.

A *c-clustered equation* over the set $X$ of variables is an expression of either of the types

$$R_i(x_{i,1},\dots,x_{i,n_i}) \ \ (i \in I) \ \ \vdash \ \ R(t_1,\dots,t_n),$$
$$R_i(x_{i,1},\dots,x_{i,n_i}) \ \ (i \in I) \ \ \vdash \ \ t_1 = t_2,$$

where (a) $I$ is a set, (b) $x_{i,k} \in X$ for all $i, k$, (c) $t_1, \dots, t_n$ are $\Sigma$-terms over $X$, (d) $R_i$ $(i \in I)$ and $R$ are relation symbols in $\mathscr{S}$ with respective arities $n_i$ and $n$, and (e) the set $X$ can be expressed as a disjoint union $X = \coprod_{j \in J} X_j$ of subsets of cardinality $|X_j| < c$ such that for every $i \in I$, the variables $x_{i,1}, \dots, x_{i,n_i}$ all lie in the same set $X_j$.

$$R_i(x_{i,1}, \ldots, x_{i,n_i}) \ \ (i \in I) \ \ \vdash \ \ R(t_1, \ldots, t_n), \tag{4.3}$$

$$R_i(x_{i,1}, \ldots, x_{i,n_i}) \ \ (i \in I) \ \ \vdash \ \ t_1 = t_2, \tag{4.4}$$

## Definition.

Let $A$ be a $\widehat{\Sigma}$-algebra over $\mathscr{C}$.

1. The algebra $A$ *satisfies* the $c$-clustered equation (4.3) if for every map $h\colon X \to A$,

   $$(R_i)_A(h(x_{i,1}), \ldots, h(x_{i,n_i})) \text{ for all } i \in I \qquad \text{implies} \qquad R_A(h^\sharp(t_1), \ldots, h^\sharp(t_n)).$$

   Here $h^\sharp\colon T_\Sigma X \to A$ denotes the unique $\Sigma$-algebra morphism extending $h$.

2. Similarly, $A$ *satisfies* the $c$-clustered equation (4.4) if for every map $h\colon X \to A$,

   $$(R_i)_A(h(x_{i,1}), \ldots, h(x_{i,n_i})) \text{ for all } i \in I \qquad \text{implies} \qquad h^\sharp(t_1) = h^\sharp(t_2).$$

## Examples.

1. For $\mathscr{C} = \mathrm{GMet}$ the $c$-clustered equations are of the form

$$x_i =_{\varepsilon_i} y_i \;\; (i \in I) \;\; \vdash \;\; t_1 =_\varepsilon t_2 \qquad \text{or} \qquad x_i =_{\varepsilon_i} y_i \;\; (i \in I) \;\; \vdash \;\; t_1 = t_2,$$

where (a) $I$ is a set, (b) $x_i, y_i \in X$ for all $i \in I$, (c) $t_1, t_2$ are $\Sigma$-terms over $X$, (d) $\varepsilon_i, \varepsilon \in [0, 1]$, and (e) the set $X$ is a disjoint union $X = \coprod_{j \in J} X_j$ of subsets of cardinality $|X_j| < c$ such that for every $i \in I$, the variables $x_i$ and $y_i$ lie in the same set $X_j$. For ordinary metric spaces, these equations correspond to the $c$-clustered equations introduced by Milius and Urbat [MU19].

2. For $\mathscr{C} = \mathrm{Pos}$ the $c$-clustered equations are of the form

$$x_i \leq y_i \;\; (i \in I) \;\; \vdash \;\; t_1 \leq t_2 \qquad \text{or} \qquad x_i \leq y_i \;\; (i \in I) \;\; \vdash \;\; t_1 = t_2,$$

subject to the conditions (a)−(c) and (e) as in the example above.

**Definition.**

A class of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ is a *c-variety* if it is axiomatizable by $c$-clustered equations.

**Variety Theorem.**

A class of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ is a $c$-variety iff it is closed under $c$-reflexive quotients, subalgebras, and products.

### Example.

For $\mathscr{C} = \mathsf{Pos}$, the cardinal number $c = 2$, and $\widehat{\Sigma}$ obtained by taking for every operation symbol the product lifting, we obtain Bloom's classical variety theorem [Bloom76] for ordered algebras.
For all other choices of $c$ and $\widehat{\Sigma}$, we obtain to a family of new variety theorems for $c$-varieties of ordered algebras.

## Example.

For $\mathscr{C}$ = metric spaces and again the product lifting for every operation symbol, we obtain a refinement of the variety theorem by Mardare et al. [MPP17]: a class of quantitative algebras is axiomatizable by $c$-clustered equations iff it is closed under $c$-reflexive quotients, subalgebras, and products.
For $\mathscr{C} =$ GMet and arbitrary liftings, we obtain a family of new variety theorems for generalized quantitative algebras.

How to prove the Variety Theorem?
Instantiate Milius and Urbat's Abstract Variety Theorem!

Fix a category $\mathscr{A}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$, a full subcategory $\mathscr{A}_0 \hookrightarrow \mathscr{A}$, and a class $\mathscr{X}$ of objects of $\mathscr{A}$.
Informally, we think of $\mathscr{A}$ as a category of algebraic structures, of $\mathscr{A}_0$ as the subcategory of those algebras over which varieties are formed, and of $\mathscr{X}$ as the class of term algebras over which equations are formed.

**Definition.**

The class $\mathscr{X}$ determines a class of quotients in $\mathscr{A}$ defined by

$$\mathcal{E}_{\mathscr{X}} = \{\, e \in \mathcal{E} : \text{every } X \in \mathscr{X} \text{ is projective w.r.t. } e \,\}.$$

An $\mathcal{E}_{\mathscr{X}}$*-quotient* is a quotient represented by a morphism in $\mathcal{E}_{\mathscr{X}}$.

## Definition.

An *abstract equation* is an $\mathcal{E}$-morphism $e\colon X \twoheadrightarrow E$ where $X \in \mathscr{X}$ and $E \in \mathscr{A}_0$. An object $A \in \mathscr{A}_0$ *satisfies* the abstract equation $e$ if every morphism $h\colon X \to A$ factorizes through $e$. A class $\mathcal{V}$ of objects of $\mathscr{A}_0$ is an *abstract variety* if it is axiomatizable by abstract equations

The following theorem, which is a special case of a result by Milius and Urbat [MU19], characterizes abstract varieties by their closure properties:

## Abstract Variety Theorem.

Suppose that the category $\mathscr{A}$ is $\mathcal{E}$-co-well-powered and has products, that $\mathscr{A}_0 \hookrightarrow \mathscr{A}$ is closed under products and subobjects, and that every object of $\mathscr{A}_0$ is an $\mathcal{E}_{\mathscr{X}}$-quotient of some object of $\mathscr{X}$. Then for every class $\mathcal{V}$ of objects of $\mathscr{A}_0$: $\mathcal{V}$ is an abstract variety iff $\mathcal{V}$ is closed under $\mathcal{E}_{\mathscr{X}}$-quotients, subobjects, and products.

## Example.

The classical Birkhoff Variety Theorem corresponds to the instantiation:

- $\mathscr{A} = \mathscr{A}_0 = \Sigma$-algebras for a finitary algebraic signature $\Sigma$;
- $(\mathcal{E}, \mathcal{M})$ = (surjective, injective);
- $\mathscr{X}$ = all free (term) algebras $T_\Sigma X$, where $X$ is a set of variables. Then $\mathcal{E}_{\mathscr{X}} = \mathcal{E}$.

## Our Instantiation.

Let $c > 1$ be a cardinal number. A structure $X \in \mathrm{Str}(\mathscr{S})$ is called *c-clustered* if it can be expressed as a coproduct $X = \coprod_{j \in J} X_j$ where $|X_j| < c$ for each $j \in J$. We instantiate the Abstract Variety Theorem to the following data:

- $\mathscr{A} = \mathbf{Alg}(\widehat{\Sigma})$ and $\mathscr{A}_0 = \mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ for a lifted signature $\widehat{\Sigma}$;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjections}, \text{ embeddings})$;
- $\mathscr{X} =$ all free algebras $T_{\widehat{\Sigma}} X$ where $X \in \mathrm{Str}(\mathscr{S})$ is a $c$-clustered structure. Then $\mathcal{E}_{\mathscr{X}} = \{c\text{-reflexive quotients}\}$.

# Exactness Property

In the classical case, it is well-known that for every $\Sigma$-algebra $A$, surjective $\Sigma$-algebra morphisms $e\colon A \twoheadrightarrow B$ are in bijective correspondence with congruence relations on $A$, which are equivalence relations respected by the operations $\sigma_A\colon A^n \to A$.
We establish a corresponding exactness property for $\widehat{\Sigma}$-algebras, which turns out to be more involved and slightly subtle.
For notational simplicity we assume that the signature $\Sigma$ is finitary.

## Definition.

1. Given a $\widehat{\Sigma}$-algebra $A$ over $\mathscr{C}$ with underlying $\mathscr{S}$-structure $(A, (R_A)_{R\in\mathscr{S}})$, a *refining structure* on $A$ is an $\mathscr{S}$-structure $(R'_A)_{R\in\mathscr{S}}$ with carrier $A$ satisfying the following properties:
   1.1 $(A, (R'_A)_{R\in\mathscr{S}})$ lies in $\mathscr{C}'$;
   1.2 $R_A \subseteq R'_A$ for each $R \in \mathscr{S}$;
   1.3 for each $\sigma \in \Sigma$, the operation $\sigma_A$ is relation-preserving w.r.t. the relations $R'_A$ and $L_\sigma(R'_A)$:

$$L_\sigma(R'_A)((a_{i,1})_{i<n}, \dots, (a_{i,m})_{i<n}) \implies R'_A(\sigma_A((a_{i,1})_{i<n}), \dots, \sigma_A((a_{i,m})_{i<n}))$$

   for every $R \in \mathscr{S}$, where $n$ is a the arity of $\sigma$, $m$ is the arity of $R$, and $a_{i,j} \in A$.
2. A *congruence* on $A$ is an equivalence relation $\equiv$ on $A$ such that, for all $\sigma \in \Sigma$ of arity $n$ and all $a_i, a'_i \in A$, $i = 1, \dots, n$, we have

$$a_i \equiv a'_i \ \ (i < n) \quad \implies \quad \sigma_A(a_1, \dots, a_n) \equiv \sigma_A(a'_1, \dots, a'_n).$$

## Definition.

A *compatible pair* $((R'_A)_{R\in\mathscr{S}}, \equiv)$ on $A$ is given by a refining structure $(R'_A)_{R\in\mathscr{S}}$ and a congruence $\equiv$ on $A$ satisfying the following conditions:

1. For each $n$-ary $R \in \mathscr{S}$ and $a_i, a'_i \in A$, $i = 1, \dots, n$, we have

$$a_i \equiv a'_i \ \ (i < n) \quad \implies \quad \big( R'_A(a_1, \dots, a_n) \iff R'_A(a'_1, \dots, a'_n) \big).$$

2. For all axioms of type (4.2) in Ax and $h\colon X \to A$,

$$(R'_i)_A(h(x_{i,1}), \dots, h(x_{i,n_i})) \text{ for all } i \in I \text{ implies } h(x_1) \equiv h(x_2).$$

We let $\mathcal{E}_{\leftrightarrow}$ denote the class of all quotients in $\mathrm{Str}(\mathscr{S})$ that both preserve and reflect relations.

**Theorem.**

Suppose that $\hat{\Sigma}$ is a lifted signature satisfying $L_\sigma(\mathcal{E}_{\leftrightarrow}) \subseteq \mathcal{E}_{\leftrightarrow}$ for all $\sigma \in \Sigma$. Then for $A \in \mathbf{Alg}(\widehat{\Sigma})$ the complete lattices of $\mathscr{C}$-quotients of $A$ and compatible pairs on $A$ are isomorphic.

# Conclusions and Future Work

We have investigated clustered algebraic equations over relational structures, which generalizes and unifies a number of related notions that naturally appear in algebraic reasoning over metric spaces or posets.

Our approach highlights the clear separation between algebraic and relational aspects.

Future Work: A natural next step will be to derive a relevant complete deduction system. A further direction to investigate is how to characterize our equational theories by properties of the corresponding free-algebra monads. Another potential direction is to investigate clustered equations over relational structures via Lawvere theories with arities.

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Thank you for your attention!

The next slides contain some additional details.

## Example (continued).

We can regard generalized metric spaces as relational structures as follows. Consider the relational signature $\mathscr{S} = \{ =_\varepsilon : \varepsilon \in [0,1] \}$ where $\mathrm{ar}(=_\varepsilon) = 2$ for each $\varepsilon \in [0,1]$. Let Ax be the corresponding subset of the following Horn clauses, where $\varepsilon, \varepsilon' \in [0,1]$:

$$
\begin{aligned}
&\vdash x =_0 x && && \text{(Refl}'\text{)}\\
x =_0 y &\vdash x = y && && \text{(Pos}'\text{)}\\
x =_\varepsilon y &\vdash y =_\varepsilon x && && \text{(Sym}'\text{)}\\
x =_\varepsilon y,\ y =_{\varepsilon'} z &\vdash x =_{\varepsilon+\varepsilon'} z && (\varepsilon + \varepsilon' \leq 1) && \text{(Tri}'\text{)}\\
x =_\varepsilon y,\ y =_{\varepsilon'} z &\vdash x =_{\max\{\varepsilon,\varepsilon'\}} z && && \text{(Max}'\text{)}\\
x =_\varepsilon y &\vdash x =_{\varepsilon'} y && (\varepsilon < \varepsilon') && \text{(Up)}\\
x =_{\varepsilon'} y\ \ (\varepsilon' > \varepsilon) &\vdash x =_\varepsilon y && && \text{(Arch)}
\end{aligned}
$$

An $\mathscr{S}$-structure $(A, (=_\varepsilon)_{\varepsilon\in[0,1]})$ satisfying Ax then gives rise to a generalized metric space $(A, d)$ with the generalized metric defined by $d(a, a') := \inf\{\varepsilon : a =_\varepsilon a'\}$. In the opposite direction, a generalized metric space $(A, d)$ defines an $\mathscr{S}$-structure $(A, (=_\varepsilon)_{\varepsilon\in[0,1]})$ where $a =_\varepsilon a'$ holds iff $d(a, a') \leq \varepsilon$. Then $\mathscr{C} = \mathrm{GMet}$.

## Examples.

For the signature $\mathscr{S} = \{=_\varepsilon : \varepsilon \in [0,1]\}$ and $\mathscr{C} = \mathsf{GMet}$ we obtain the quantitative $\widehat{\Sigma}$-algebras by Mio et al. [MSV22]. In *op. cit.* and in [MSV23] the authors consider two non-trivial liftings which are motivated by applications in quantitative term rewriting and machine learning:

1. The *Lipschitz lifting* $L_\sigma^{\mathsf{Lip},\alpha}$ for a fixed parameter $\alpha \in [1,\infty)$ maps $A \in \mathsf{Str}(\mathscr{S})$ to $A^n$ equipped with the relations $(a_i)_{i<n} =_\varepsilon (a_i')_{i<n}$ iff $a_i =_{\varepsilon/\alpha} a_i'$ for all $i < n$. Then the operation $\sigma_A : A^n \to A$ of a quantitative $\widehat{\Sigma}$-algebra $A$ is an $\alpha$-Lipschitz map w.r.t. the product metric $d$ on $A^n$, which is defined by $d((a_i)_{i<n},(a_i')_{i<n}) := \sup_{i<n} d_A(a_i,a_i')$.

2. The *Łukaszyk–Karmowski* lifting $L_\sigma^{\mathsf{LK},p}$, for a fixed parameter $p \in (0,1)$ and a binary operation symbol $\sigma \in \Sigma$, sends $A \in \mathsf{Str}(\mathscr{S})$ to $A^2$ equipped with the relations defined by $(a_1,a_2) =_\varepsilon (a_1',a_2')$ iff there exist $\varepsilon_{ij} \in [0,1]$ $(i,j = 1,2)$ such that $a_1 =_{\varepsilon_{11}} a_1'$, $a_1 =_{\varepsilon_{12}} a_2'$, $a_2 =_{\varepsilon_{21}} a_1'$, $a_2 =_{\varepsilon_{22}} a_2'$ and $\varepsilon = p^2\varepsilon_{11} + p(1-p)\varepsilon_{12} + (1-p)p\varepsilon_{21} + (1-p)^2\varepsilon_{22}$. Then given a quantitative $\widehat{\Sigma}$-algebra $A$ the operation $\sigma_A : A^2 \to A$ is nonexpansive w.r.t. the Łukaszyk–Karmowski distance.

# Examples.

For the signature $\mathscr{S} = \{\leq\}$ and $\mathscr{C} = \mathrm{Pos}$ we obtain various notions of *ordered algebras*, i.e. algebras carried by a poset.

1. The discrete lifting and the product lifting correspond to ordered algebras with arbitrary or monotone operations, respectively. The latter are standard ordered algebras due to Bloom.
2. These two liftings admit a common generalization: for a fixed subset $S \subseteq \{1, \ldots, n\}$ and $\sigma \in \Sigma$, let $L_\sigma^S$ be the lifting that sends $A \in \mathrm{Str}(\mathscr{S})$ to $A^n$ with the relation $(a_i)_{i<n} \leq (a_i')_{i<n}$ iff $a_i \leq a_i'$ for every $i \in S$. An operation $\sigma_A \colon A^n \to A$ is then monotone in precisely the coordinates from $S$.
3. The *lexicographic lifting* $L_\sigma^{\mathrm{lex}}$ sends $A \in \mathrm{Str}(\mathscr{S})$ to $A^n$ with $(a_i)_{i<n} \leq (a_i')_{i<n}$ if either $(a_i)_{i<n} = (a_i')_{i<n}$, or $a_k \leq a_k'$ for $k = \min\{i < n : a_i \neq a_i'\}$. An operation $\sigma_A \colon A^n \to A$ is then monotone w.r.t. the lexicographic ordering on $A^n$.

Furthermore, combinations of the above items are easily conceivable, e.g. we may specify ordered algebras with a monotone operation $\sigma_A \colon A^5 \to A$ where the order on $A^5$ is lexicographic in the first two coordinates, coordinatewise in the last two, and discrete in the third.