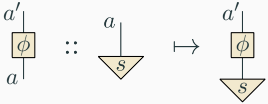
A Profunctorial Semantics for Quantum Supermaps

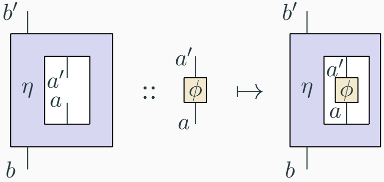
James Hefford and Matt Wilson

ACT 2024

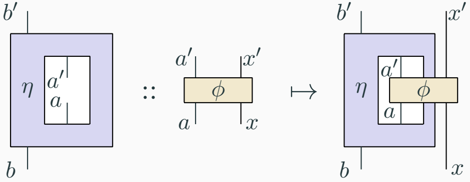
What is a Supermap?



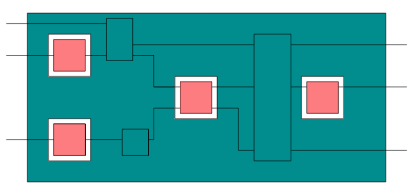
What is a Supermap?



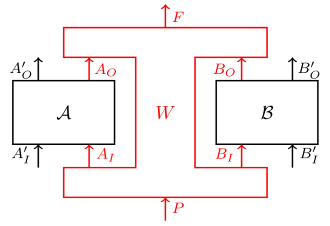
What is a Supermap?



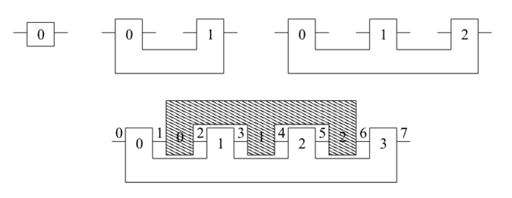
Quantum Supermaps



2008: Chiribella, D'Ariano, Perinotti



2020: Araújo, Feix, Navascués, Brukner



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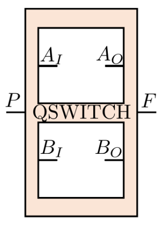
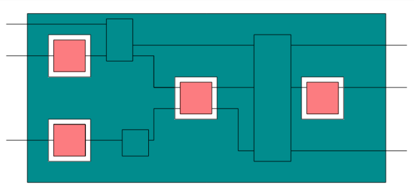


2021: Yokojima, Quintino, Soeda, Murao

Case 1: holes in circuits / concrete networks / combs



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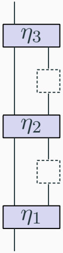
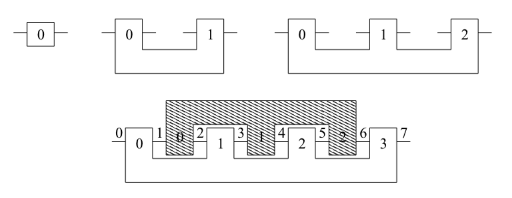
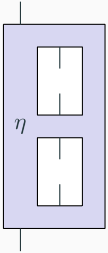


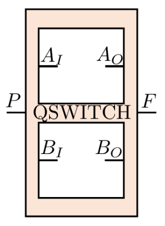
Case 2: abstract (black-box) definite causal order



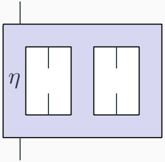
2009: Chiribella, D'Ariano, Perinotti



Case 3: abstract (black-box) indefinite causal order



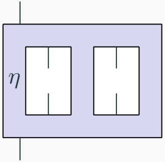
2021: Yokojima, Quintino, Soeda, Murao



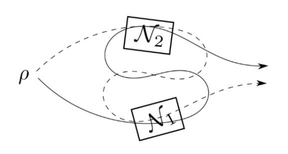
Case 3: abstract (black-box) indefinite causal order

There exist maps with no decomposition as a comb:

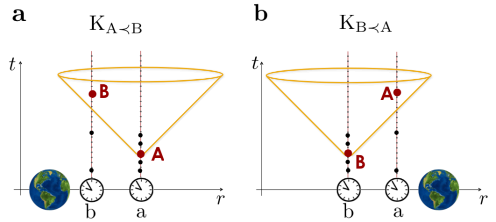
- Quantum Switch (2013: Chiribella, D'Ariano, Perinotti, Valiron)
- Lugano process (2014: Baumeler, Feix, Wolf)
- OCB process (2012: Oreshkov, Costa, Brukner)
- Grenoble process (2021: Wechs, Dourdent, Abbott, Branciard)



Case 3: abstract (black-box) indefinite causal order

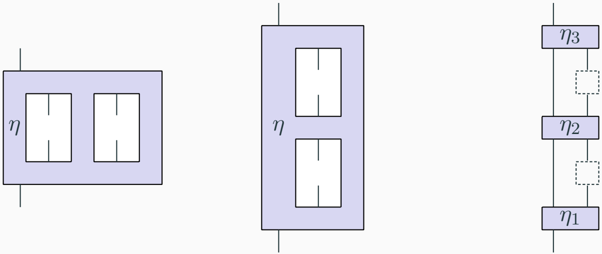


2018: Ebler, Salek, Chiribella



2020: Zych, Costa, Pikovski, Brukner

Three Cases



Indefinite Causal Order $(- \otimes_{\mathcal{C}} -)$

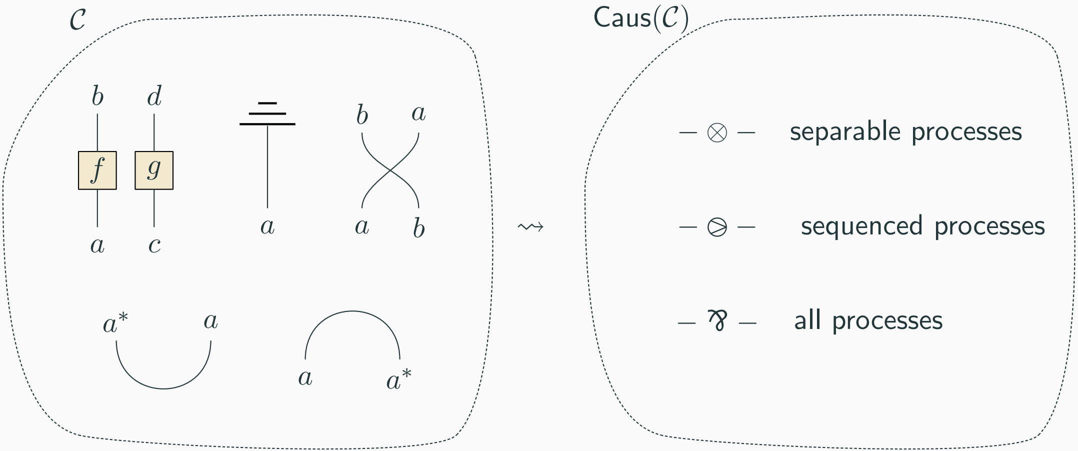
Abstract Definite Order $(- \ogreaterthan -)$

Concrete Networks (Representability)

Current Approaches

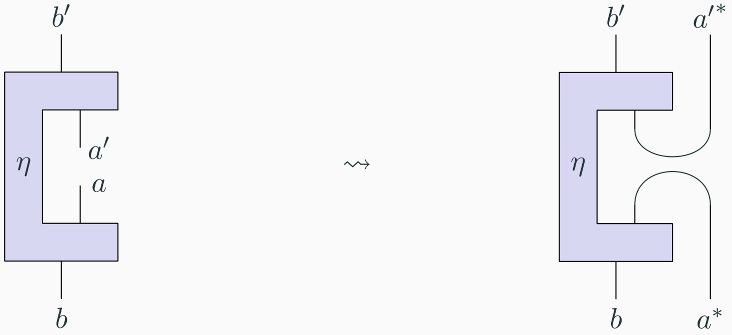
1. Caus-construction
2. Coend optics
3. Locally-applicable transformations

Caus-construction



2019: Kissinger, Uijlen; 2022 & 2024: Simmons, Kissinger

Caus-construction



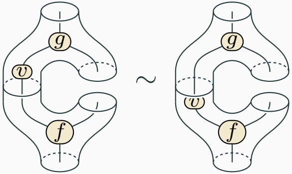
BUT: Hilb, Unitaries, Isometries, many GPTs are not compact closed!

Coend Optics

Category Optic($\mathcal{C}$)

- objects are pairs $\boldsymbol{a} := (a, a')$ of objects of $\mathcal{C}$
- hom-sets are

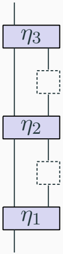
$$\text{Optic}(\mathcal{C})(\boldsymbol{a}, \boldsymbol{b}) := \int^{x} \mathcal{C}(b, x \otimes a) \times \mathcal{C}(x \otimes a', b').$$



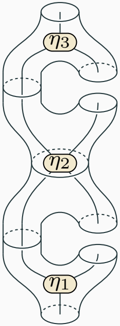
2008: Pastro, Street; 2018: Riley; 2020: Román
2020: Clarke, Elkins, Gibbons, Loregian, Milewski, Pillmore, Román

Coend Optics

Can model case 1:

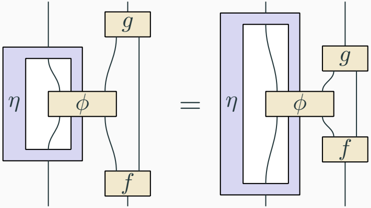


$\rightsquigarrow$



Not clear how to handle cases 2 and 3

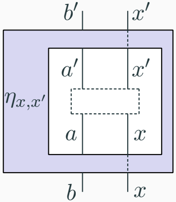
Locally-applicable Transformations



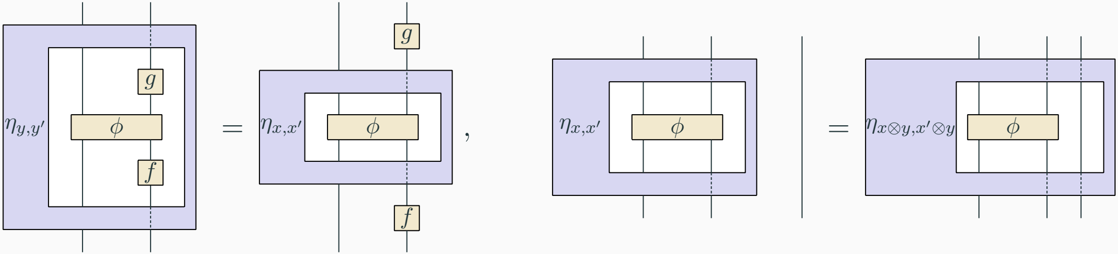
2022: Wilson, Chiribella, Kissinger; 2022: Wilson, Chiribella

Locally-applicable Transformations

A *locally-applicable transformation* $\eta : \boldsymbol{a} \to \boldsymbol{b}$ is a family of functions:



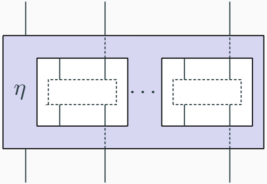
such that,



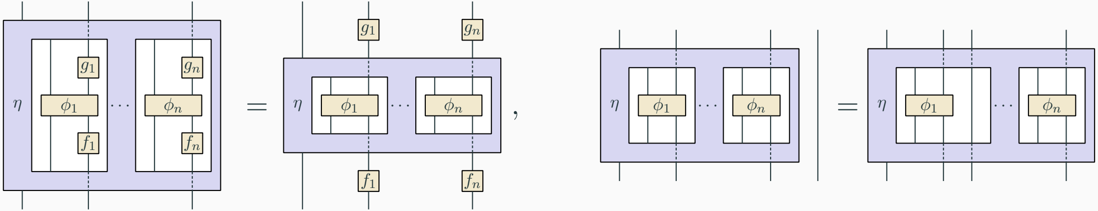
2022: Wilson, Chiribella, Kissinger; 2022: Wilson, Chiribella

Locally-applicable Transformations

A *multi-partite* locally-applicable transformation $\eta : \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \to \boldsymbol{b}$ is a family of functions:



such that,



2022: Wilson, Chiribella, Kissinger; 2022: Wilson, Chiribella

Locally-applicable Transformations

Pros:

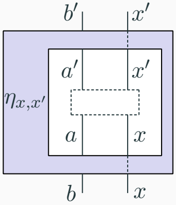
- Can model case 3:
  - $\eta : \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \to \boldsymbol{b}$ are the indefinitely-causally ordered quantum supermaps on quantum channels
- Very few underlying assumptions: no compact closure!

Cons:

- Algebraically intractable
- What is going on categorically?
- Cases 1 and 2?

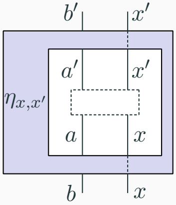
Single-Party Supermaps

Locally-applicable Transformations: Naturality



$$\phi \in \mathcal{C}(a \otimes x, a' \otimes x')$$
$$\big\downarrow \eta_{x,x'}$$
$$\eta_{x,x'}(\phi) \in \mathcal{C}(b \otimes x, b' \otimes x')$$

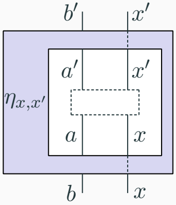
Locally-applicable Transformations: Naturality



$$\phi \in \mathcal{C}(a \otimes x, a' \otimes x') \xmapsto{\eta_{x,x'}} \eta_{x,x'}(\phi) \in \mathcal{C}(b \otimes x, b' \otimes x')$$

- $\mathcal{C}(a \otimes -, a' \otimes =) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathsf{Set}$ is an endoprofunctor $\mathcal{C} \nrightarrow \mathcal{C}$

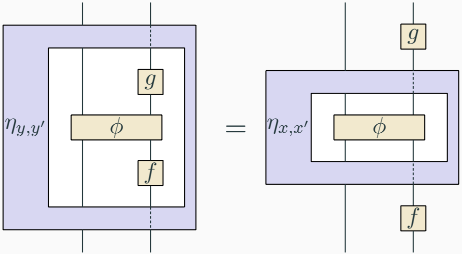
Locally-applicable Transformations: Naturality



$$\phi \in \mathcal{C}(a \otimes x, a' \otimes x') \overset{\eta_{x,x'}}{\longmapsto} \eta_{x,x'}(\phi) \in \mathcal{C}(b \otimes x, b' \otimes x')$$

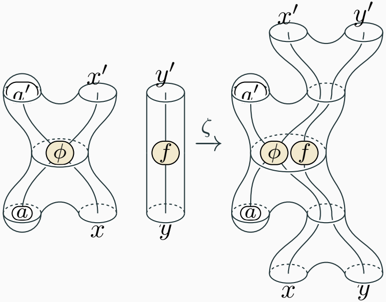
- $\mathcal{C}(a \otimes -, a' \otimes =) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$ is an endoprofunctor $\mathcal{C} \nrightarrow \mathcal{C}$
- $\eta_{x,x'}$ are components of $\eta : \mathcal{C}(a \otimes -, a' \otimes =) \Rightarrow \mathcal{C}(b \otimes -, b' \otimes =)$

Locally-applicable Transformations: Naturality

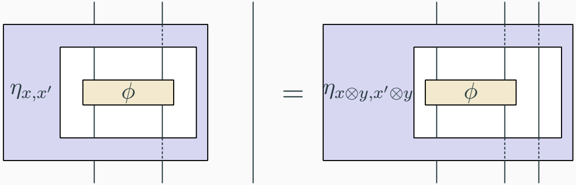


$$
\begin{array}{ccc}
\mathcal{C}(a \otimes x, a' \otimes x') & \xrightarrow{\mathcal{C}(1 \otimes f, 1 \otimes g)} & \mathcal{C}(a \otimes y, a' \otimes y') \\
\eta_{xx'} \downarrow & & \downarrow \eta_{yy'} \\
\mathcal{C}(b \otimes x, b' \otimes x') & \xrightarrow[\mathcal{C}(1 \otimes f, 1 \otimes g)]{} & \mathcal{C}(b \otimes y, b' \otimes y')
\end{array}
$$

Locally-applicable Transformations: Strength

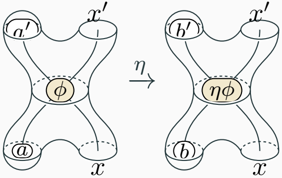


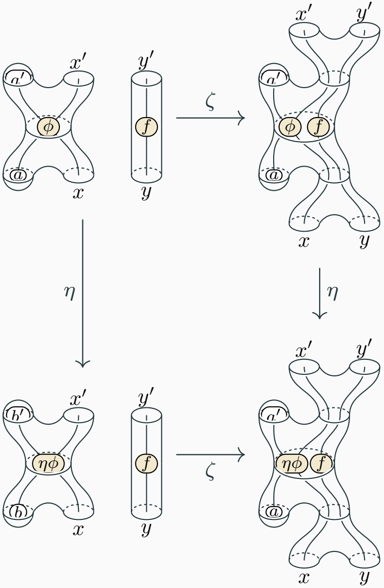
Locally-applicable Transformations: Strength



$$\begin{array}{ccc}
\mathcal{C}(a\otimes x, a'\otimes x')\times\mathcal{C}(y,y') & \xrightarrow{\zeta_{xx'yy'}} & \mathcal{C}(a\otimes x\otimes y, a'\otimes x'\otimes y') \\
\Big\downarrow{\scriptstyle \eta_{xx'}\times 1} & & \Big\downarrow{\scriptstyle \eta_{x\otimes y,x'\otimes y'}} \\
\mathcal{C}(b\otimes x, b'\otimes x')\times\mathcal{C}(y,y') & \xrightarrow[\zeta_{xx'yy'}]{} & \mathcal{C}(b\otimes x\otimes y, b'\otimes x'\otimes y')
\end{array}$$

Locally-applicable Transformations: Strength





Locally-applicable Transformations: Categorically

Definition
A single-party locally-applicable transformation is a strong natural transformation

$$\eta : \mathcal{C}(a \otimes -, a' \otimes =) \Rightarrow \mathcal{C}(b \otimes -, b' \otimes =).$$

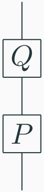
Multi-Party Supermaps

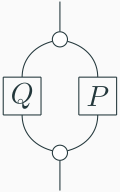
Strong Profunctors

$\mathsf{StProf}(\mathcal{C})$ is the category with:

- Objects: **strong** endoprofunctors $P : \mathcal{C} \nrightarrow \mathcal{C}$
- Morphisms: **strong** natural transformations $\eta : P \Rightarrow Q$.

Strong Profunctors

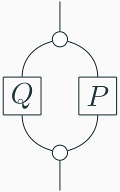
$$Q \ogreaterthan P =$$  $$= \int^{c} Q(-,c) \times P(c,-)$$

$$Q \otimes_{\mathcal{C}} P =$$  $$/ \sim \; = \left(\int^{abcd} \mathcal{C}(-, a \otimes b) \times Q(a,c) \times P(b,d) \times \mathcal{C}(c \otimes d, -) \right) / \sim$$

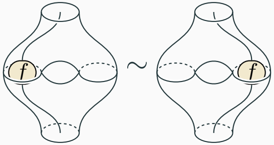
2008: Pastro, Street; 2016: Garner, López Franco; 2023: Earnshaw, H, Román

Theorem (Pastro & Street)
$\mathsf{StProf}(\mathcal{C}) \cong [\mathsf{Optic}(\mathcal{C}), \mathsf{Set}]$

$$Q \otimes_{\mathcal{C}} P =$$



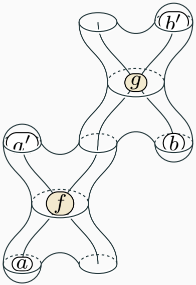
$$/ \sim \; = \int^{\boldsymbol{ab}} Q(\boldsymbol{a}) \times P(\boldsymbol{b}) \times \mathsf{Optic}(\mathcal{C})(\boldsymbol{a} \otimes \boldsymbol{b}, -)$$



2008: Pastro, Street; 2016: Garner, López Franco; 2023: Earnshaw, H, Román

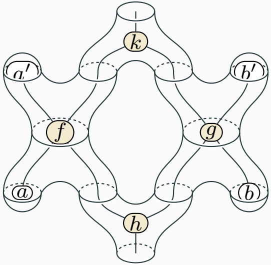
Generalised Spaces of Maps

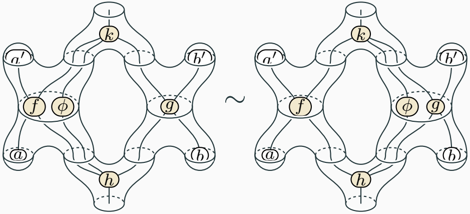
$$\mathcal{C}(- \otimes b, = \otimes b') \oslash \mathcal{C}(a \otimes -, a' \otimes =)$$



Generalised Spaces of Maps

$$\mathcal{C}(a \otimes -, a' \otimes =) \otimes_{\mathcal{C}} \mathcal{C}(- \otimes b, = \otimes b')$$

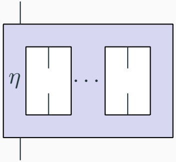




Indefinitely Causally Ordered Supermaps

Theorem
The multi-partite locally-applicable transformations $\eta : \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \to \boldsymbol{b}$ *are the morphisms*

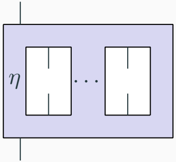
$$\eta : \bigotimes_{i=1}^{n}{}_{\mathcal{C}} \, \mathcal{C}(a_i \otimes -, a_i' \otimes =) \to \mathcal{C}(b \otimes -, b' \otimes =)$$



Indefinitely Causally Ordered Supermaps

Theorem
The quantum supermaps on the non-signalling channels are the morphisms

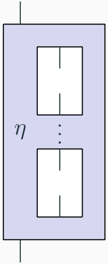
$$S : \bigotimes_{\mathsf{CPTP}}^{i} \mathsf{CPTP}(a_i \otimes -, a_i' \otimes =) \to \mathsf{CPTP}(b \otimes -, b' \otimes =).$$



Definitely Causally Ordered Supermaps

Definition
The definitely causally ordered supermaps are the morphisms

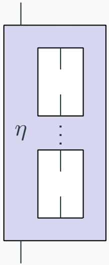
$$\bigodot_{i=1}^{n} \mathcal{C}(a_i \otimes -, a_i' \otimes =) \to \mathcal{C}(c \otimes -, c' \otimes =)$$



Definitely Causally Ordered Supermaps

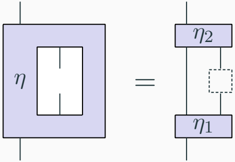
Theorem
The definitely causally ordered quantum supermaps are the morphisms

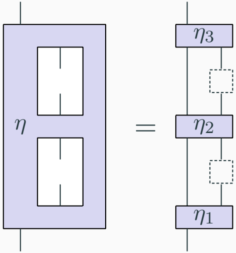
$$\bigodot_{i=1}^{n} \mathsf{CPTP}(a_i \otimes -, a_i' \otimes =) \to \mathsf{CPTP}(c \otimes -, c' \otimes =)$$



Decomposition and Duality

Decompositions of Supermaps

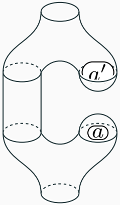




Decompositions of Supermaps

$$y : \mathsf{Optic}(\mathcal{C})^{\mathrm{op}} \to [\mathsf{Optic}(\mathcal{C}), \mathsf{Set}] \cong \mathsf{StProf}(\mathcal{C})$$

$$(a, a') \mapsto y_{a,a'} = \mathsf{Optic}(\mathcal{C})((a, a'), -) =$$



Decompositions of Supermaps

A symmetric monoidal category $\mathcal{C}$ has a 1-arity supermap decomposition theorem if

$$\mathsf{StProf}(\mathcal{C})\big(\mathcal{C}(a \otimes -, a' \otimes =), \mathcal{C}(b \otimes -, b' \otimes =)\big) \cong \mathsf{StProf}(\mathcal{C})\big(y_{b,b'}, y_{a,a'}\big)$$

$\mathcal{C}$ has an n-arity supermap decomposition theorem if

$$\mathsf{StProf}(\mathcal{C})\big(\varogreaterthan_i \mathcal{C}(a_i \otimes -, a_i' \otimes =), \mathcal{C}(b \otimes -, b' \otimes =)\big) \cong \mathsf{StProf}(\mathcal{C})(y_{b,b'}, \varogreaterthan_i y_{a_i,a_i'})$$

Decompositions of Supermaps

Theorem
The category CPTP *has an* n*-arity supermap decomposition theorem for every* n*.*

Duality

$(\mathcal{C}, \otimes, i)$ closed monoidal category with internal-hom $[-,-]$.

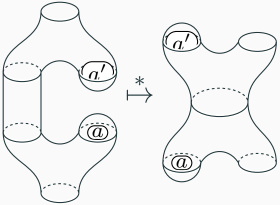
We have *weak duals*, $a^* := [a, i]$.

- $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$,
- $a^{**} \ncong a$ in general,
- models of *tensorial* logic,
- $*$-autonomy and linear logic proceeds when $a^{**} \cong a$.

2010: Melliès, Tabareau

Duality

Lemma
The weak dual of $y_{\boldsymbol{a}}$ *is* $\mathcal{C}(a \otimes -, a' \otimes =)$.



Duality

Proposition

A symmetric monoidal category $\mathcal{C}$ has a 1-arity decomposition theorem if and only if

$$\mathcal{C}(a \otimes -, a' \otimes =)^* \cong y_{\boldsymbol{a}}, \quad \textit{or equivalently,} \quad y_{\boldsymbol{a}}^{**} \cong y_{\boldsymbol{a}}.$$

Furthermore, $\mathcal{C}$ has an n-arity supermap decomposition theorem if and only if

$$(\oslash_i y^*_{\boldsymbol{a}_i})^* \cong \oslash_i y_{\boldsymbol{a}_i}.$$

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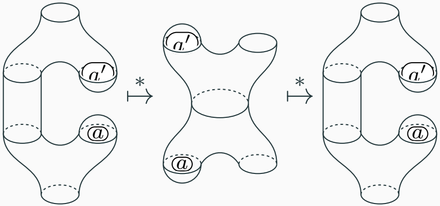

Duality

- We can define a functorial par,

$$- \parr - := (-^* \otimes -^*)^*$$

- $- \parr -$ is not generally a tensor of $\mathsf{StProf}(\mathcal{C})$.
- Strength, $P^{**} \otimes Q \to (P \otimes Q)^{**}$.
- Distributor of linear logic,

$$(P \parr Q) \otimes R = (P^* \otimes Q^*)^* \otimes R \to (P^* \otimes (Q \otimes R)^*)^* = P \parr (Q \otimes R).$$

Summary

- Formalised locally-applicable transformations
  - United with optics
  - A framework for supermaps over any smc $\mathcal{C}$
- Extended to include definite and indefinite causal orderings
- Identified decomposition theorems as representablility over optics

Some Thank Yous

With many thanks to:

- Mario Román
- Cole Comfort