

Completeness of graphical languages for finite dimensional Hilbert spaces

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ZX-calculus

Spiders

$$\begin{array}{c} m \\ \vdots \\ \text{---} \\ \alpha \\ \text{---} \\ \vdots \\ n \end{array} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |0\rangle^{\otimes n} \langle 0|^{\otimes m} + e^{i\alpha} |1\rangle^{\otimes n} \langle 1|^{\otimes m}$$

$$\begin{array}{c} m \\ \vdots \\ \text{---} \\ \alpha \\ \text{---} \\ \vdots \\ n \end{array} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |+\rangle^{\otimes n} \langle +|^{\otimes m} + e^{i\alpha} |-\rangle^{\otimes n} \langle -|^{\otimes m}$$

Computational basis states

$$\text{---} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |0\rangle \quad = \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

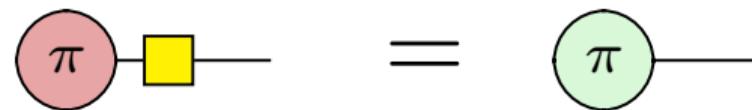
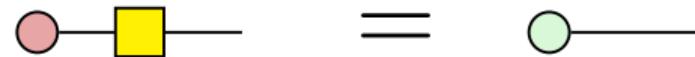
$$\pi \text{---} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |1\rangle \quad = \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Superposition

$$\text{---} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |+\rangle \quad = \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{---} \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad |-\rangle \quad = \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hadamard box



Hadamard box

$$\text{Red circle} \otimes \text{Yellow square} = \text{Green circle}$$

$$\text{Red circle with } \pi \otimes \text{Yellow square} = \text{Green circle with } \pi$$

$$\text{Yellow square} \mapsto |\cdot\rangle\langle 0| + |\cdot\rangle\langle 1|$$

Hadamard matrix

$$\text{---} \boxed{} \text{---} \quad \mapsto \quad |+\rangle \langle 0| + |-\rangle \langle 1|$$

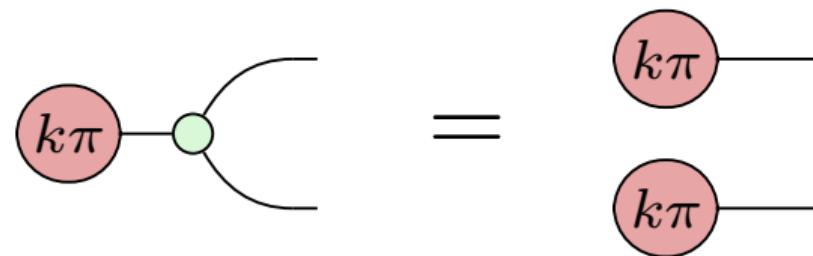
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Z-spider

$$\begin{array}{c} \text{Diagram 1: } \text{A red circle connected by a horizontal line to a green circle. From the green circle, two curved lines branch out downwards.} \\ = \\ \text{Diagram 2: } \text{Two separate horizontal lines, each ending in a red circle.} \end{array}$$
$$\begin{array}{c} \text{Diagram 3: } \text{A red circle containing the symbol } \pi \text{ connected by a horizontal line to a green circle. From the green circle, two curved lines branch out downwards.} \\ = \\ \text{Diagram 4: } \text{Two separate horizontal lines, each ending in a red circle containing the symbol } \pi. \end{array}$$

Z-spider



Z-spider

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \xrightarrow{\llbracket \cdot \rrbracket} |0\rangle\langle 0,0| + |1\rangle\langle 1,1|$$

X-spider

$$\begin{array}{c} j\pi \\ \text{---} \\ k\pi \end{array} = \boxed{(j \oplus k)\pi} \text{---}$$

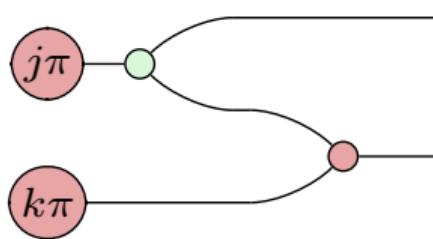
Composition

$$[(\text{---}) \otimes (\text{---} \textcircled{\textcolor{red}{r}} \text{---})] \circ [(\text{---} \textcircled{\textcolor{green}{l}} \text{---}) \otimes (\text{---})]$$

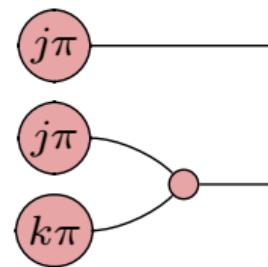
$$= \left[\begin{array}{c} \text{---} \\ \text{---} \textcircled{\textcolor{red}{r}} \text{---} \\ \text{---} \end{array} \right] \circ \left[\begin{array}{c} \text{---} \\ \text{---} \textcircled{\textcolor{green}{l}} \text{---} \\ \text{---} \end{array} \right]$$

$$= \text{---} \textcircled{\textcolor{green}{l}} \text{---} \text{---} \textcircled{\textcolor{red}{r}} \text{---} = \text{---} \textcircled{\textcolor{green}{l}} \text{---} \text{---} \textcircled{\textcolor{red}{r}} \text{---}$$

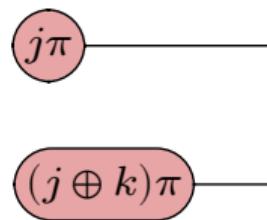
CNOT gate



=



=



Phases

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \bigcirc \alpha \begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\llbracket \cdot \rrbracket} |0\rangle\langle 0| + e^{i\alpha} |1\rangle\langle 1|$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \bigcirc \alpha \begin{array}{c} \text{---} \\ \text{---} \end{array} \xrightarrow{\llbracket \cdot \rrbracket} |+\rangle\langle +| + e^{i\alpha} |-\rangle\langle -|$$

Phases

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} \alpha \text{---} \begin{array}{c} \mathbb{I} \cdot \mathbb{I} \\ \mapsto \end{array} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} \alpha \text{---} \begin{array}{c} \mathbb{I} \cdot \mathbb{I} \\ \mapsto \end{array} \frac{1}{2} \begin{bmatrix} 1 + e^{i\alpha} & 1 - e^{i\alpha} \\ 1 - e^{i\alpha} & e^{i\alpha} \end{bmatrix}$$

Quantum gates

$$\text{NOT} = \begin{array}{c} \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$= \begin{array}{c} \text{---} \\ \pi \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$\text{CNOT} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \\ \oplus \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ -\frac{\pi}{2} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$\text{HAD} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \square \end{array}$$

$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \frac{\pi}{4} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

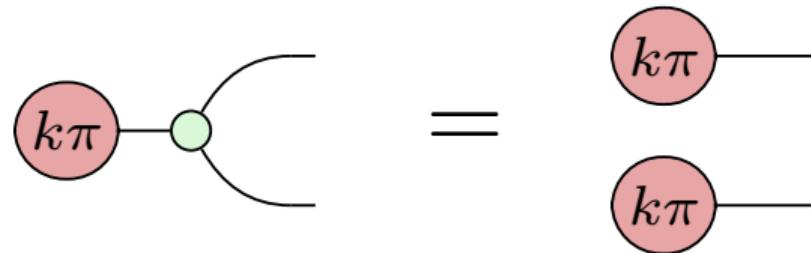
$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ -\frac{\pi}{4} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

Theorem (Universality)

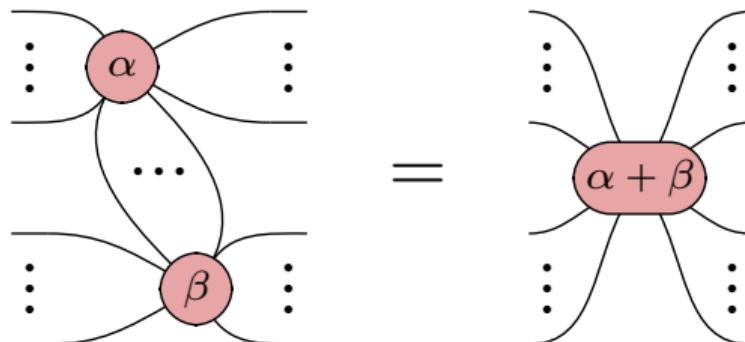
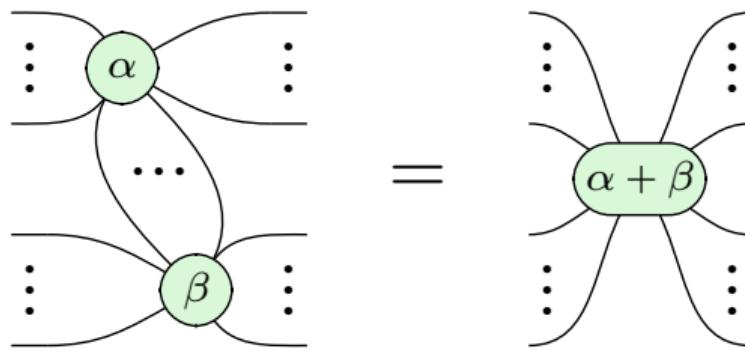
Any linear map between qubits can be expressed in terms of ZX diagrams.

Rewrite rules

Copy



Fusion

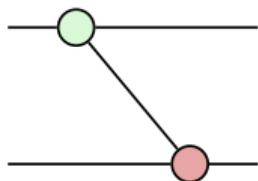


Color

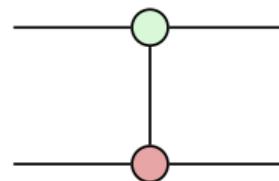
$$\begin{array}{c} \text{Diagram 1: } \\ \text{Left: A green circle labeled } \alpha \text{ with four yellow square inputs and two curved black outputs connecting to vertical dots.} \\ \text{Right: An equals sign followed by a red circle labeled } \alpha \text{ with two curved black outputs connecting to vertical dots.} \end{array}$$

$$\begin{array}{c} \text{Diagram 2: } \\ \text{Left: A red circle labeled } \alpha \text{ with four yellow square inputs and two curved black outputs connecting to vertical dots.} \\ \text{Right: An equals sign followed by a green circle labeled } \alpha \text{ with two curved black outputs connecting to vertical dots.} \end{array}$$

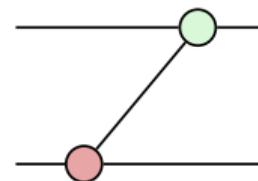
Only Connectivity Matters



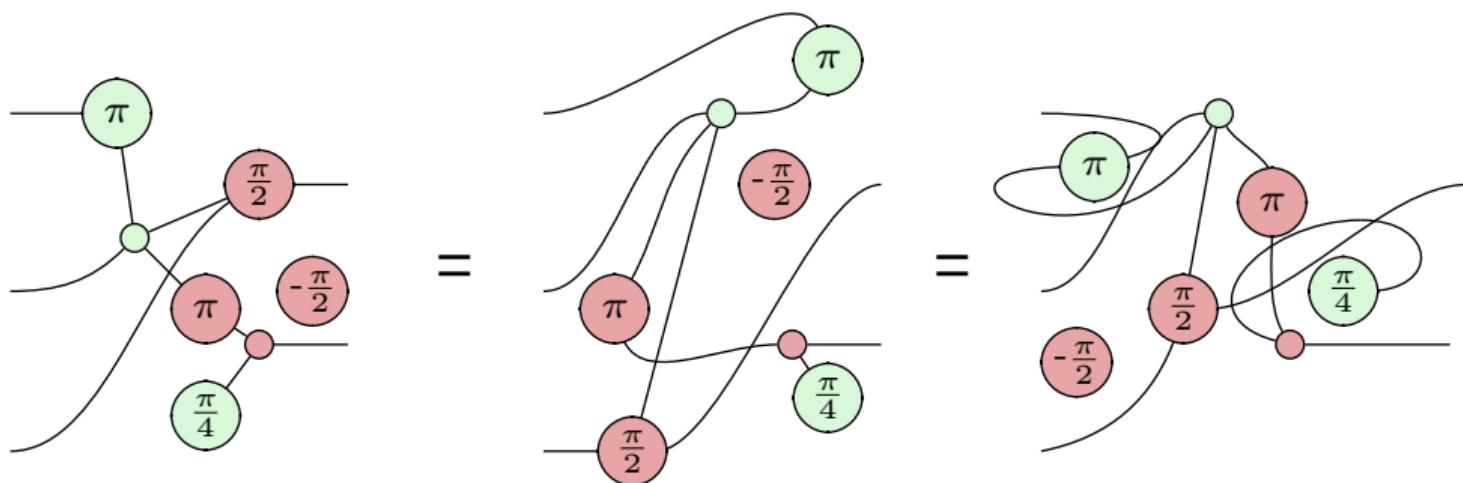
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Only Connectivity Matters



Axioms

$$\begin{array}{ccc} \text{Diagram showing two green circles labeled } \alpha \text{ and } \beta \text{ with multiple outgoing lines, connected by a horizontal line with an equals sign.} & & \\ (\text{FUSION}) & = & \text{Diagram showing a single green circle labeled } \alpha + \beta \text{ with multiple outgoing lines.} \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing a green dot on a line, followed by an equals sign.} & & \text{Diagram showing a red dot on a line, followed by an equals sign.} \\ (\text{Z-ELIM}) & = & (\text{X-ELIM}) \\ \text{Diagram showing a yellow square on a line, followed by an equals sign.} & & \text{Diagram showing three circles: green } \frac{\pi}{2}, red, green } \frac{\pi}{2} \text{ connected by lines.} \\ (\text{EULER}) & = & \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing a green circle labeled } \alpha \text{ with four yellow squares attached to its top and bottom, connected by a horizontal line with an equals sign.} & & \\ (\text{COLOUR}) & = & \text{Diagram showing a red circle labeled } \alpha \text{ with multiple outgoing lines.} \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing three red circles labeled } \alpha_1, \alpha_2, \alpha_3 \text{ connected by lines, followed by an equals sign.} & & \text{Diagram showing three circles: red } \alpha, green \pi, green \gamma \text{ connected by lines.} \\ (\ast) & = & \text{Diagram showing three green circles labeled } \beta_1, \beta_2, \beta_3 \text{ connected by lines.} \end{array}$$

$$\begin{array}{ccc} \text{Diagram showing two red dots and one green dot connected by lines, followed by an equals sign.} & & \text{Diagram showing a red dot on a line.} \\ (\text{COPY}) & = & \end{array}$$

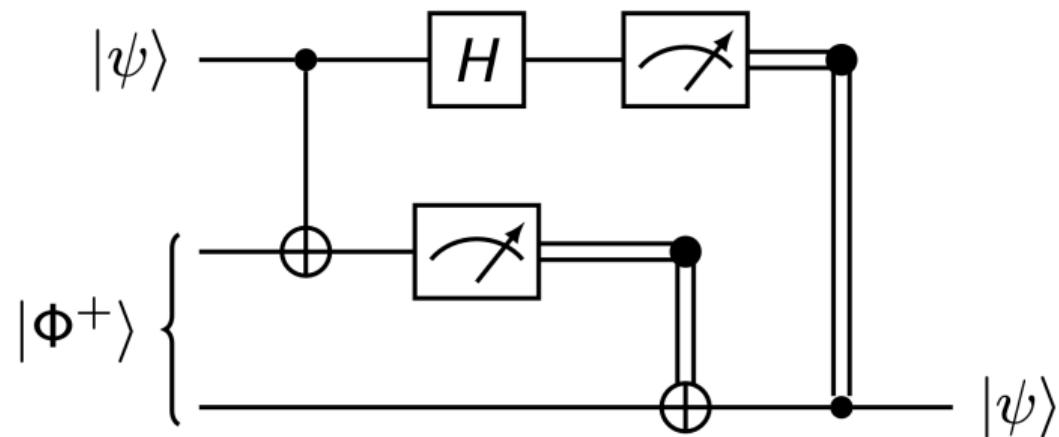
$$\begin{array}{ccc} \text{Diagram showing two red dots and one green dot connected by lines, followed by an equals sign.} & & \text{Diagram showing two circles: green } \frac{\pi}{4}, red } -\frac{\pi}{4} \text{ connected by lines, followed by an equals sign.} \\ (\text{BIGEBRA}) & = & \text{Diagram showing a red dot connected to a green dot, with a dotted square next to it.} \end{array}$$

Theorem (Completeness)

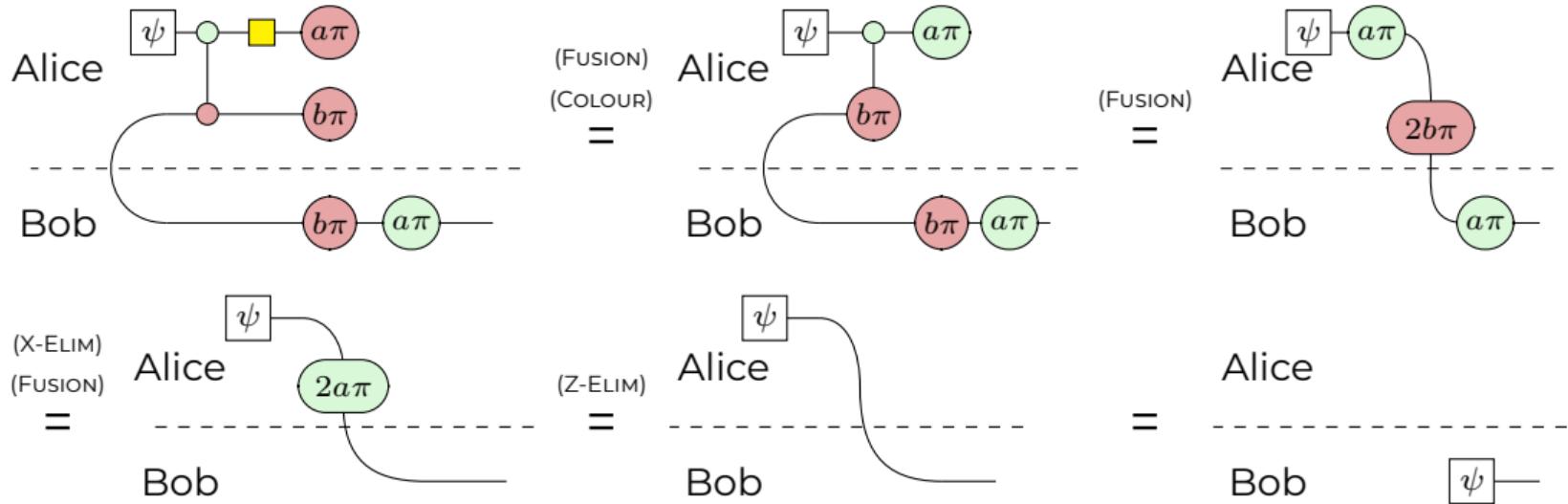
Any equation that holds for linear maps between qubits can be derived in ZX-calculus.

Example

Quantum Teleportation



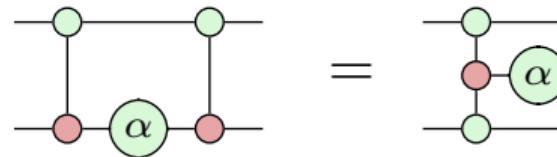
Quantum Teleportation



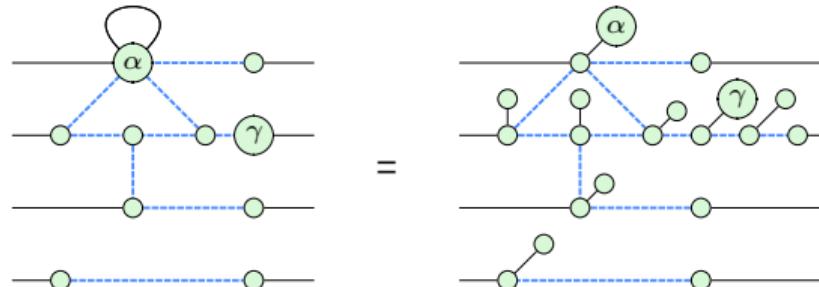
Extensions

Applications: ZX-calculus

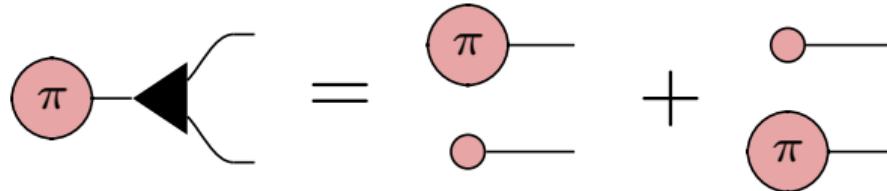
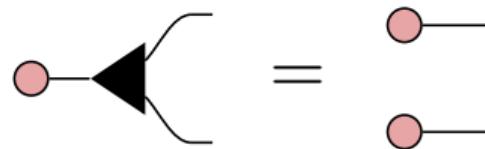
Quantum Circuit Optimisation



Measurement-Based Quantum Computing



W node



ZW-calculus

Summation

$$\begin{array}{c} \text{---} \nearrow \nwarrow \\ \square \boxed{a} \\ \square \boxed{b} \\ \text{---} \searrow \swarrow \end{array} = \text{---} \boxed{a + b} \text{---}$$

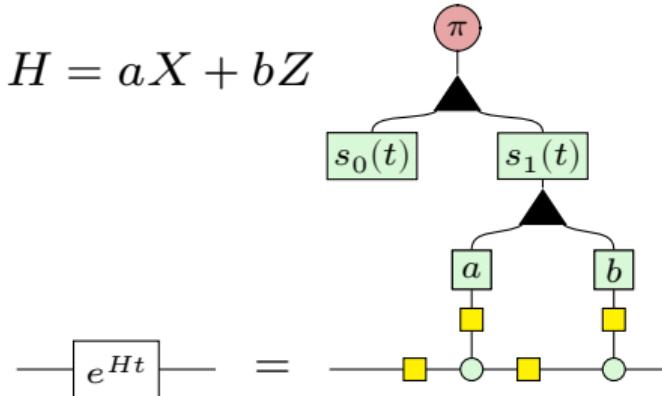
Linear Optical Quantum Computing

$$\begin{array}{c} \text{---} \nearrow \nwarrow \\ \square \quad \square \\ \square \quad \square \\ \text{---} \searrow \swarrow \end{array} = \begin{array}{c} \text{---} \nearrow \nwarrow \\ \square \quad \square \\ \square \quad \square \\ \text{---} \searrow \swarrow \end{array}$$

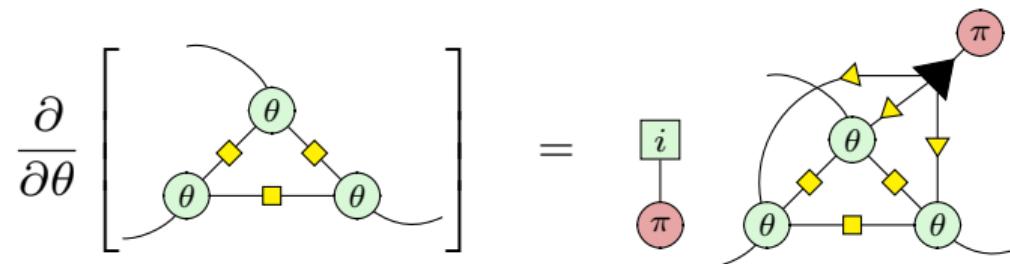
ZXW-calculus

Hamiltonians

$$H = aX + bZ$$



Differentiation and integration



Completeness of qufinite ZXW calculus, a graphical language for finite-dimensional quantum theory

Finite-dimensional Hilbert spaces

Definition

FHilb is the category of finite-dimensional Hilbert spaces.

Definition

FHilb_d is the subcategory of **FHilb**, where Hilbert spaces have dimensions of d^n .

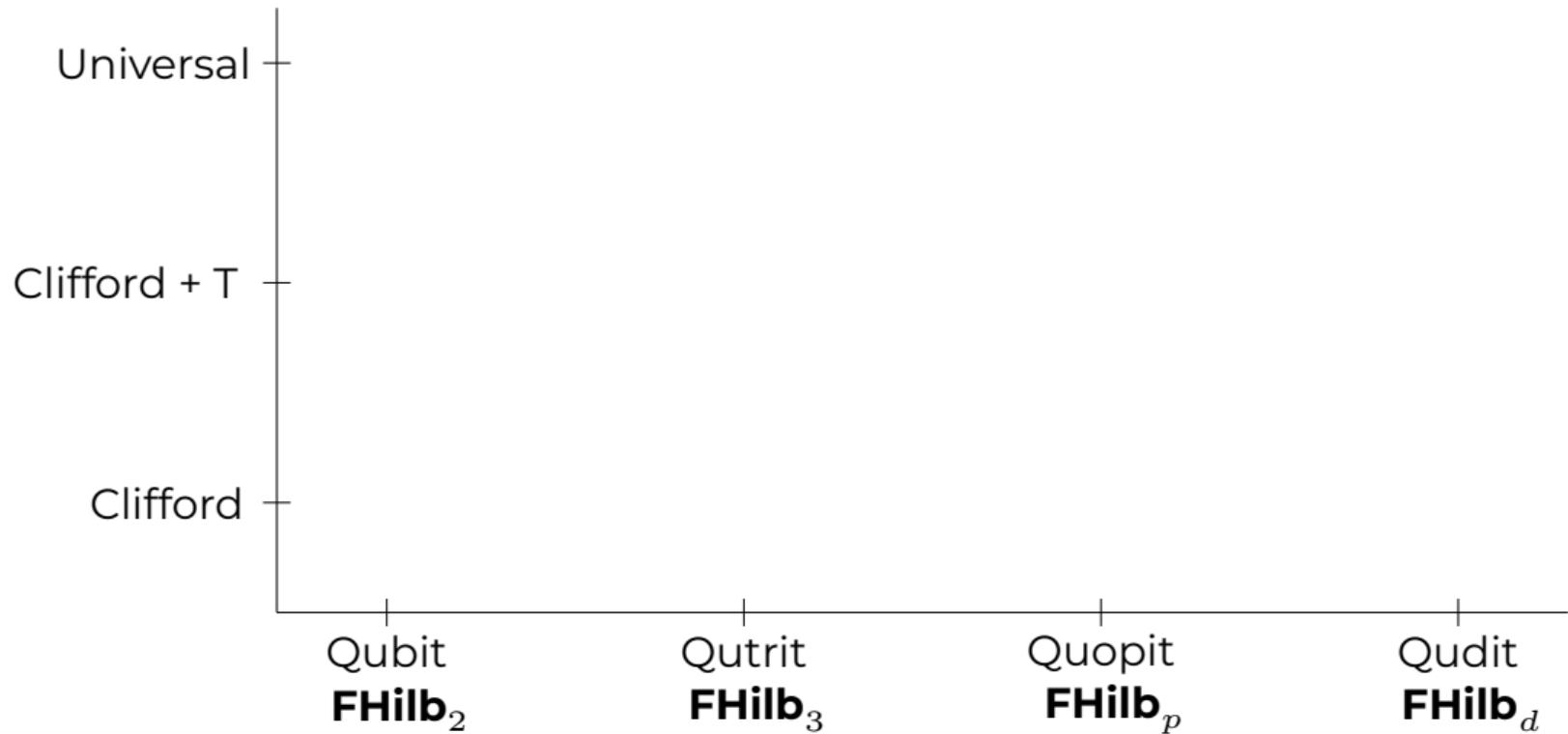
Qubits:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

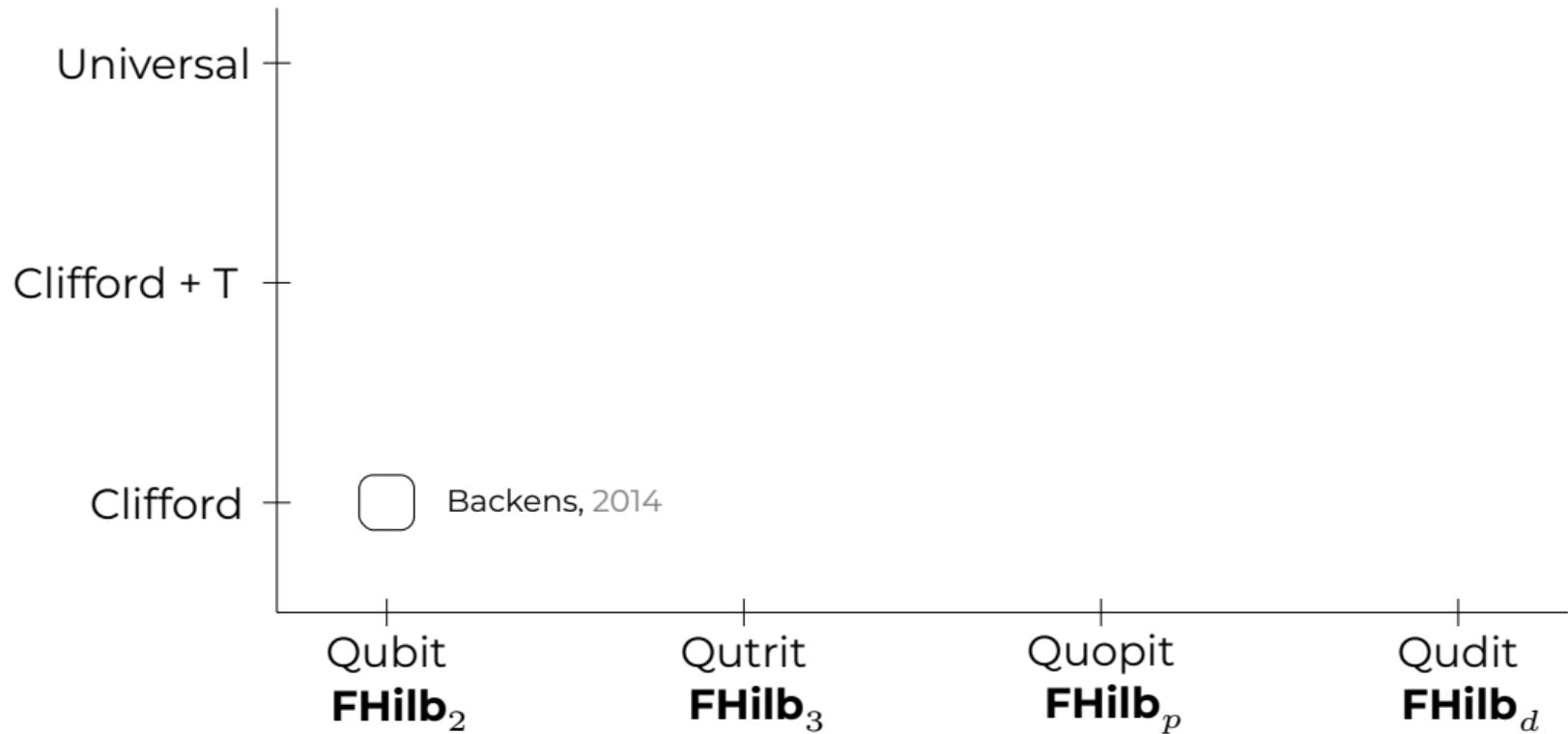
Qudits:

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \dots + a_{d-1}|d-1\rangle$$

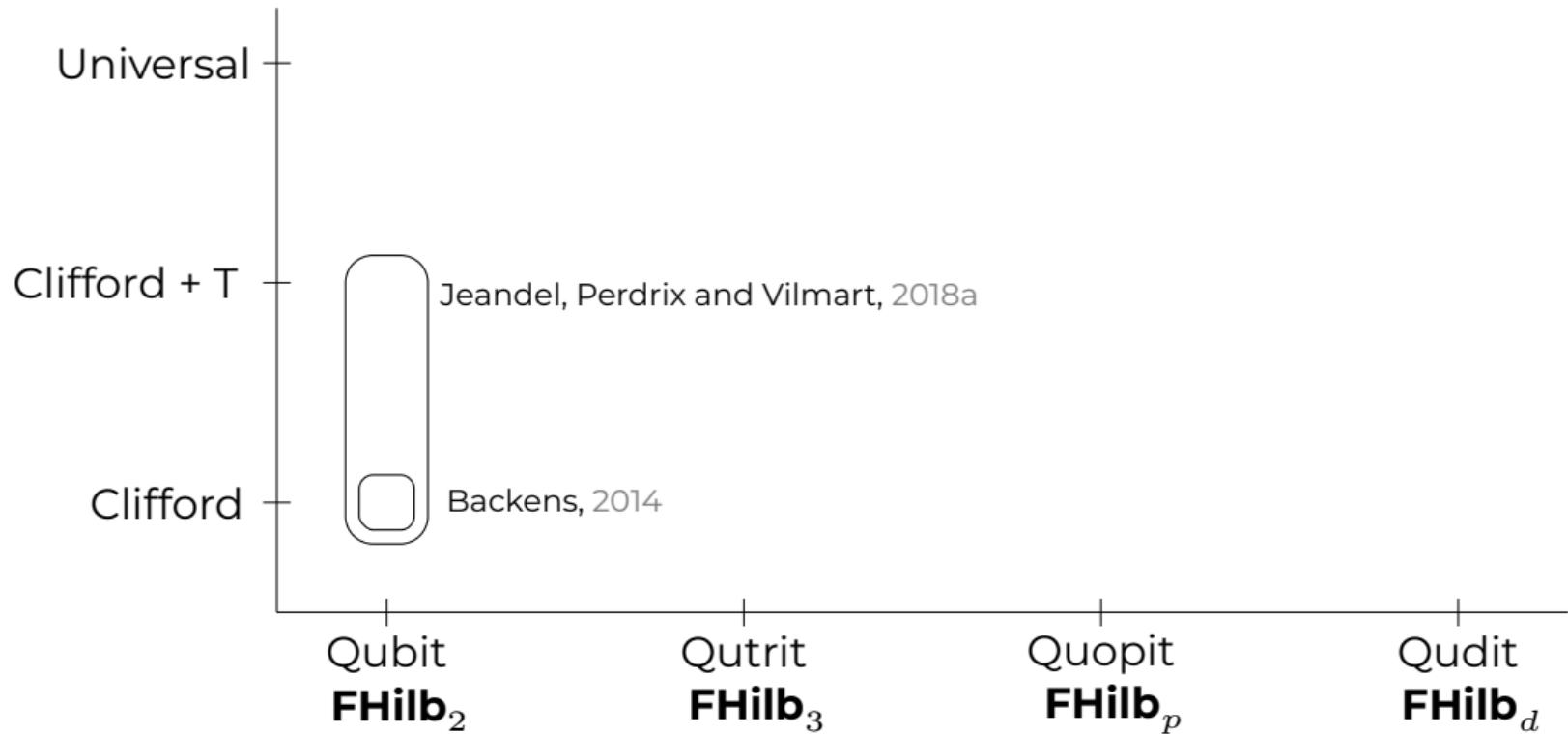
History of Completeness



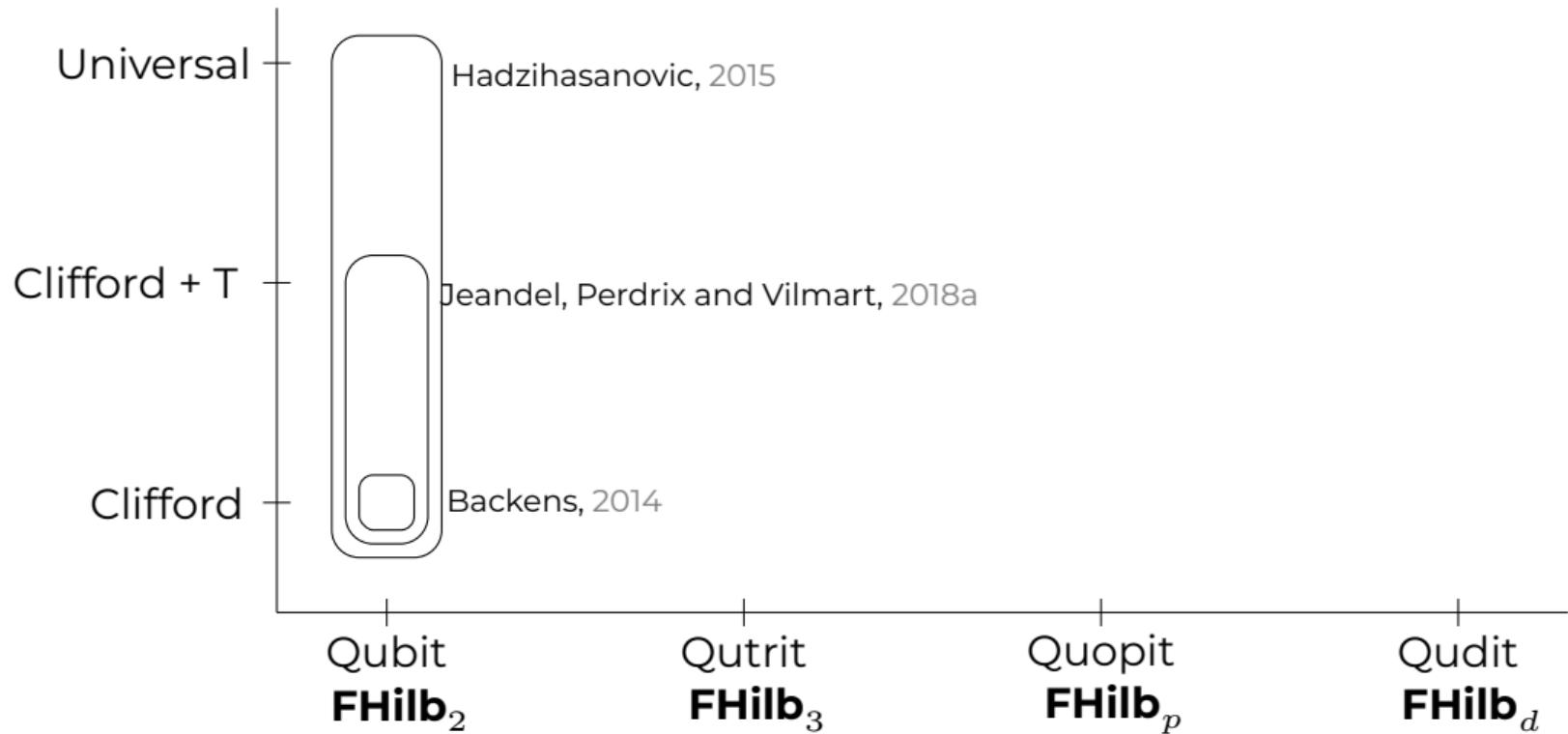
History of Completeness



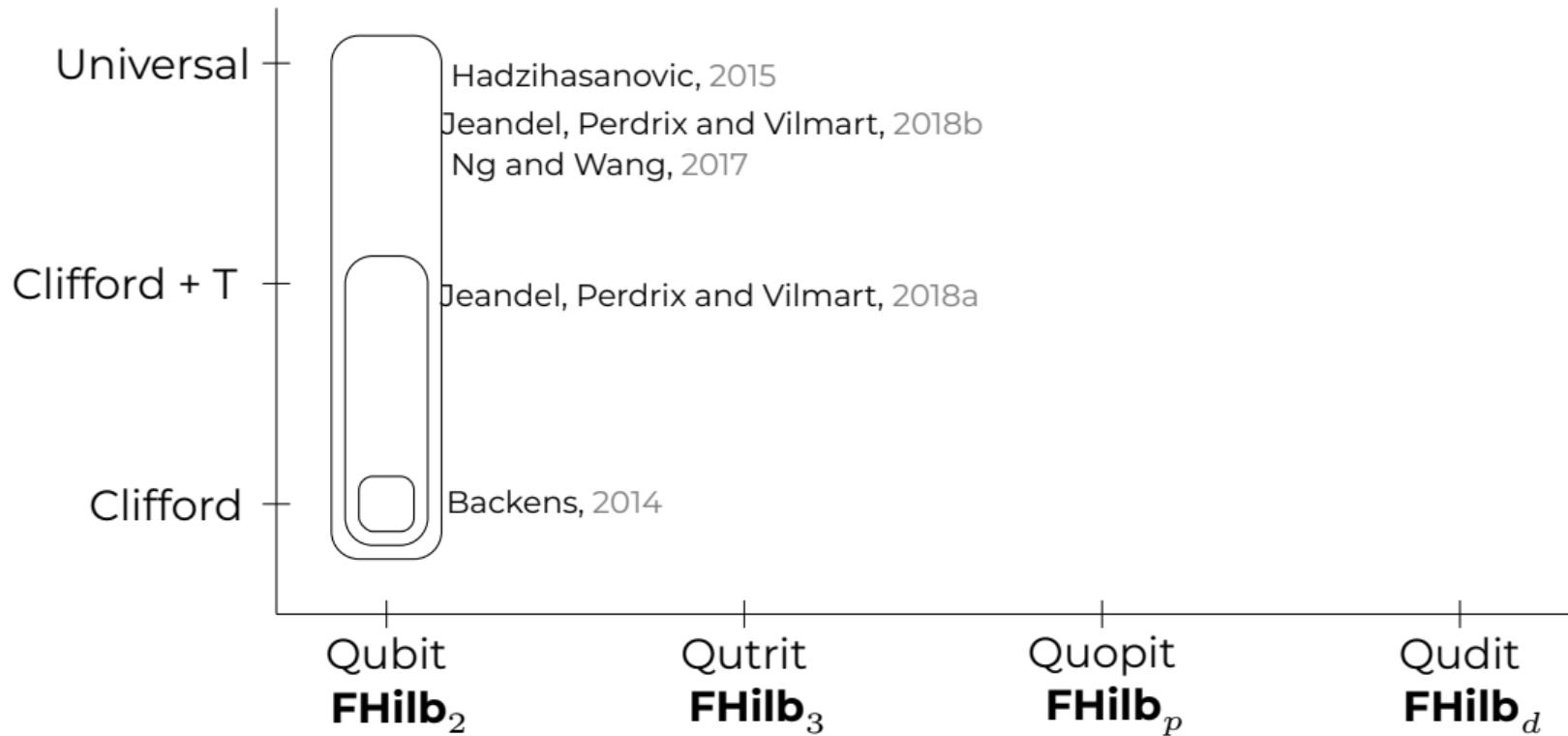
History of Completeness



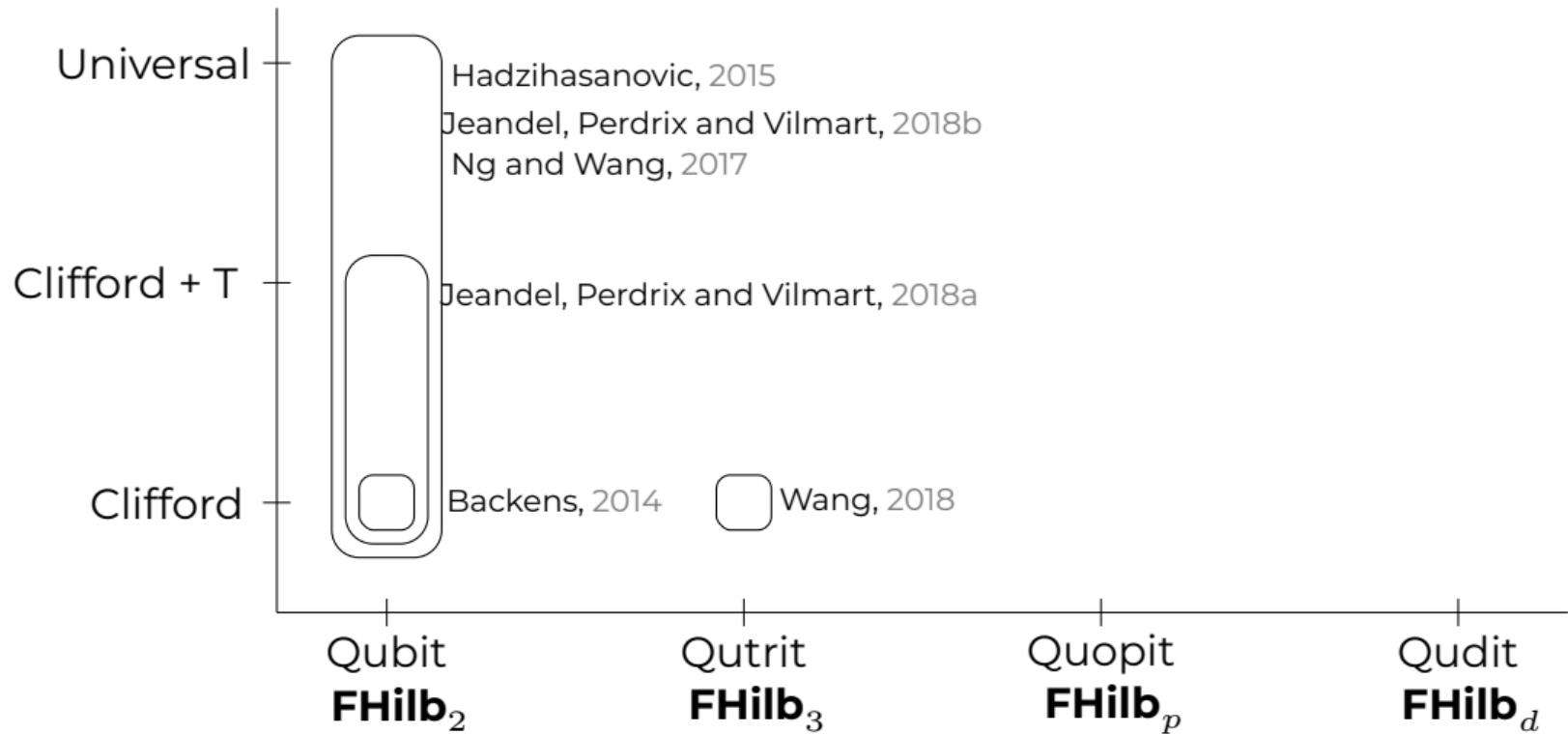
History of Completeness



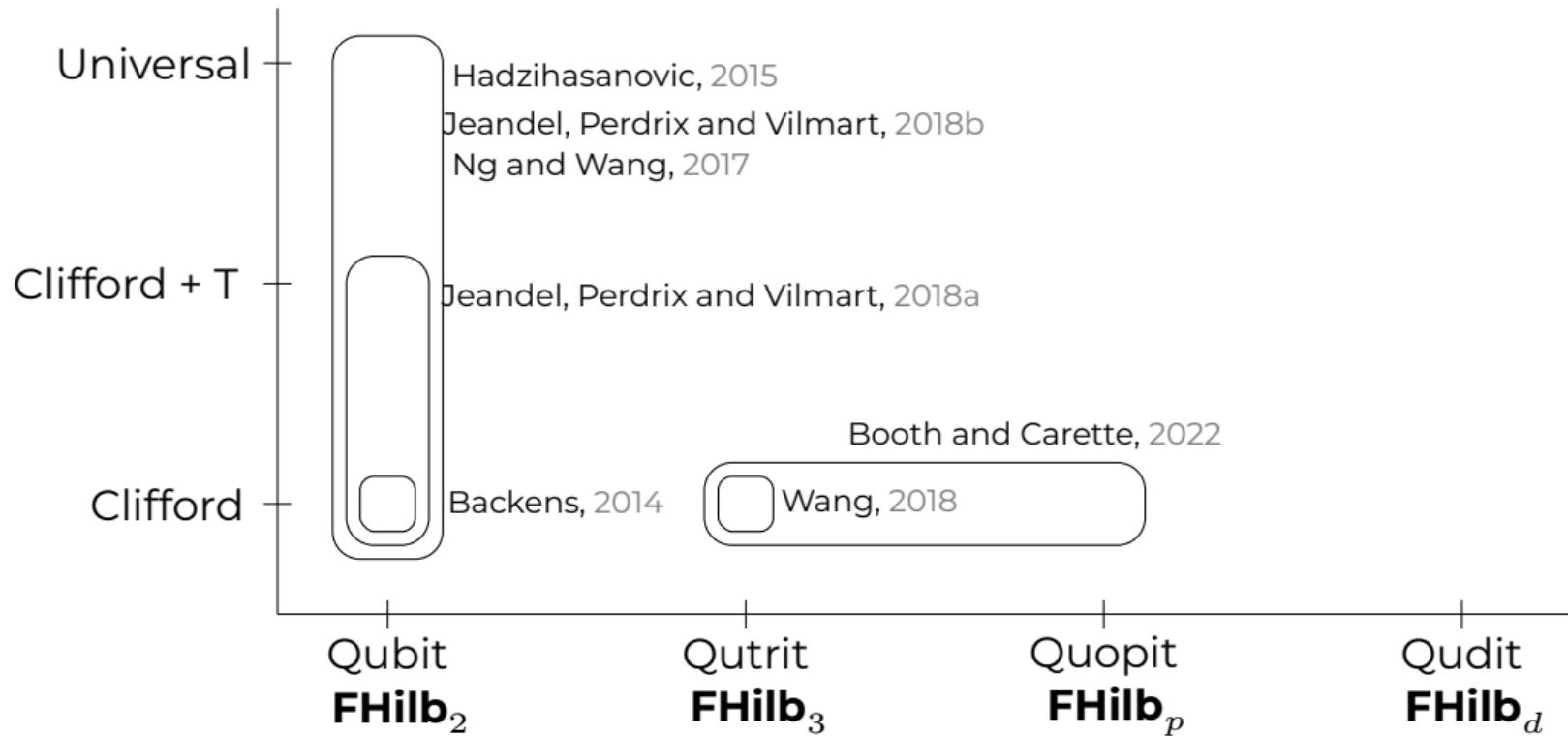
History of Completeness



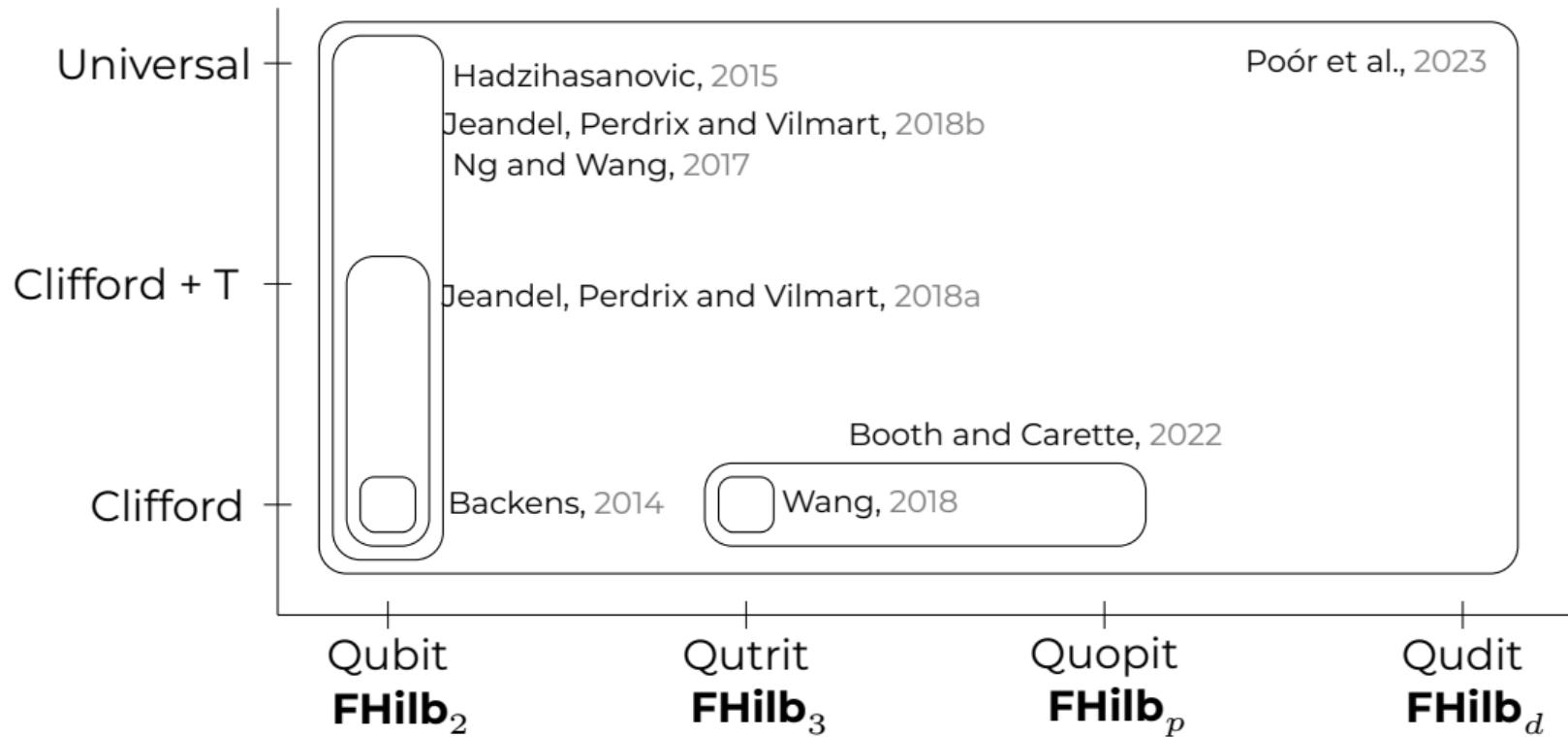
History of Completeness



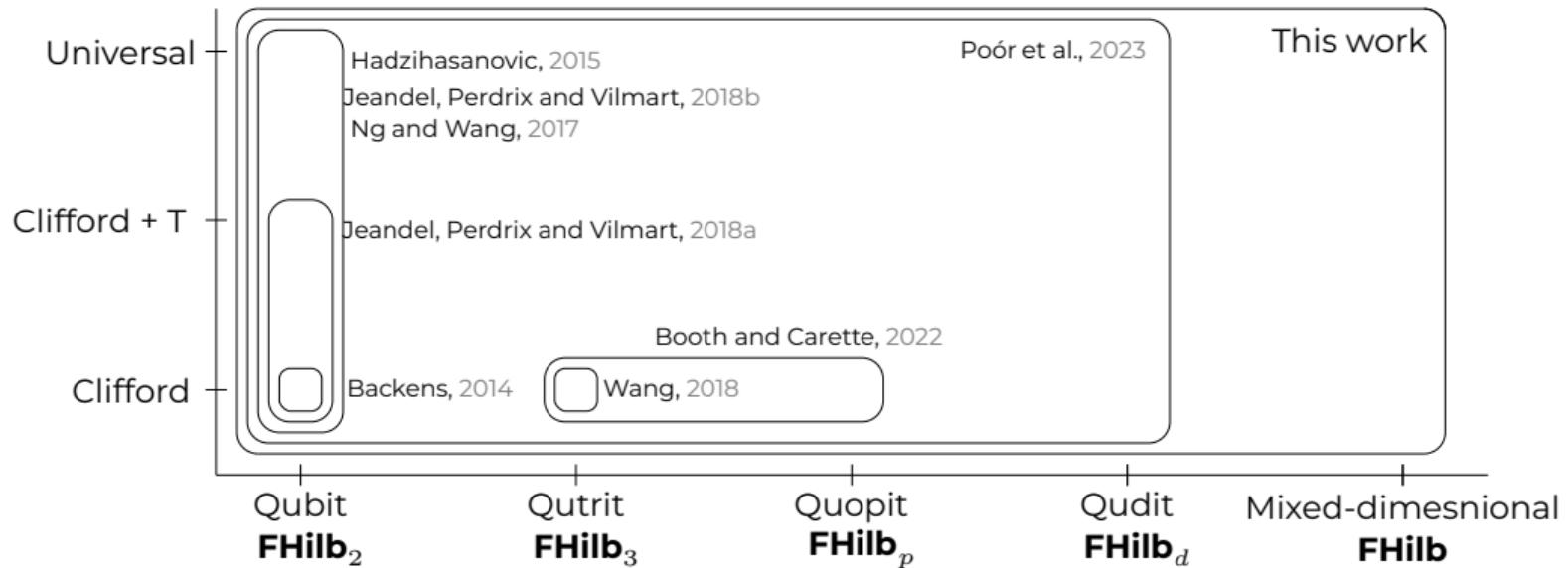
History of Completeness



History of Completeness



History of Completeness



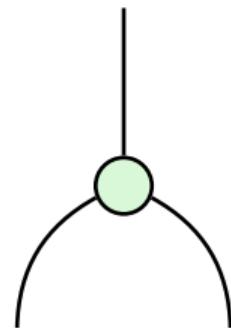
The qufinite ZXW-calculus

Standard bases

For $0 \leq j < d$,

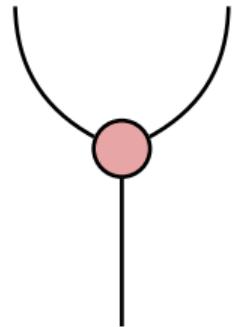
$$\begin{array}{ccc} K_j & \longmapsto & |d-j\rangle \end{array}$$

Z spider

 $\llbracket \cdot \rrbracket \rightarrow$

$|k\rangle \mapsto |k, k\rangle$

X spider

 $\llbracket \cdot \rrbracket$ $|i, j\rangle \mapsto |i + j \bmod d\rangle$

Notation: The multiplier

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad := \quad \begin{array}{c} | \\ \text{---} \\ \text{---} \\ m \\ \dots \\ \text{---} \\ | \end{array} \quad \xrightarrow{[\cdot]} \quad |k\rangle \mapsto |m \cdot k \bmod d\rangle,$$

Generator: W node

 $\llbracket \cdot \rrbracket$

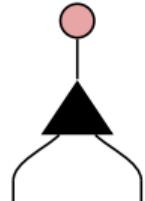
$$\llbracket \cdot \rrbracket \mapsto |00\rangle\langle 0| + \sum_{i=1}^{d-1} (|i0\rangle + |0i\rangle)\langle i|$$

Generator: W node

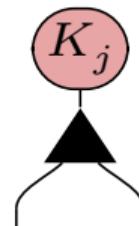
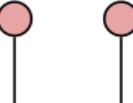


$\llbracket \cdot \rrbracket$

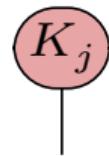
$$|00\rangle\langle 0| + \sum_{i=1}^{d-1} (|i0\rangle\langle i| + |0i\rangle\langle i|)$$



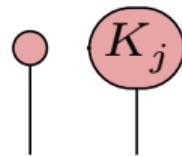
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+



Understanding the Z box

Z spider:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \alpha \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}, \quad \text{where } \alpha \in \mathbb{R}.$$

Z box:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} a \quad \xrightarrow{\llbracket \cdot \rrbracket} \quad \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \quad \text{where } a \in \mathbb{C}.$$

Understanding the qudit Z box

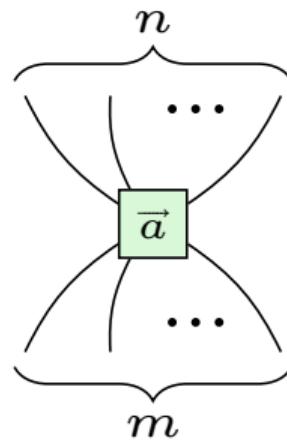
Qubit Z box: for $a \in \mathbb{C}$,

$$\begin{array}{c} | \\ \square \\ | \end{array} \xrightarrow{\mathbb{E}[\cdot]} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

Qudit Z box: for $\vec{a} = (a_1, a_2, \dots, a_{d-1}) \in \mathbb{C}^{d-1}$,

$$\begin{array}{c} | \\ \square \\ | \end{array} \xrightarrow{\mathbb{E}[\cdot]} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{d-1} \end{bmatrix}$$

Generator: Z box

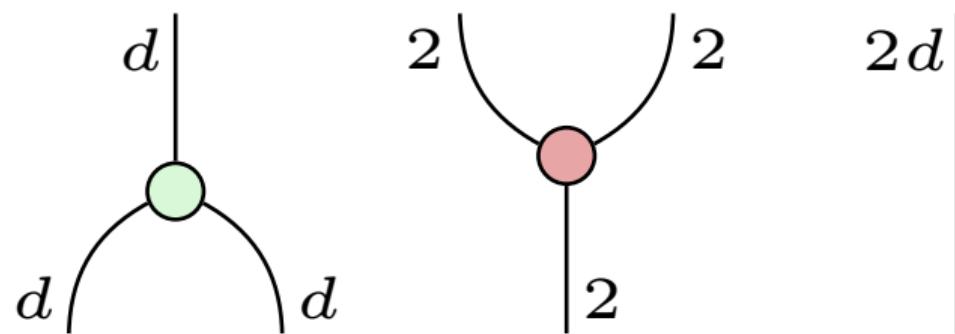

$$\mapsto \sum_{j=0}^{d-1} a_j |j\rangle^{\otimes m} \langle j|^{\otimes n},$$

where $\vec{a} = (a_1, \dots, a_{d-1}) \in \mathbb{C}^{d-1}$

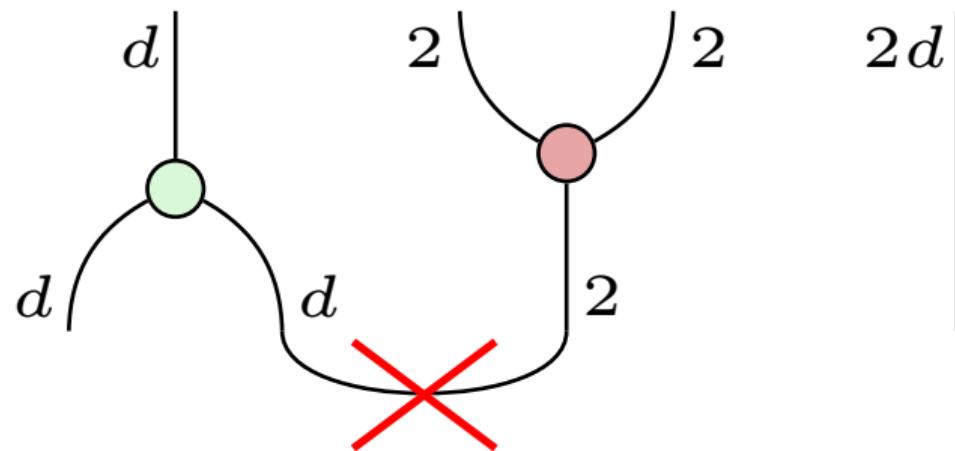
and $a_0 := 1$

Mixed-dimensional generators

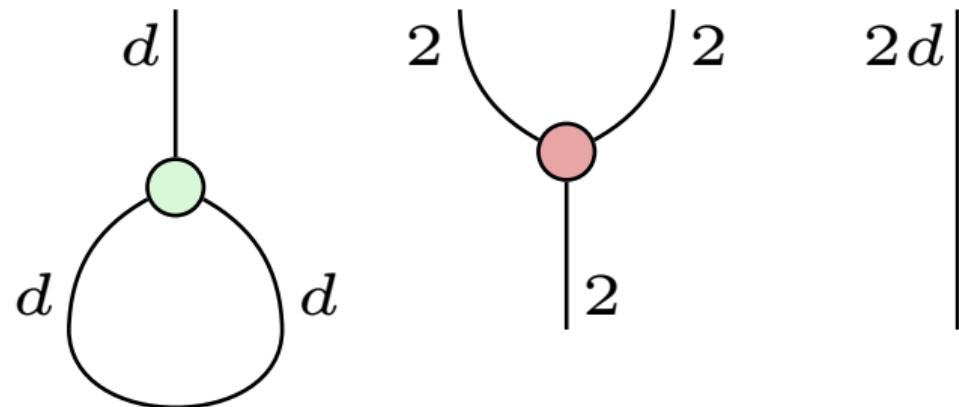
Mixed dimensions



Mixed dimensions



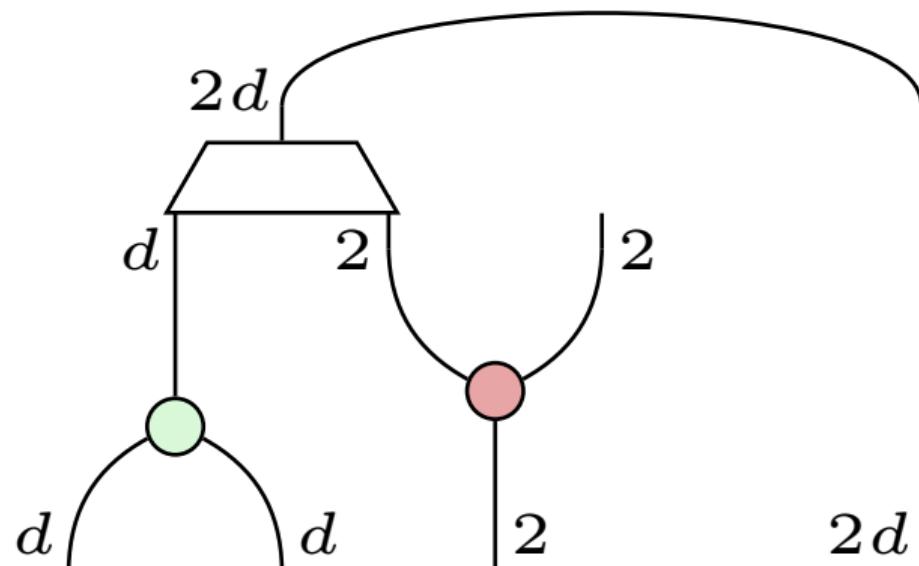
Mixed dimensions



Dimension splitter

$$\begin{array}{c} | \\ m \quad n \\ \hline \end{array} \quad \mapsto \quad \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |i, j\rangle \langle in + j|.$$

Interacting diemensions



Mixed-dimensional Z box

$$\llbracket \cdot \rrbracket \mapsto \sum_{j=0}^{\min\{d_i\}_i - 1} a_j |j, \dots, j\rangle \langle j, \dots, j|,$$

where $\vec{a} = (a_1, \dots, a_{\min\{d_i\}_i - 1}) \in \mathbb{C}^{d-1}$

and $a_0 := 1$

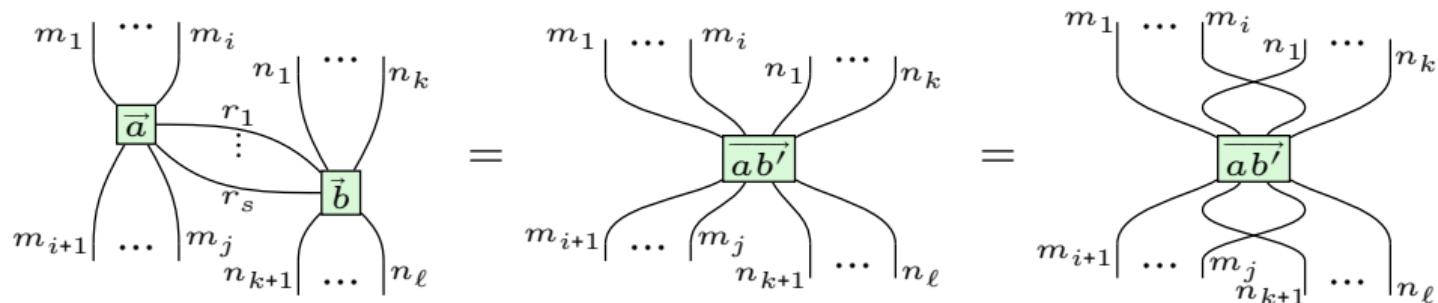
Mixed-dimensional copy

$$\begin{array}{c} K_{-j} \\ \text{---} \\ m \\ \text{---} \\ \text{---} \end{array} = \begin{cases} \begin{array}{cc} K_{-j} & K_{-j} \\ | & | \\ n_1 & n_2 \end{array}, & \text{if } j < n_1, n_2 \\ 0, & \text{otherwise} \end{cases}$$

Rule: Mixed-dimensional copy

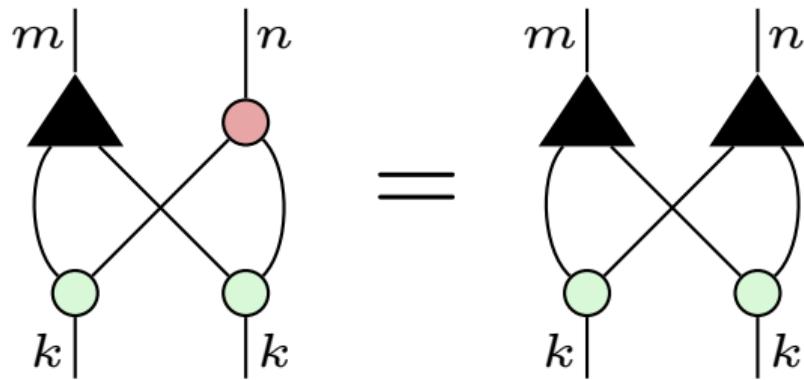
$$\begin{array}{c} K_{-j} \\ \text{---} \\ \text{---} \end{array} m \quad = \quad \begin{array}{ccc} K_{-j} & \boxed{1_N} & K_{-j} \\ \text{---} & \text{---} & \text{---} \\ n_1 & | & n_2 \end{array}$$

Rule: Mixed-dimensional fusion

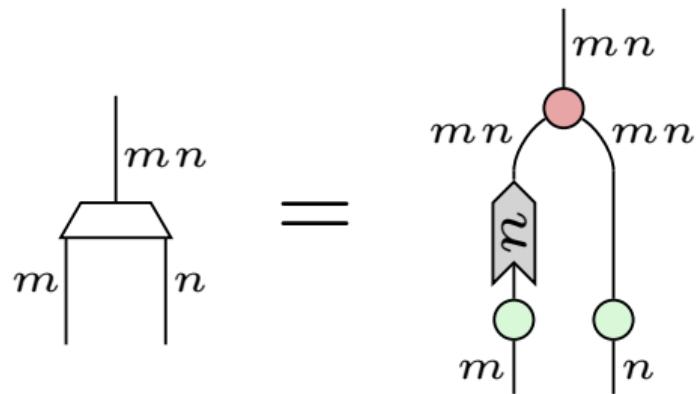


where $M = \min\left\{\min_{t=1}^j m_t, \min_{t=1}^\ell n_t, \min_{t=1}^s r_t\right\}$, $\overrightarrow{ab'} = (a_1 b_1, \dots, a_{M-1} b_{M-1})$.

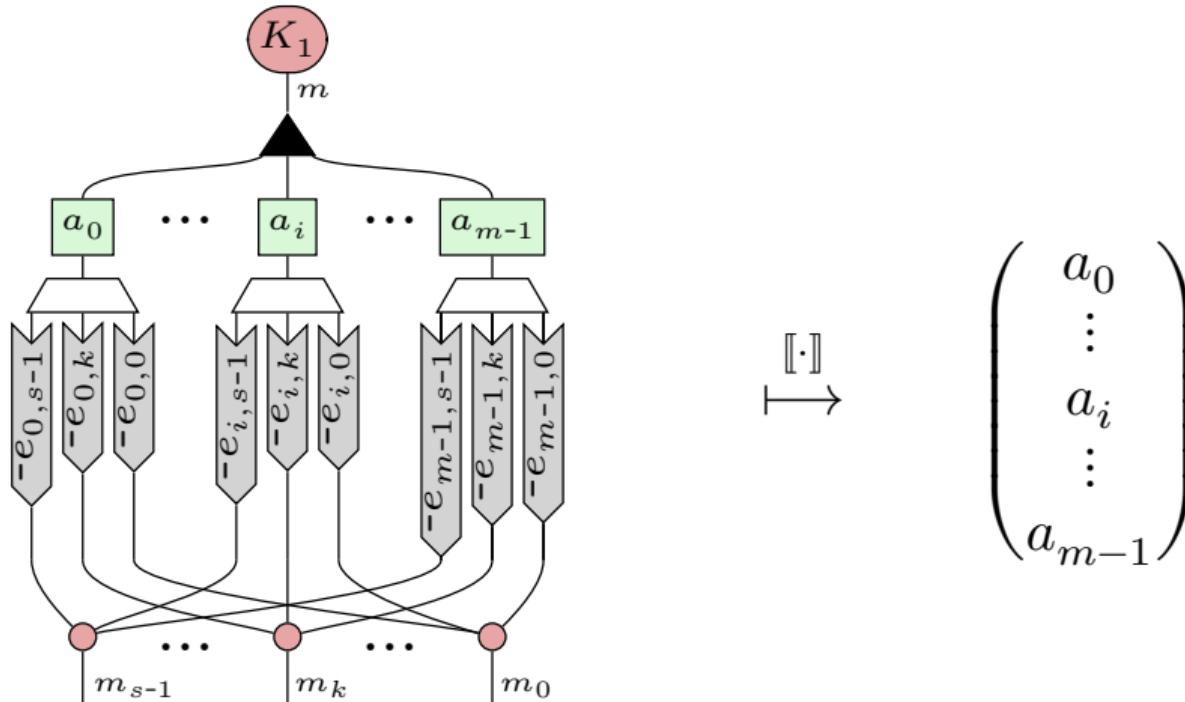
Rule: Trialgebra



Rule: Dimension Splitter



A Normal Form



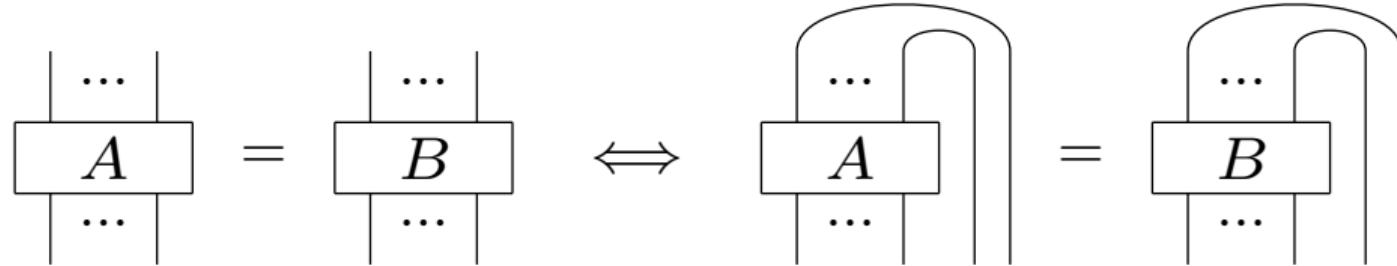
where $m = \prod m_i$

Proposition

The interpretation functor $\llbracket \cdot \rrbracket$ is full.

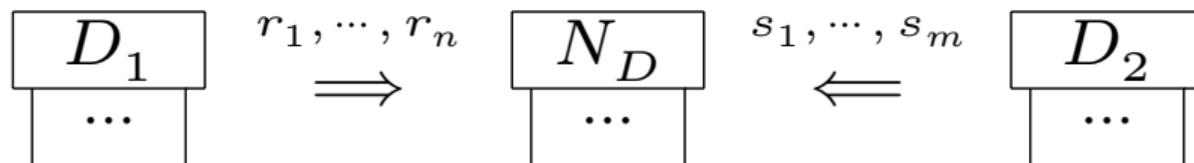
Completeness proof

Map-state duality

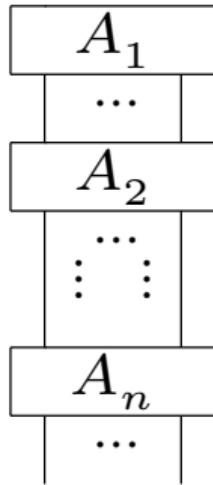


Completeness using a normal form

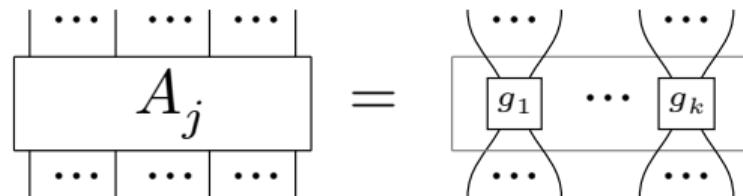
If $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$, then:



Structure of states



with

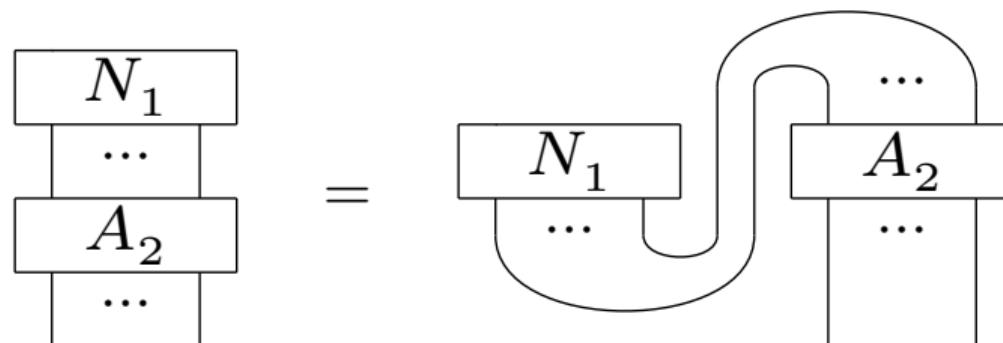


where g_1, \dots, g_k are generators.

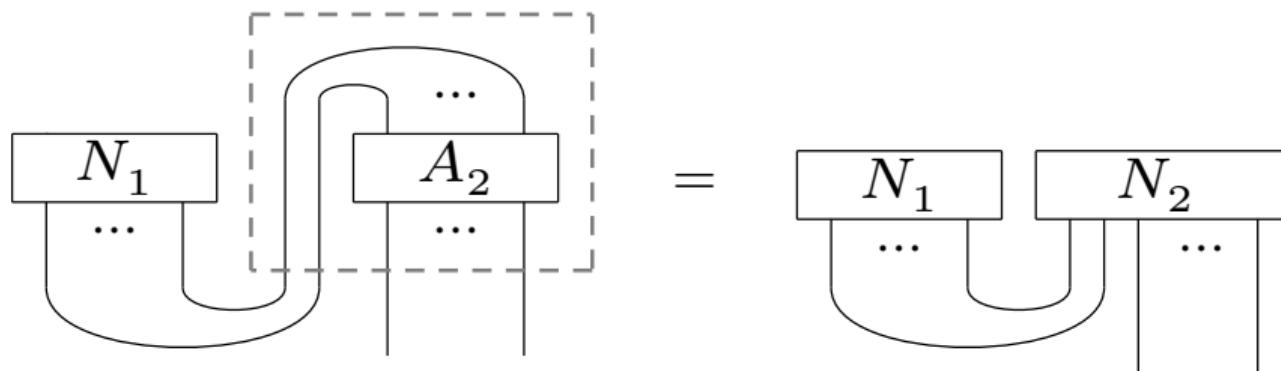
State \Rightarrow normal form I.

$$\begin{array}{c} A_1 \\ \vdots \end{array} = \begin{array}{c} N_1 \\ \vdots \end{array}$$

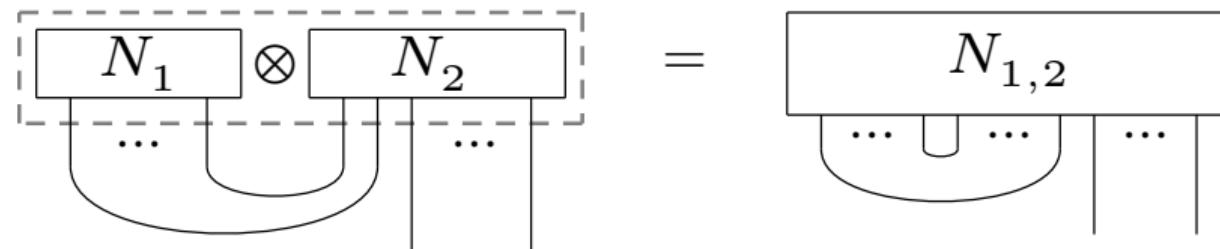
State \Rightarrow normal form II.



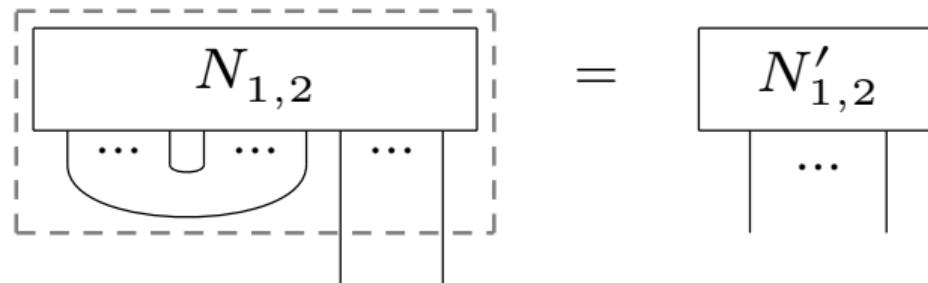
State \Rightarrow normal form III.



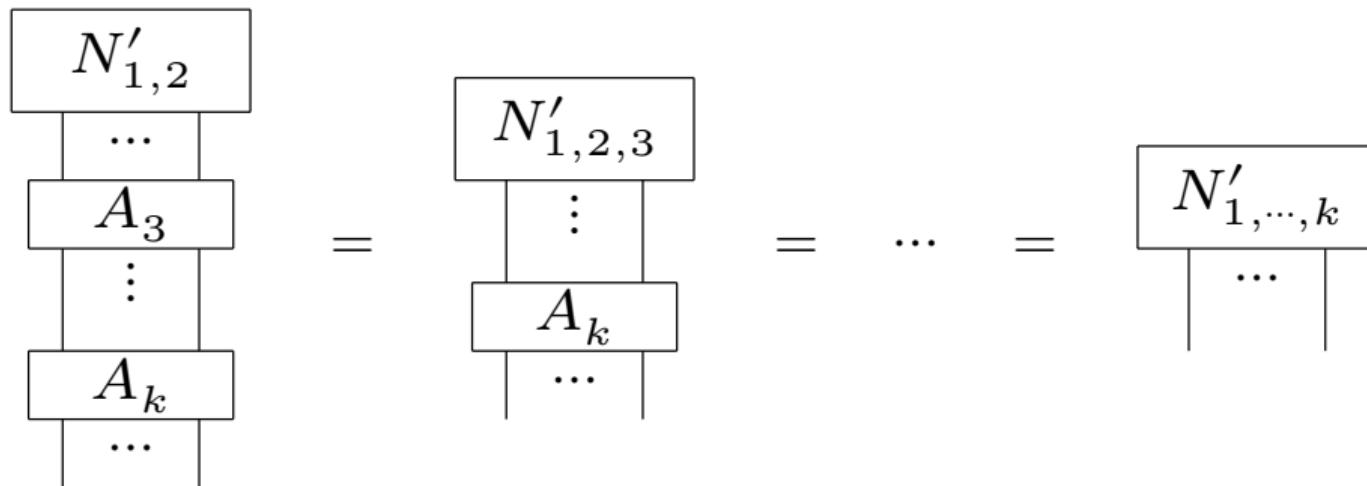
State \Rightarrow normal form IV.



State \Rightarrow normal form V.



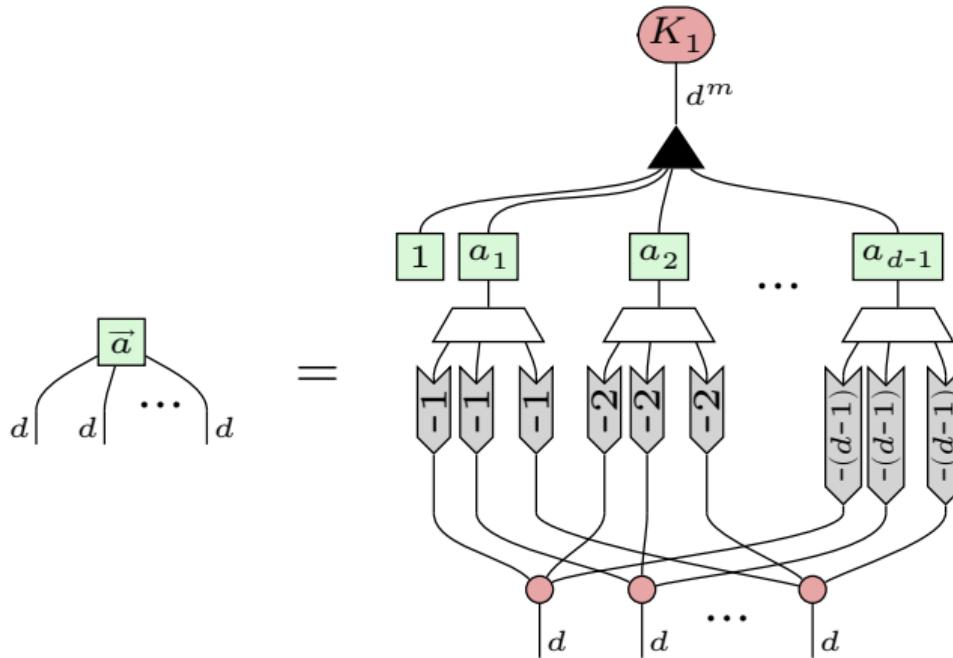
State \Rightarrow normal form VI.



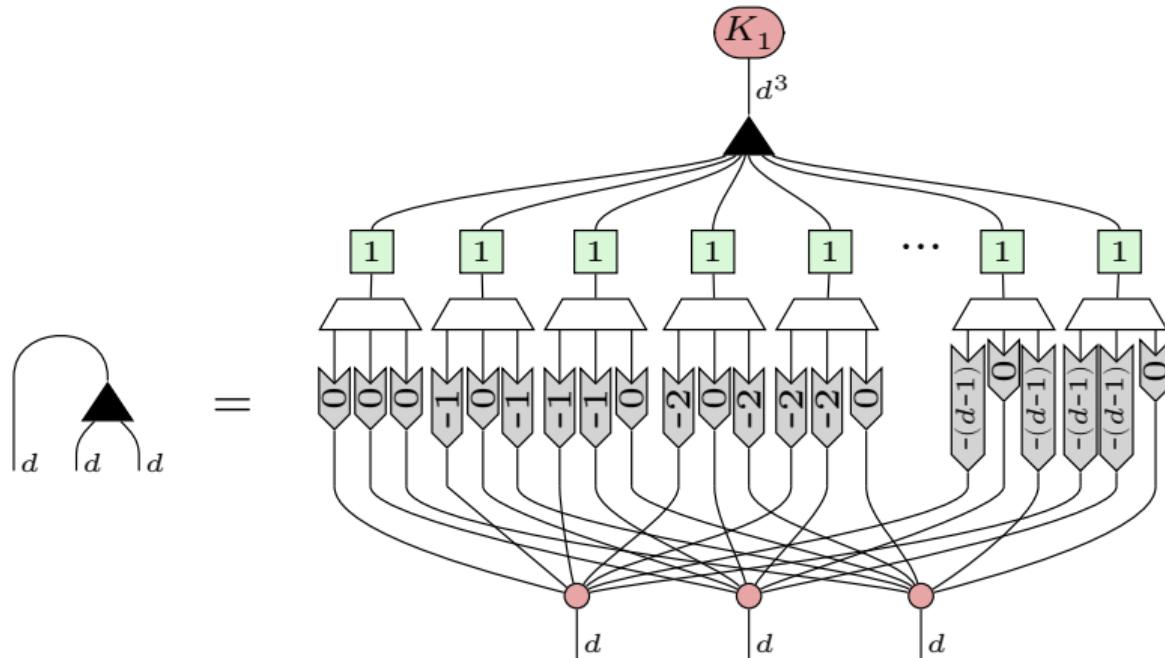
Summary: state \Rightarrow normal form

- Generators
- Tensor product of two normal forms
- Partial-traced normal form

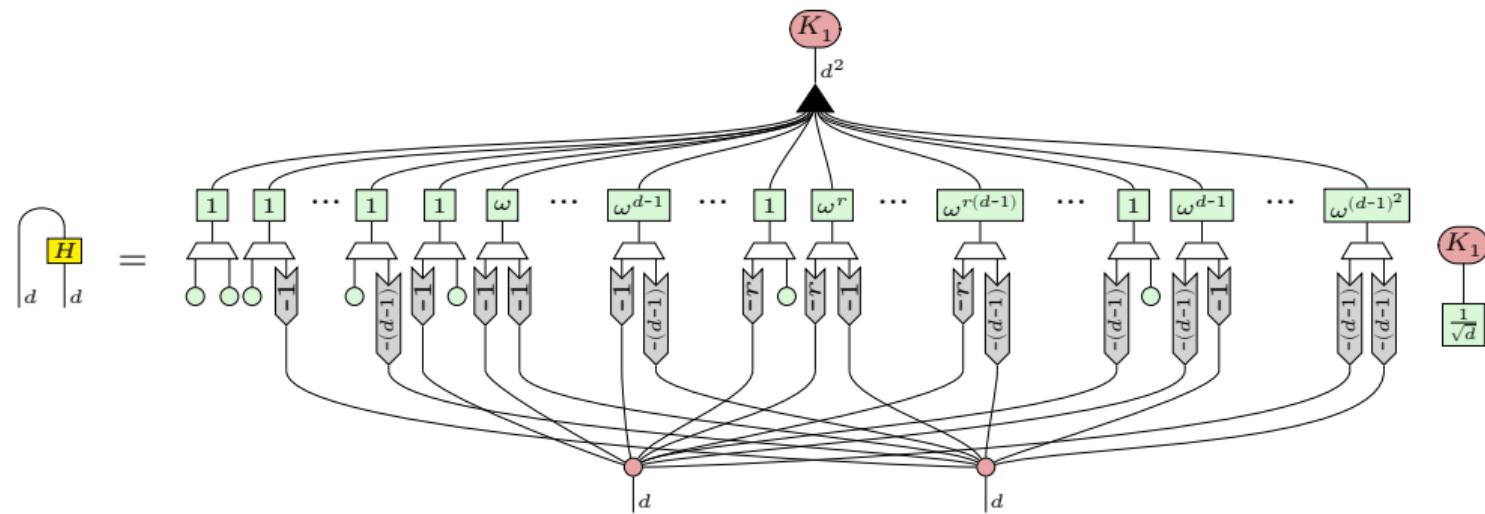
Lemma: Z box



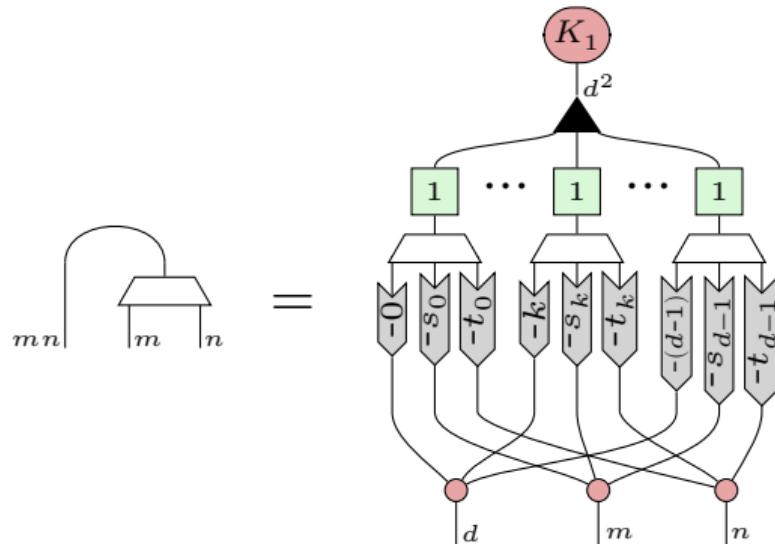
Lemma: W node



Lemma: Hadamard box

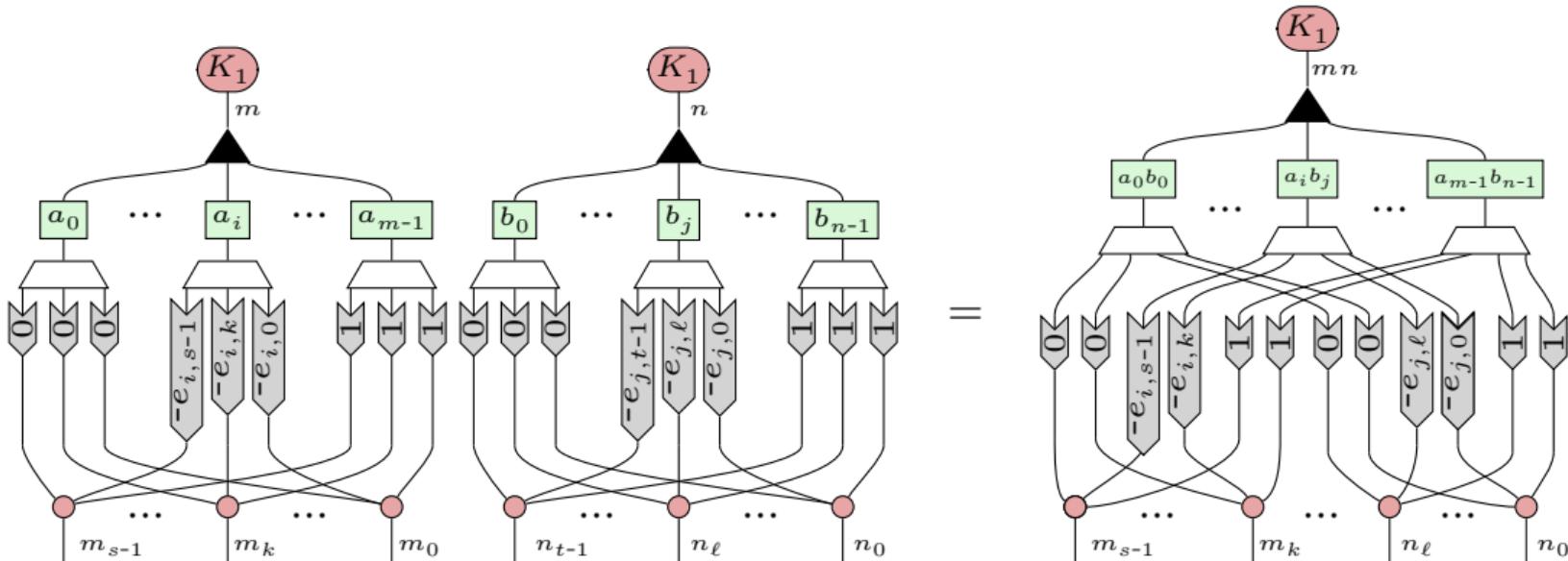


Lemma: Dimension splitter



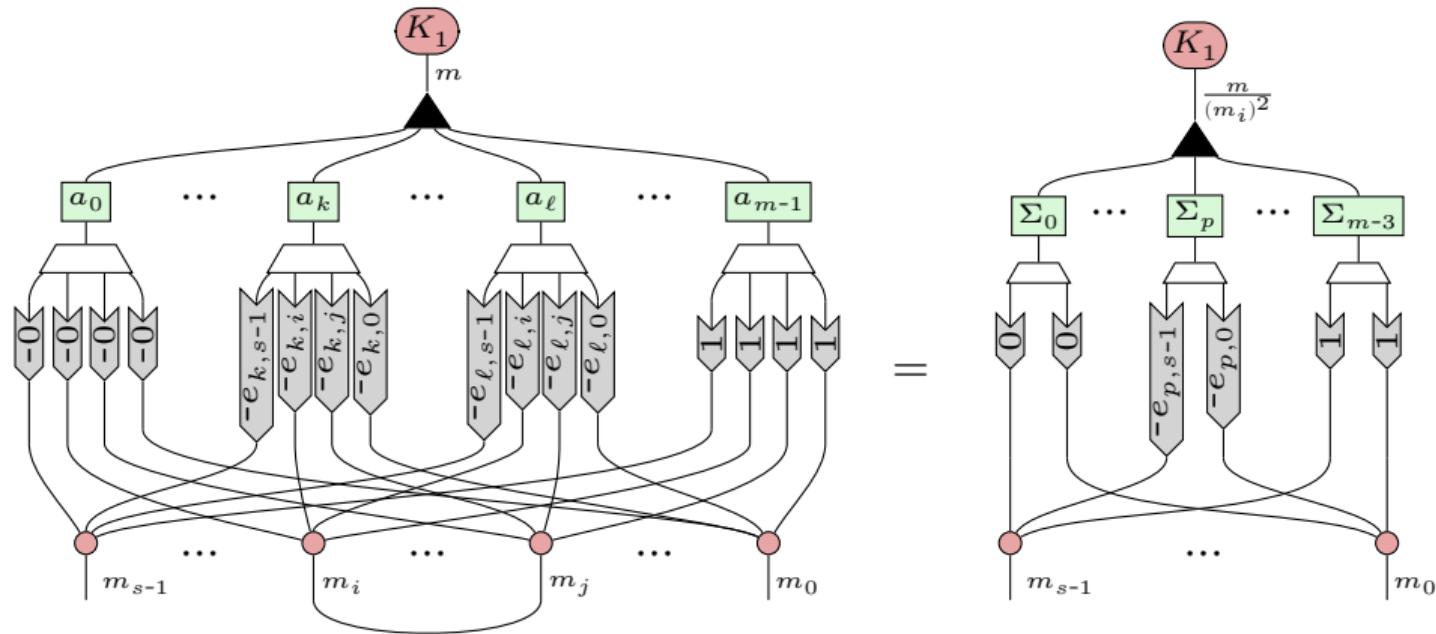
where $k = s_k n + t_k, 0 \leq k \leq mn - 1$.

Lemma: Tensor product



where $M = d^m - 1$, $N = d^n - 1$.

Lemma: Partial trace



where $m_i = m_j$ and Σ_k corresponds to the elements of the partial trace over s and t indices.

Theorem (Completeness)

The qufinite ZXW calculus is complete for finite-dimensional Hilbert spaces.

Corollary

*The category **ZXW** is monoidally equivalent to the category **FHilb**.*

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Two categories are monoidally equivalent if

- there is a monoidal functor between them and
- the functor is full, faithful, and
- essentially surjective on objects

(Heunen and Vicary, 2019)

Corollary

The category **ZXW** is monoidally equivalent to the category **FHilb**.

Proof.

- The interpretation functor $\llbracket \cdot \rrbracket$ is a monoidal functor.

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- For any object $H \in \mathbf{FHilb}$, we have an object $(\dim(H), \circ) \in \mathbf{ZXW}$ such that $H \cong \mathbb{C}^{\dim(H)} = \llbracket (\dim(H), \circ) \rrbracket$; hence, $\llbracket \cdot \rrbracket$ is essentially surjective on objects.

Corollary

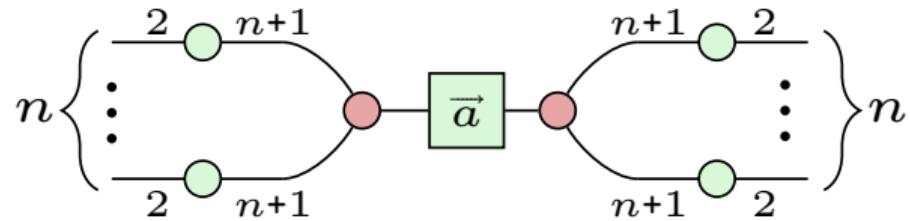
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Applications

Symmetrizer of spin- $\frac{n}{2}$



where $\vec{a} = \left(\frac{1}{\binom{n}{1}}, \dots, \frac{1}{\binom{n}{k}}, \dots, \frac{1}{\binom{n}{n}} \right)$

Penrose Spin Calculus: ZX for $SU(2)$

Quanlong Wang¹

Richard D. P. East

Razin A. Shaikh^{1,2}

Lia Yeh^{1,2}

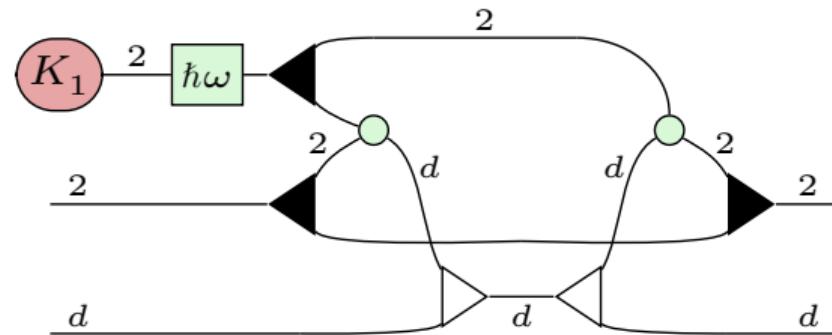
Boldizsár Poór¹

¹Quantinuum, 17 Beaumont Street, Oxford, OX1 2NA, United Kingdom

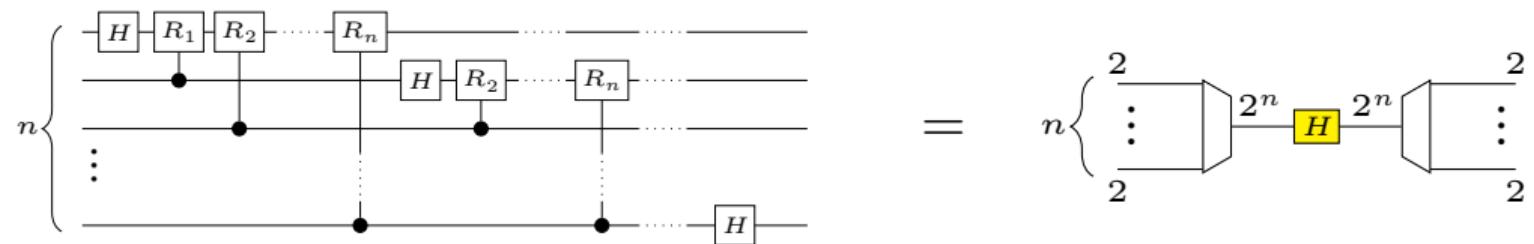
²University of Oxford, United Kingdom

We introduce the Penrose spin calculus as an elevation of Penrose's diagrams and associated Binor calculus to the level of a formal diagrammatic language. By leveraging the mixed-dimensional ZX calculus, a complete language for finite dimensional Hilbert spaces, we formulate a diagrammatic language for $SU(2)$ representation theory in quantum informational terms. Using this language we firstly articulate the classic angular moment relations

Jaynes-Cummings



Quantum programming language: QFT



ZX-calculus is Complete for Finite-Dimensional Hilbert Spaces

Boldizsár Poór¹

Razin A. Shaikh^{1,2}

Quanlong Wang¹

¹Quantinuum, 17 Beaumont Street, Oxford, OX1 2NA, United Kingdom

²University of Oxford, United Kingdom

The ZX-calculus is a graphical language for reasoning about quantum computing and quantum information theory. As a complete graphical language, it incorporates a set of axioms rich enough to derive any equation of the underlying formalism. While completeness of the ZX-calculus has been established for qubits and the Clifford fragment of prime-dimensional qudits, universal completeness beyond two-level systems has remained unproven until now. In this paper, we present a proof establishing the completeness of finite-dimensional ZX-calculus, incorporating only the mixed-dimensional Z-spider and the qudit X-spider as generators. Our approach builds on the completeness of another graphical language, the finite-

Thank you!

Overview

1 Introduction

Qudits

Completeness

2 The qufinite ZXW-calculus

Qudit generators

Mixed-dimensional generators

Example equalities

3 Completeness proof

Proof idea

Lemmas

4 Applications

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