Completeness of graphical languages for finite dimensional Hilbert spaces

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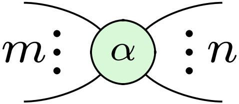
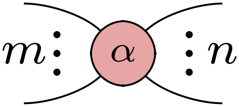
ZX-calculus

Spiders

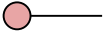


$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |0\rangle^{\otimes n} \langle 0|^{\otimes m} + e^{i\alpha} |1\rangle^{\otimes n} \langle 1|^{\otimes m}$$

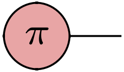


$$\overset{[\![\cdot]\!]}{\longmapsto} \quad |+\rangle^{\otimes n} \langle +|^{\otimes m} + e^{i\alpha} |-\rangle^{\otimes n} \langle -|^{\otimes m}$$

Computational basis states

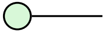


$$\xrightarrow{[\![\cdot]\!]} \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

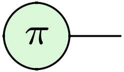


$$\xrightarrow{[\![\cdot]\!]} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Superposition

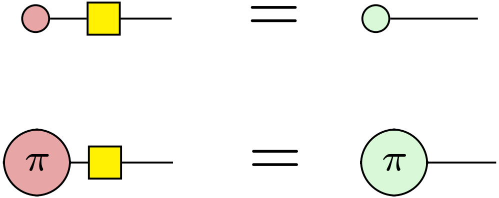


$$\xmapsto{[\![\cdot]\!]} \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

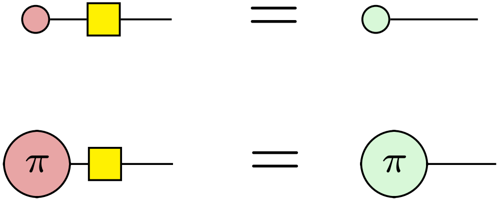


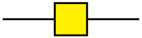
$$\xmapsto{[\![\cdot]\!]} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

Hadamard box



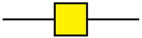
Hadamard box





$$\xmapsto{[\![\cdot]\!]} \quad |+\rangle\langle 0| + |-\rangle\langle 1|$$

Hadamard matrix

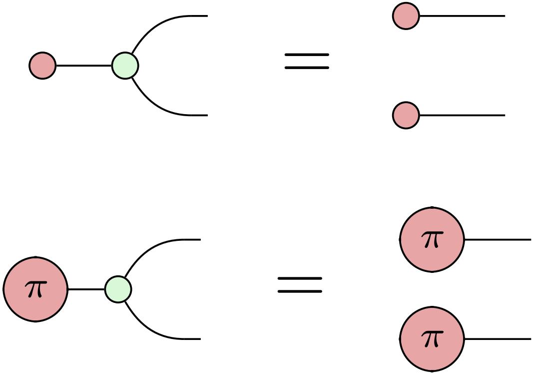


$$\xrightarrow{[\![\cdot]\!]} \quad |+\rangle\langle 0| + |-\rangle\langle 1|$$

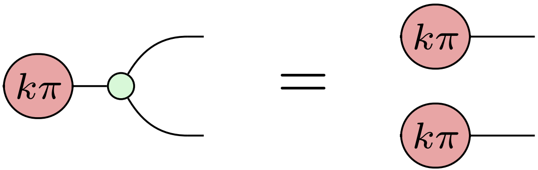
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

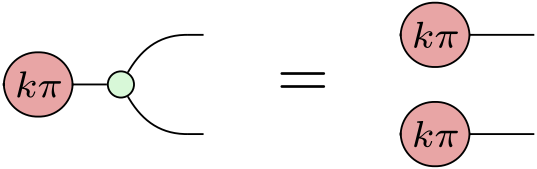
Z-spider

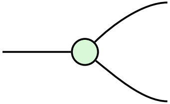


Z-spider



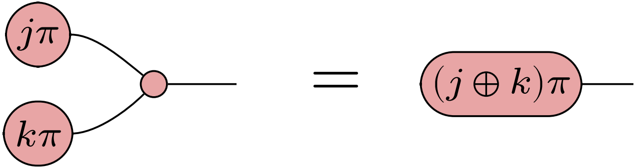
Z-spider



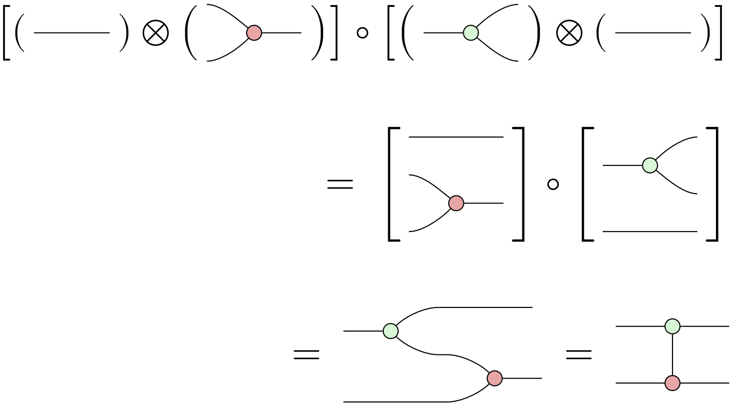


$$\xmapsto{[\![\cdot]\!]} \quad |0\rangle\langle 0,0| + |1\rangle\langle 1,1|$$

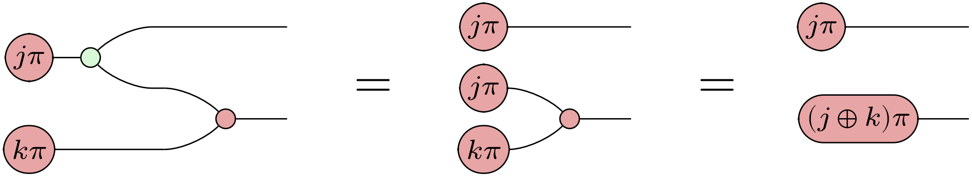
X-spider



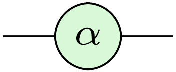
Composition



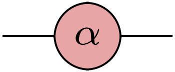
CNOT gate



Phases

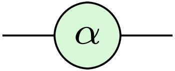


$$\xrightarrow{[\![\cdot]\!]} \quad |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1|$$

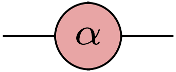


$$\xrightarrow{[\![\cdot]\!]} \quad |+\rangle\langle +| + e^{i\alpha}|-\rangle\langle -|$$

Phases

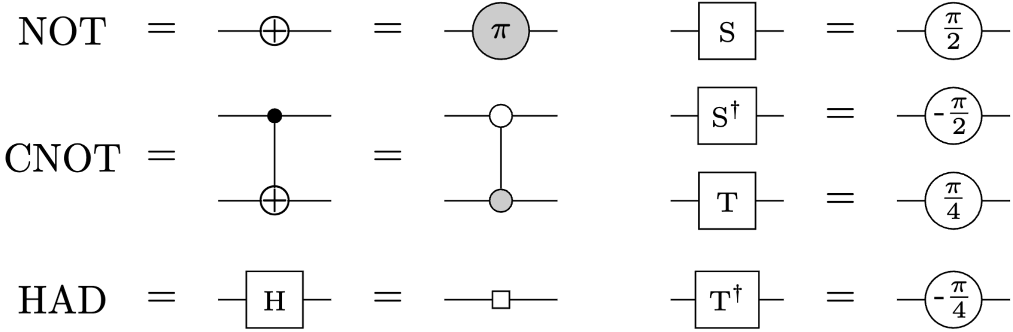


$$\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$$



$$\xmapsto{[\![\cdot]\!]} \frac{1}{2}\begin{bmatrix} 1+e^{i\alpha} & 1-e^{i\alpha} \\ 1-e^{i\alpha} & e^{i\alpha} \end{bmatrix}$$

Quantum gates

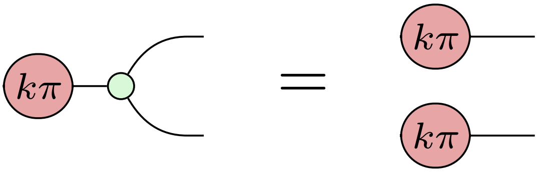


Theorem (Universality)

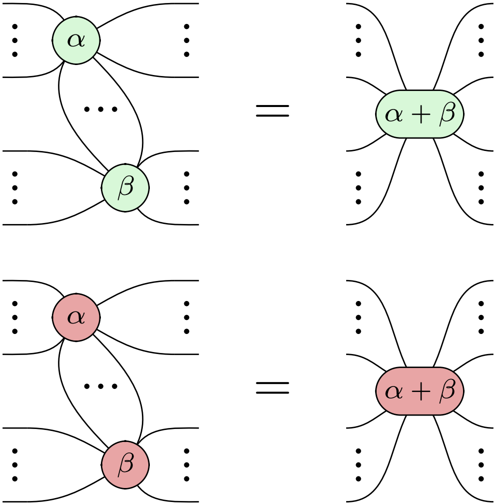
Any linear map between qubits can be expressed in terms of ZX diagrams.

Rewrite rules

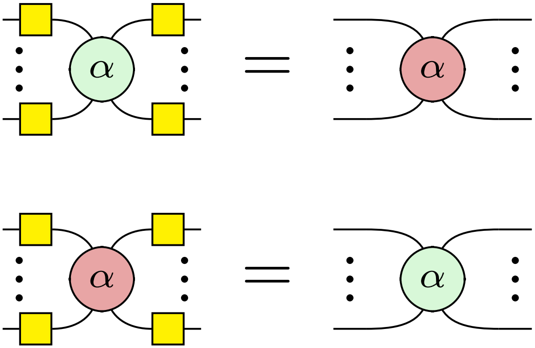
Copy



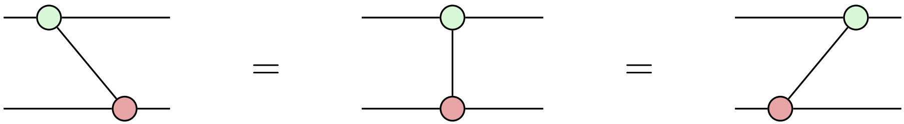
Fusion



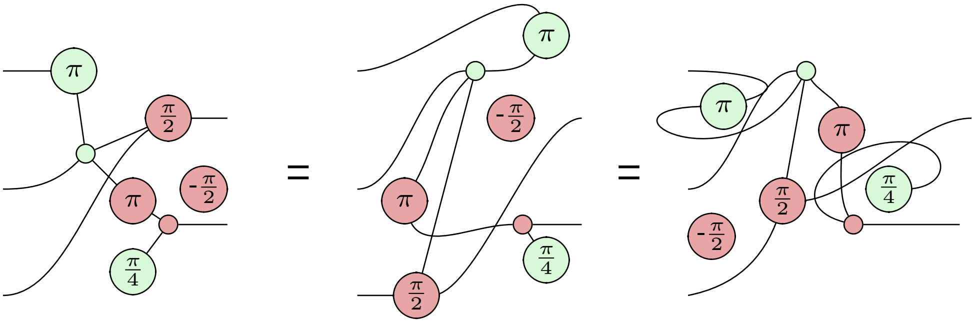
Color



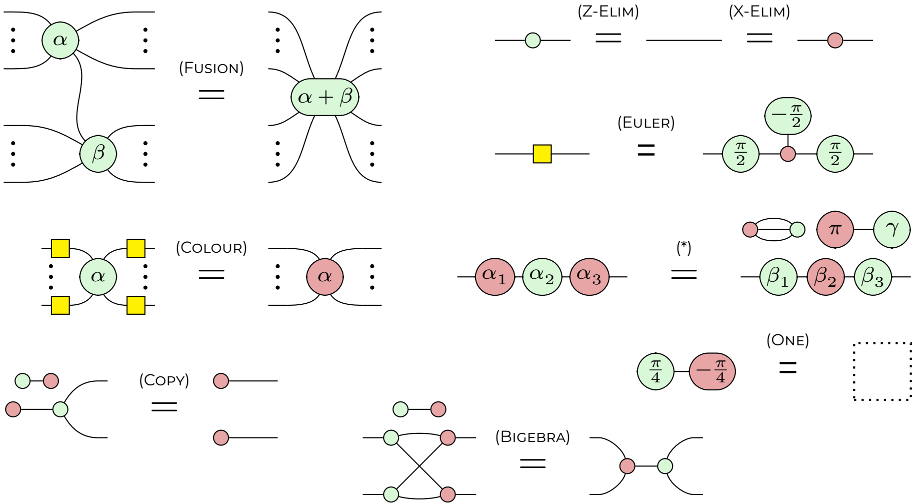
Only Connectivity Matters



Only Connectivity Matters



Axioms

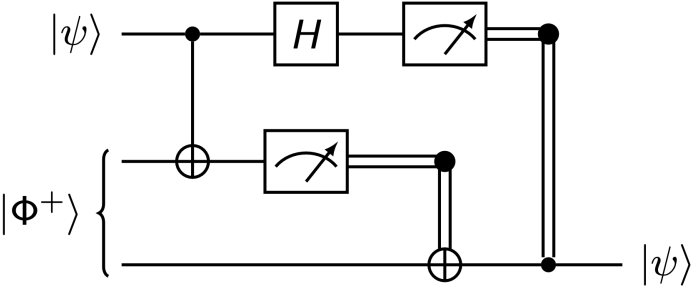


Theorem (Completeness)

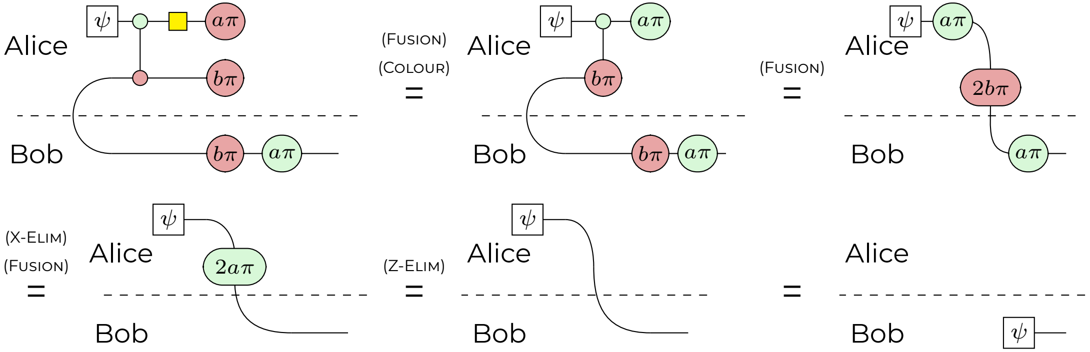
Any equation that holds for linear maps between qubits can be derived in ZX-calculus.

Example

Quantum Teleportation



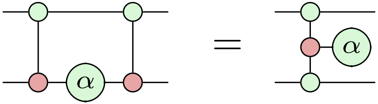
Quantum Teleportation



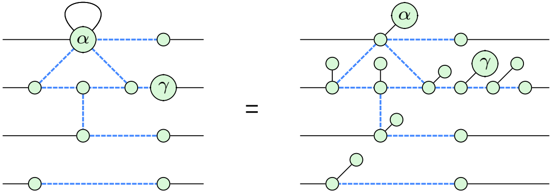
Extensions

Applications: ZX-calculus

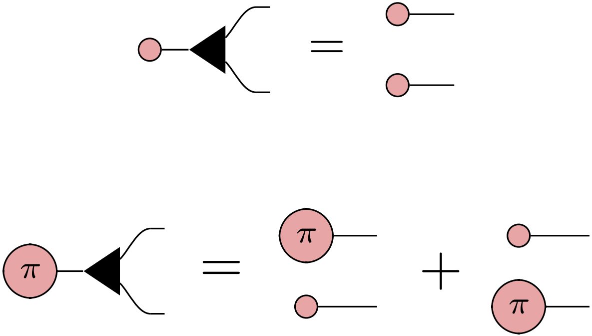
Quantum Circuit Optimisation



Measurement-Based Quantum Computing

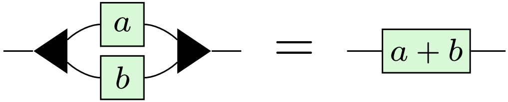


W node

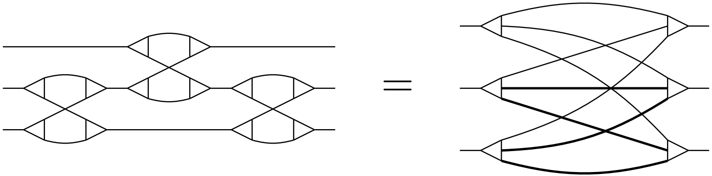


ZW-calculus

Summation

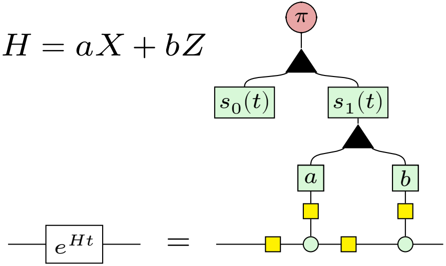


Linear Optical Quantum Computing

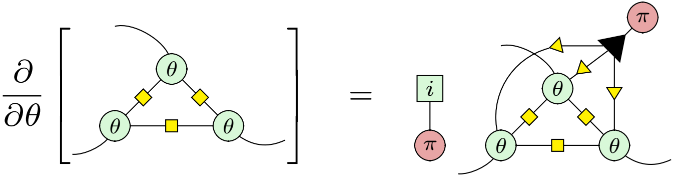


ZXW-calculus

Hamiltonians



Differentiation and integration



Completeness of qufinite ZXW calculus,

a graphical language for finite-dimensional quantum theory

Finite-dimensional Hilbert spaces

Definition

FHilb is the category of finite-dimensional Hilbert spaces.

Definition

$\textbf{FHilb}_d$ is the subcategory of **FHilb**, where Hilbert spaces have dimensions of d^n.

Qubits:

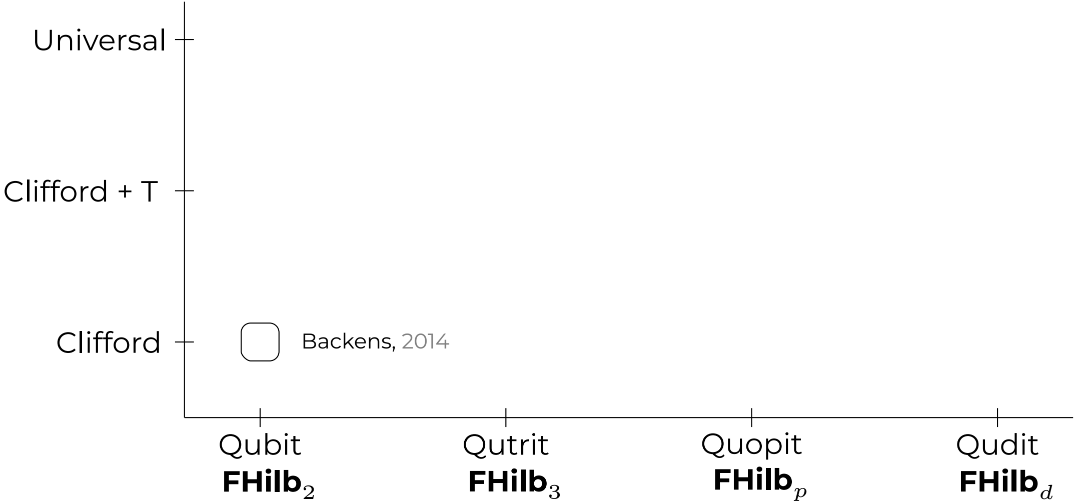
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

Qudits:

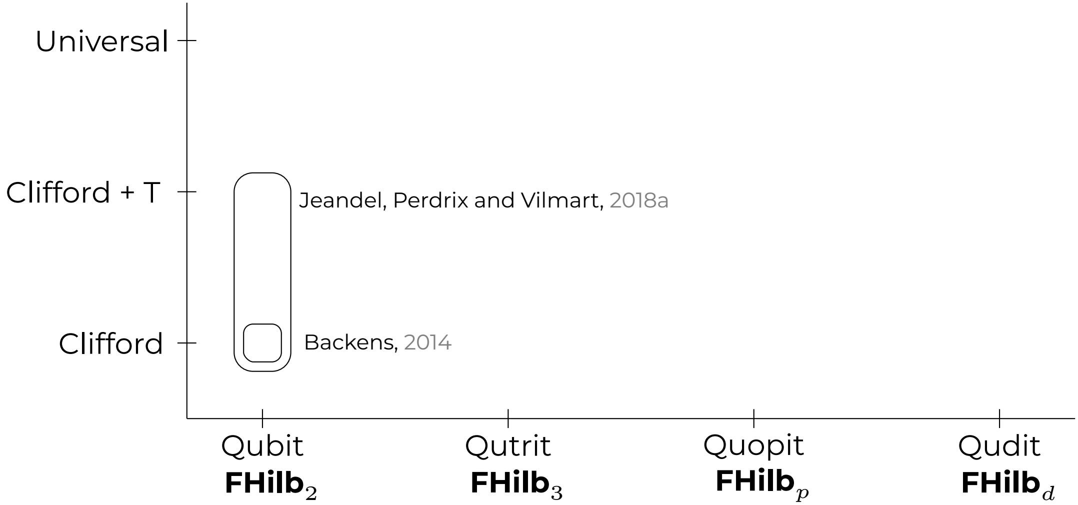
$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle + a_2\,|2\rangle + \cdots + a_{d-1}\,|d-1\rangle$$

History of Completeness

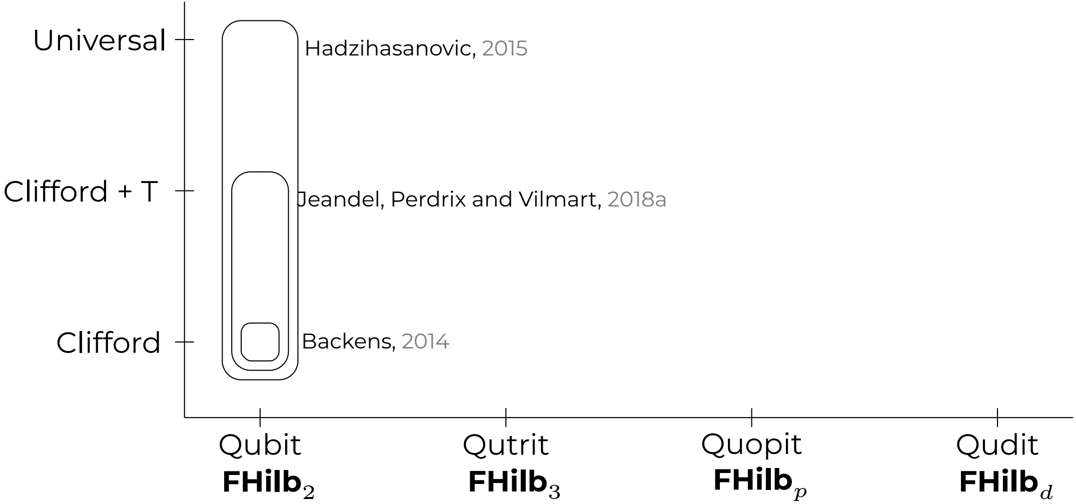
History of Completeness



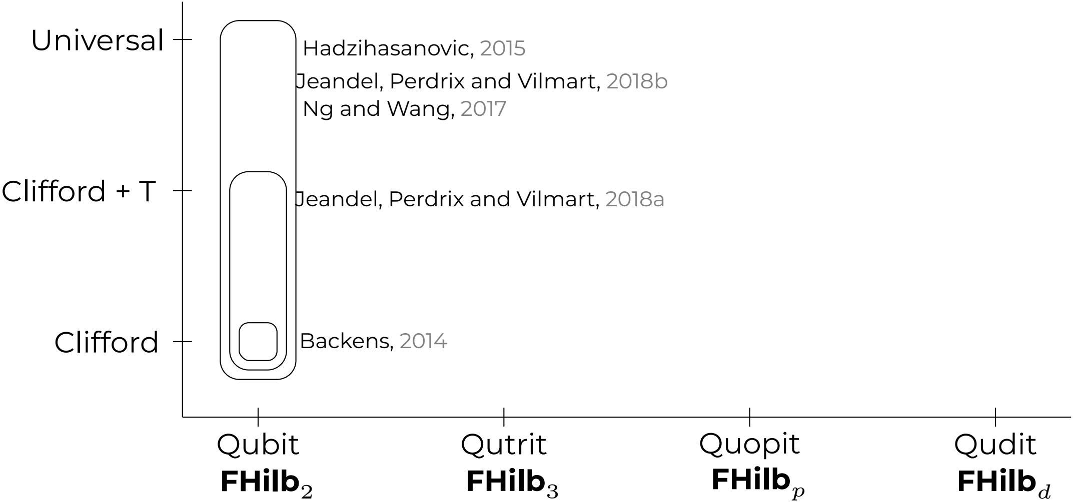
History of Completeness



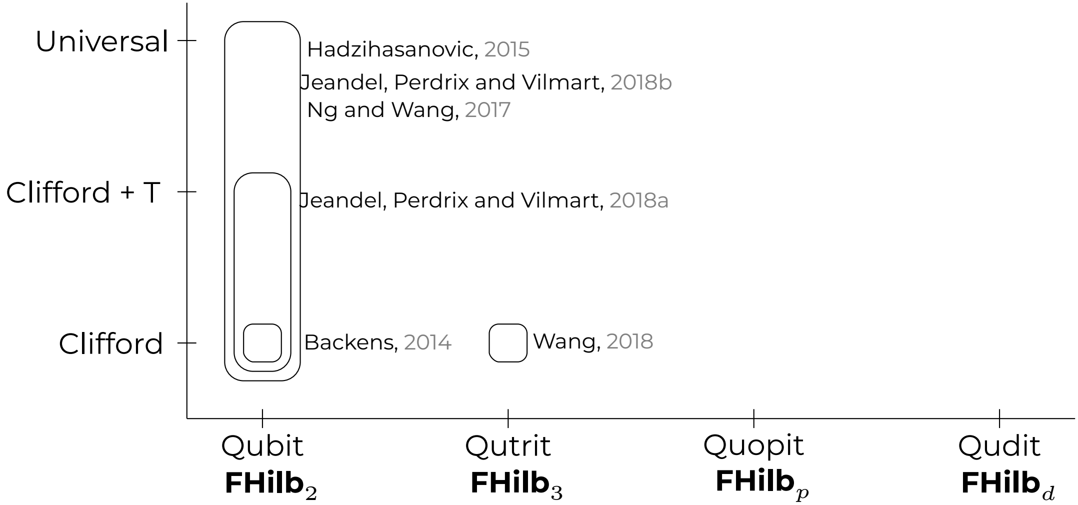
History of Completeness



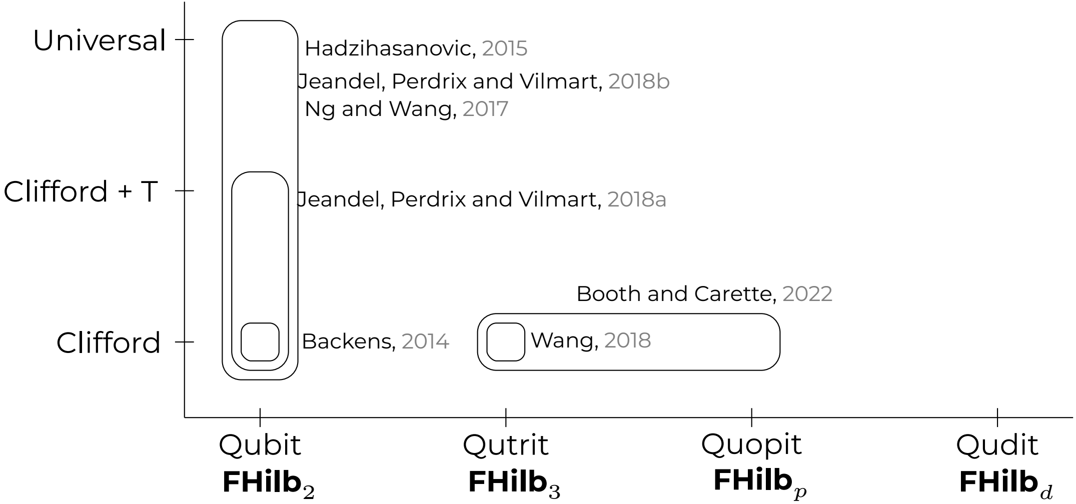
History of Completeness



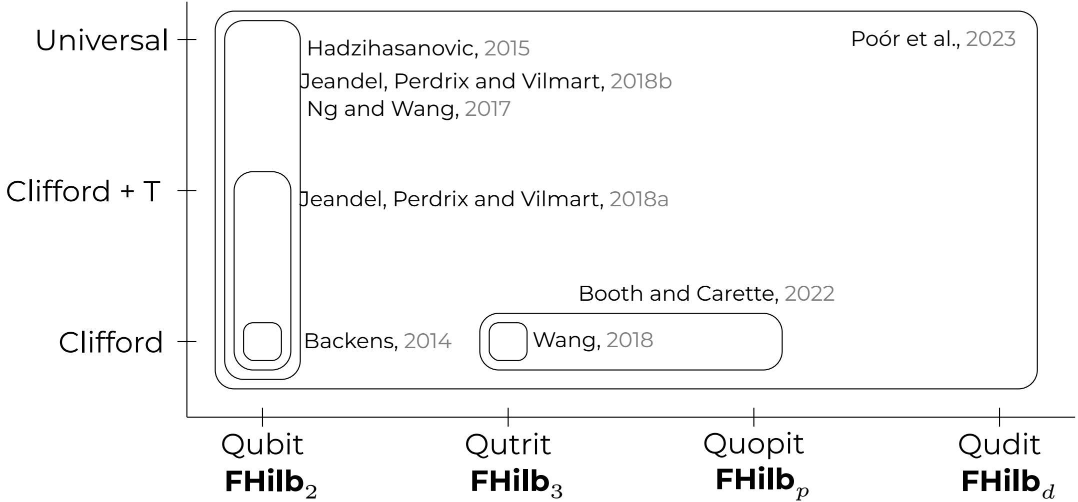
History of Completeness



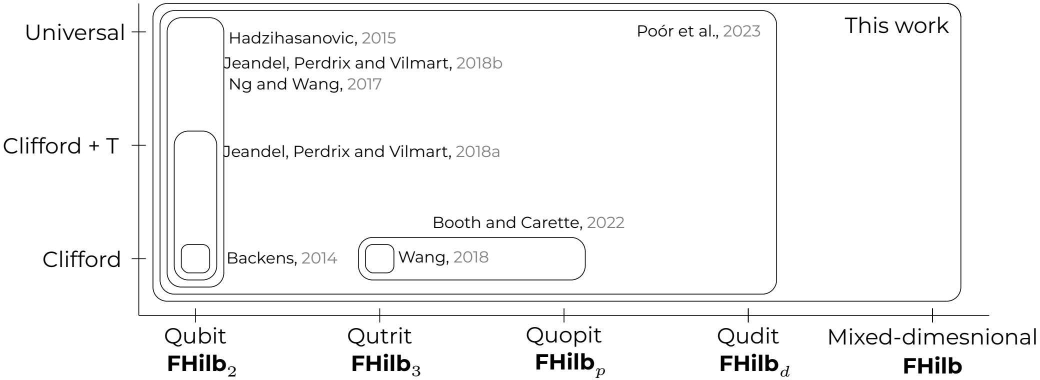
History of Completeness



History of Completeness



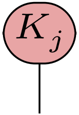
History of Completeness



The qufinite ZXW-calculus

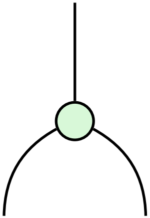
Standard bases

For $0 \leq j < d$,



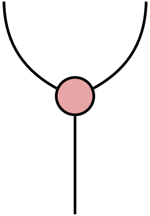
$$\xmapsto{[\![\cdot]\!]} \quad |d - j\rangle$$

Z spider



$$\xrightarrow{[\![\cdot]\!]} \quad |k\rangle \mapsto |k,\ k\rangle$$

X spider



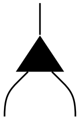
$$\xrightarrow{[\![\cdot]\!]} \quad |i,\ j\rangle \mapsto |i + j \mod d\rangle$$

Notation: The multiplier

$$:=$$

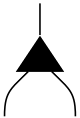
$$\xrightarrow{[\![\cdot]\!]} \quad |k\rangle \mapsto |m \cdot k \mod d\rangle \,,$$

Generator: W node

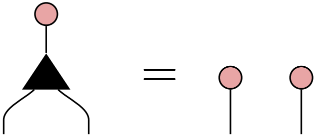


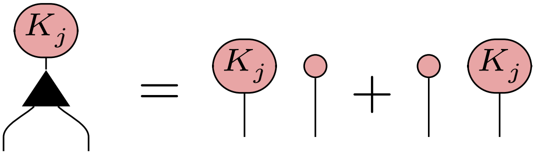
$$\xrightarrow{[\![\cdot]\!]} \quad |00\rangle\langle 0| + \sum_{i=1}^{d-1} \left(|i0\rangle + |0i\rangle\right)\langle i|$$

Generator: W node



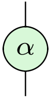
$$\xmapsto{[\![\cdot]\!]} \quad |00\rangle\langle 0| + \sum_{i=1}^{d-1}\left(|i0\rangle + |0i\rangle\right)\langle i|$$





Understanding the Z box

Z spider:

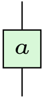
 $\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix},$ where $\alpha \in \mathbb{R}$.

Z box:

 $\xmapsto{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix},$ where $a \in \mathbb{C}$.

Understanding the qudit Z box

Qubit Z box: for $a \in \mathbb{C}$,

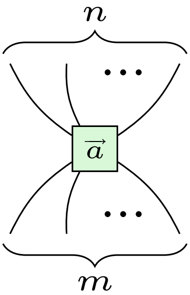


$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

Qudit Z box: for $\vec{a} = (a_1, a_2, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$,

$$\xrightarrow{[\![\cdot]\!]} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \end{bmatrix}$$

Generator: Z box



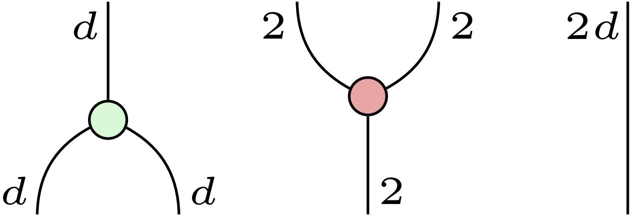
$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \sum_{j=0}^{d-1} a_j \, |j\rangle^{\otimes m} \, \langle j|^{\otimes n},$$

where $\vec{a} = (a_1, \cdots, a_{d-1}) \in \mathbb{C}^{d-1}$

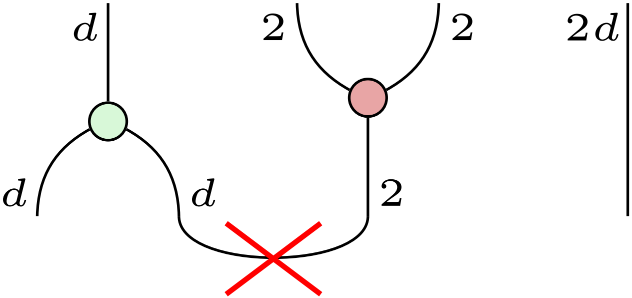
and $a_0 := 1$

Mixed-dimensional generatos

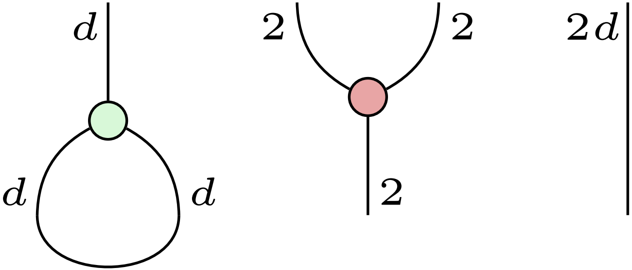
Mixed dimensions



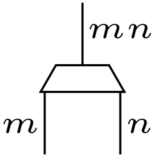
Mixed dimensions



Mixed dimensions

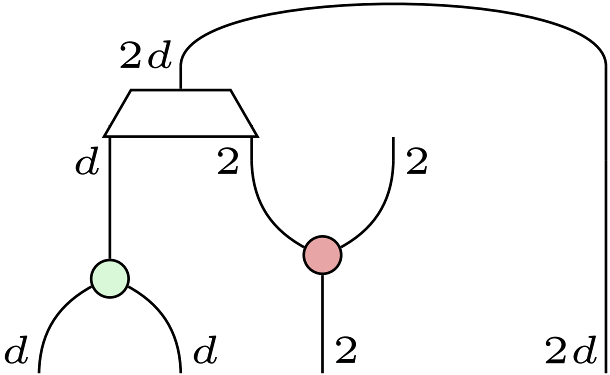


Dimension splitter

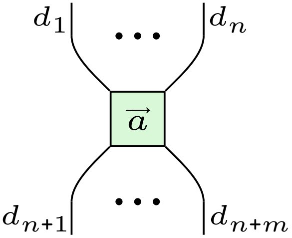


$$\xrightarrow{[\![\cdot]\!]} \quad \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |i, j\rangle \langle in + j| \,.$$

Interacting diemensions



Mixed-dimensional Z box

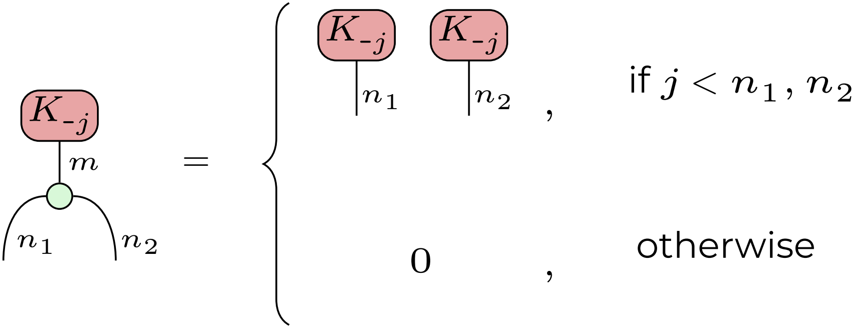


$$\overset{[\![\cdot]\!]}{\longmapsto} \quad \sum_{j=0}^{\min\{d_i\}_i-1} a_j \left|j,\cdots,j\right\rangle \left\langle j,\cdots,j\right|,$$

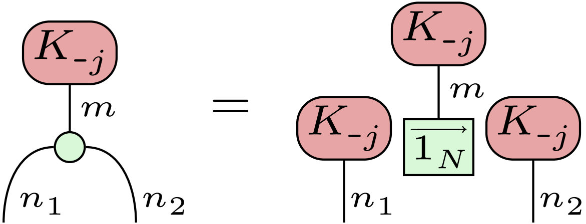
where $\vec{a} = \left(a_1, \cdots, a_{\min\{d_i\}_i-1}\right) \in \mathbb{C}^{d-1}$

and $a_0 := 1$

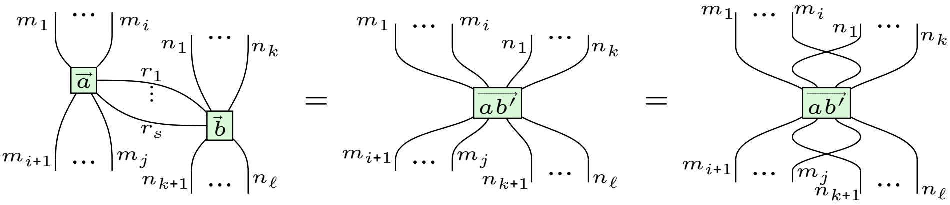
Mixed-dimensional copy



Rule: Mixed-dimensional copy

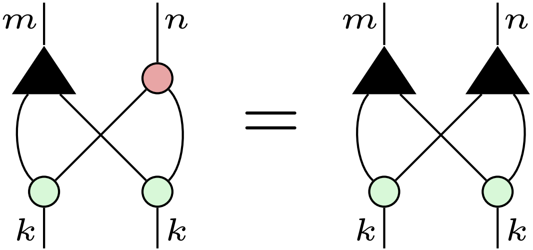


Rule: Mixed-dimensional fusion

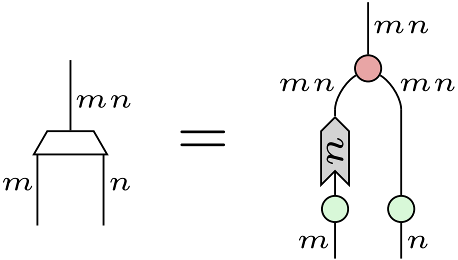


where $M = \min\{\min_{t=1}^{j} m_t, \min_{t=1}^{\ell} n_t, \min_{t=1}^{s} r_t\}$, $\overrightarrow{ab'} = (a_1 b_1, \dots, a_{M-1} b_{M-1})$.

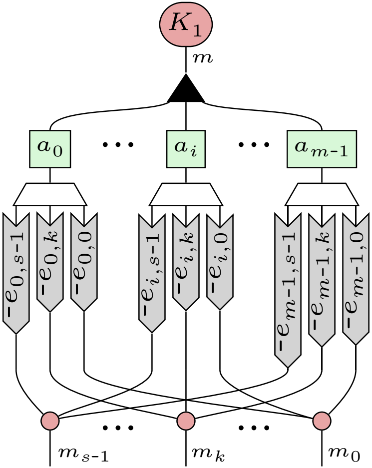
Rule: Trialgebra



Rule: Dimension Splitter



A Normal Form



$$\overset{[\![\cdot]\!]}{\longmapsto} \begin{pmatrix} a_0 \\ \vdots \\ a_i \\ \vdots \\ a_{m-1} \end{pmatrix}$$

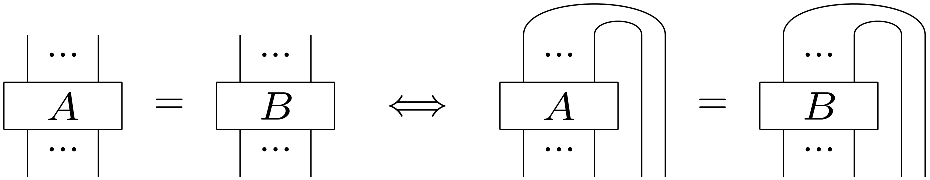
where $m = \prod m_i$

Proposition

The interpretation functor $[\![\cdot]\!]$ *is full.*

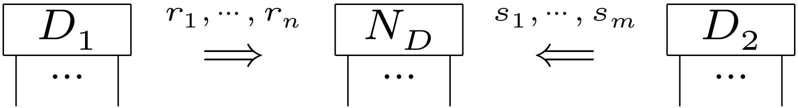
Completeness proof

Map-state duality

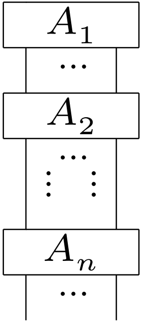


Completeness using a normal form

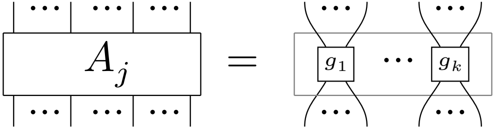
If $[\![D_1]\!] = [\![D_2]\!]$, then:



Structure of states

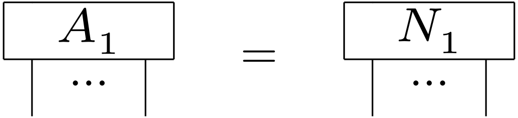


with

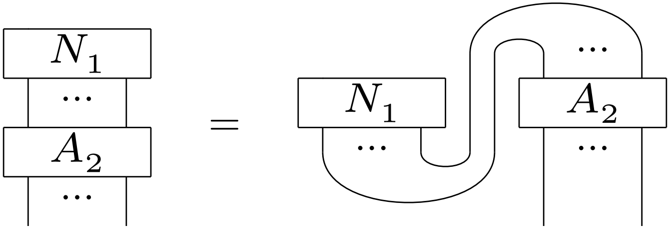


where $g_1, \cdots, g_k$ are generators.

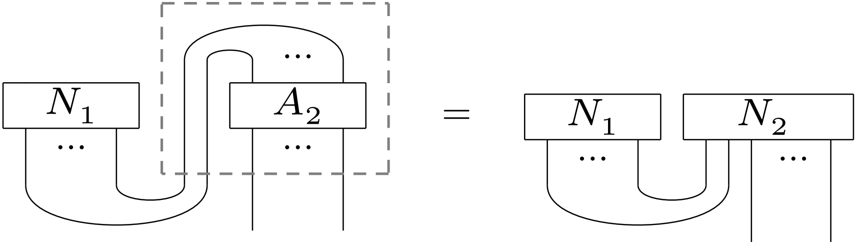
State $\Longrightarrow$ normal form I.



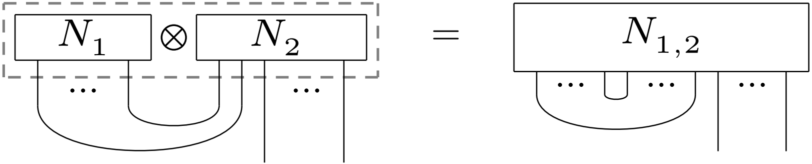
State $\Longrightarrow$ normal form II.



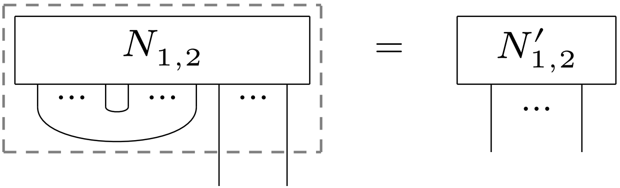
State $\Longrightarrow$ normal form III.



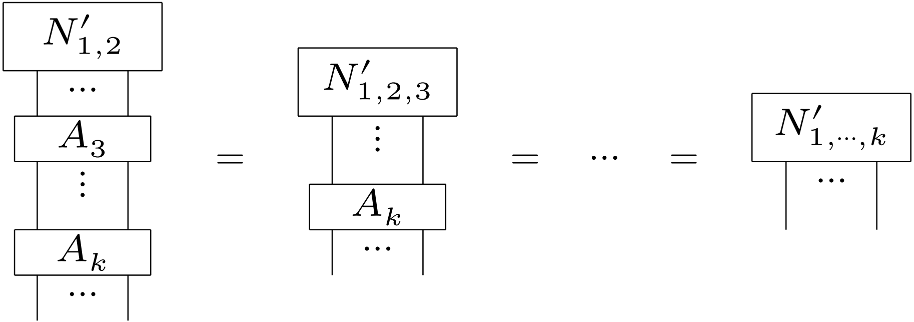
State $\Longrightarrow$ normal form IV.



State $\Longrightarrow$ normal form V.



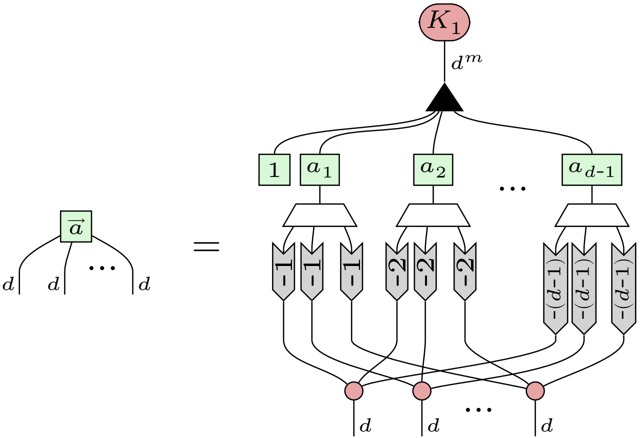
State $\Longrightarrow$ normal form VI.



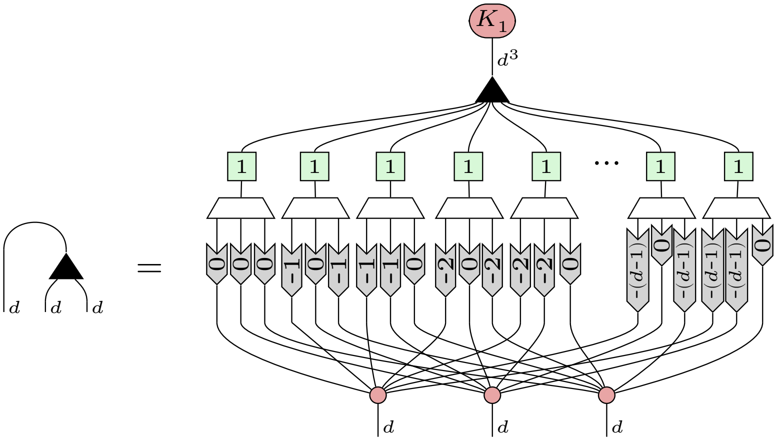
Summary: state $\implies$ normal form

- Generators
- Tensor product of two normal forms
- Partial-traced normal form

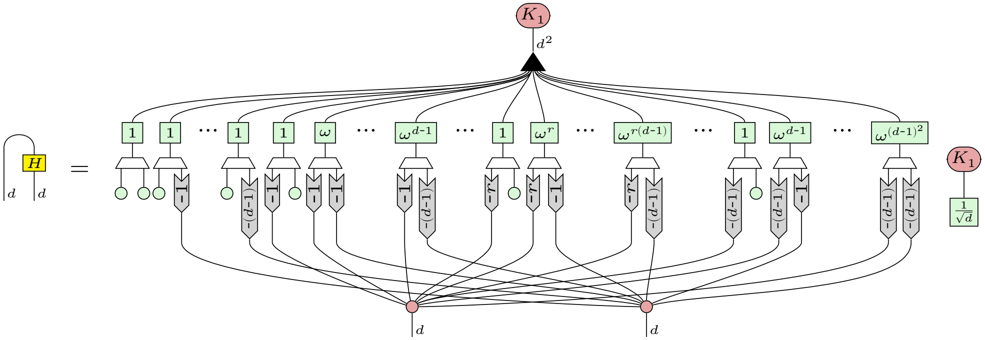
Lemma: Z box



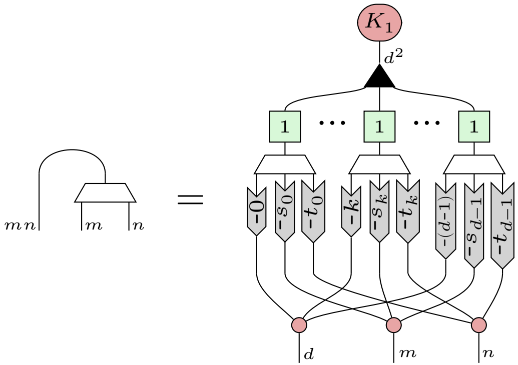
Lemma: W node



Lemma: Hadamard box

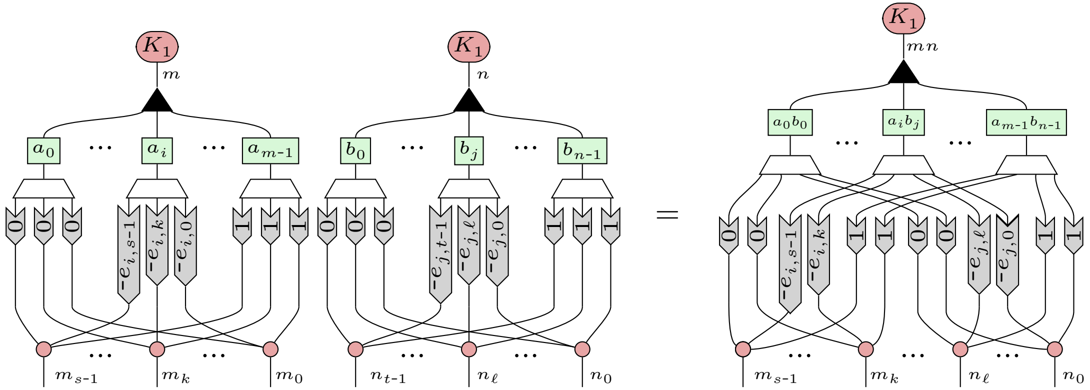


Lemma: Dimension splitter



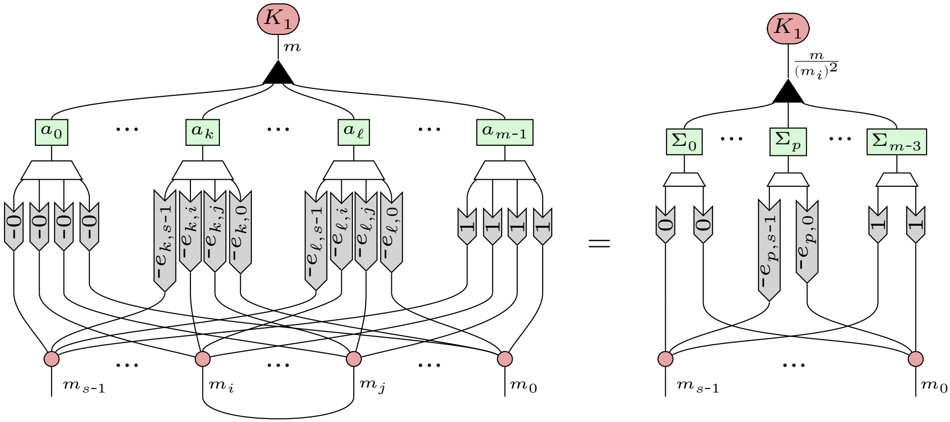
where $k = s_k n + t_k, 0 \leq k \leq mn - 1.$

Lemma: Tensor product



where $M = d^m - 1,\ N = d^n - 1.$

Lemma: Partial trace



where $m_i = m_j$ and Σ_k corresponds to the elements of the partial trace over s and t indices.

Theorem (Completeness)

The qufinite ZXW calculus is complete for finite-dimensional Hilbert spaces.

Corollary

*The category **ZXW** is monoidally equivalent to the category **FHilb**.*

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Two categories are monoidally equivalent if

- there is a monoidal functor between them and
- the functor is full, faithful, and
- essentially surjective on objects

(Heunen and Vicary, 2019)

Corollary

The category ***ZXW*** *is monoidally equivalent to the category* ***FHilb****.*

Proof.

- The interpretation functor $[\![\cdot]\!]$ is a monoidal functor.

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- For any object $H \in$ **FHilb**, we have an object $(\dim(H),) \in$ **ZXW** such that $H \cong \mathbb{C}^{\dim(H)} = [\![(\dim(H),)]\!]$; hence, $[\![\cdot]\!]$ is essentially surjective on objects.

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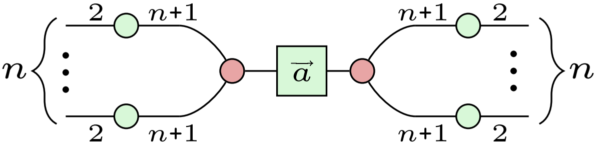
$\square$

Applications

Symmetrizer of spin-$\frac{n}{2}$



where $\vec{a} = \left(\frac{1}{\binom{n}{1}}, \cdots, \frac{1}{\binom{n}{k}}, \cdots, \frac{1}{\binom{n}{n}} \right)$

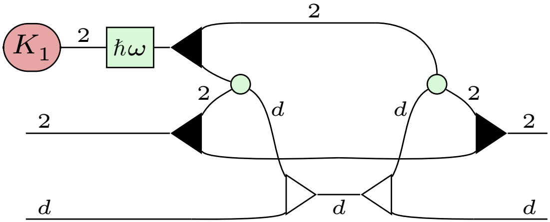
Penrose Spin Calculus: ZX for SU(2)

Quanlong Wang[1] Richard D. P. East Razin A. Shaikh[1,2] Lia Yeh[1,2]
Boldizsár Poór[1]

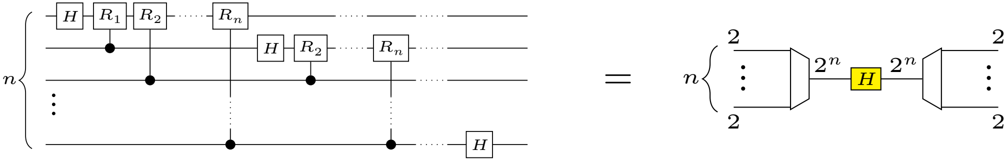
[1]Quantinuum, 17 Beaumont Street, Oxford, OX1 2NA, United Kingdom
[2]University of Oxford, United Kingdom

We introduce the Penrose spin calculus as an elevation of Penrose's diagrams and associated Binor calculus to the level of a formal diagrammatic language. By leveraging the mixed-dimensional ZX calculus, a complete language for finite dimensional Hilbert spaces, we formulate a diagrammatic language for $SU(2)$ representation theory in quantum informational terms. Using this language we firstly articulate the classic angular moment relations

Jaynes-Cummings



Quantum programming language: QFT



ZX-calculus is Complete for Finite-Dimensional Hilbert Spaces

Boldizsár Poór[1] Razin A. Shaikh[1,2] Quanlong Wang[1]

[1]Quantinuum, 17 Beaumont Street, Oxford, OX1 2NA, United Kingdom
[2]University of Oxford, United Kingdom

The ZX-calculus is a graphical language for reasoning about quantum computing and quantum information theory. As a complete graphical language, it incorporates a set of axioms rich enough to derive any equation of the underlying formalism. While completeness of the ZX-calculus has been established for qubits and the Clifford fragment of prime-dimensional qudits, universal completeness beyond two-level systems has remained unproven until now. In this paper, we present a proof establishing the completeness of finite-dimensional ZX-calculus, incorporating only the mixed-dimensional Z-spider and the qudit X-spider as generators. Our approach builds on the completeness of another graphical language, the finite-

Thank you!

Overview

1. Introduction
   - Qudits
   - Completeness
2. The qufinite ZXW-calculus
   - Qudit genarators
   - Mixed-dimensional generatos
   - Example equalities
3. Completeness proof
   - Proof idea
   - Lemmas
4. Applications

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