Effect-labelled Coalgebras for Quantum Verification

ACT 2024

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June 20, 2024

Intro

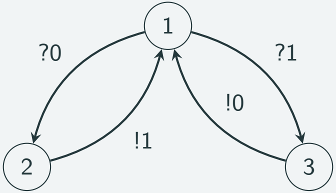
Motivation

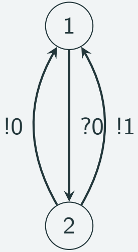
- Modelling concurrent systems is a challenging task
- Lot of different characterization, both syntactical and semantic, also in the probabilistic case
- Quantum systems are not just probabilistic! How to characterize their **observable behaviour?**
- Start from quantum theory (*effects*) and build up to computer science (*ELTS*).

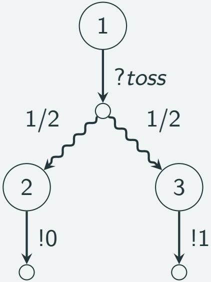
Roadmap

- We model systems which run in parallel, communicate and act according to their qubits.
- **Effect-weighted LTSs** ($\mathbb{E}$LTSs) encompass both probabilistic and quantum systems.
- We study their behavioural equivalence
- Compositional reasoning thanks to graded operators on $\mathbb{E}$LTS.

Modelling Concurrent Systems







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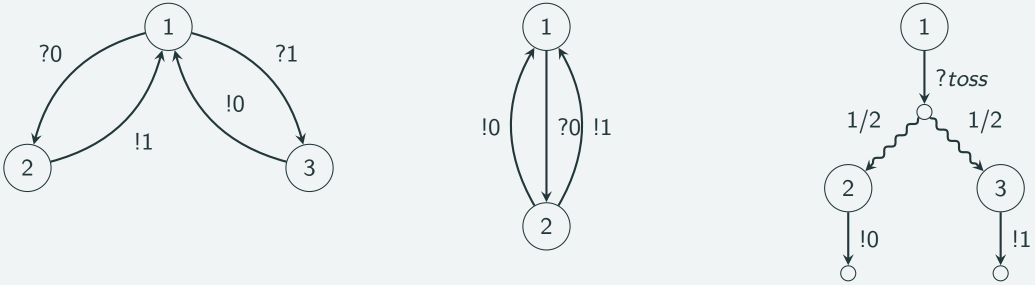

LTS:stateful computations with non-deterministic and/or probabilistic aspects

Coalgebras: $X \to \mathcal{P}(X)^L,\ X \to D(X),\ X \to \mathcal{P}(D(X))^L$

Quantum States and effects

Quantum States are density matrices, representing distributions of statevectors.

$$DM_{\mathcal{H}} = \left\{ \rho \in \mathbb{C}^{d \times d} \mid 0_d \sqsubseteq \rho, tr(\rho) = 1 \right\}$$

Examples:

$$|0\rangle\langle 0| \, , |1\rangle\langle 1| \, , |+\rangle\langle +| \, , \mathbb{I}/2$$

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Quantum Effects are functions from states to probabilities, $\mathcal{E}f_{\mathcal{H}} \simeq \mathbf{Conv}(DM_{\mathcal{H}}, [0,1])$.

They are represented as positive matrices smaller than the identity

$$\mathcal{E}f_{\mathcal{H}} = \left\{ E \in \mathbb{C}^{d\times d} \mid 0_d \sqsubseteq E \sqsubseteq \mathbb{I}_d \right\}$$

Examples:

$$|0\rangle\langle 0|, \mathbb{I}, \frac{1}{2}|1\rangle\langle 1|$$

Effect Algebras

Definition: An effect algebra is a tuple $\langle \mathbb{E}, 0, +, \cdot' \rangle$ with $\langle \mathbb{E}, 0, + \rangle$ a *partial* commutative monoid and $\cdot'$ a unary operator satisfying

- e' is the unique element in E such that $e + e' = 1$ with $1 = 0'$;
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- σ-algebras with disjoint union
- **Quantum Effects**

Effect algebras define a partial order by $a \leq b \Leftrightarrow \exists c.a + c = b$. Effect algebras and their homomorphism form the symmetric monoidal category **EA**.

Effect Distributions

Definition: Given an effect algebra $\langle \mathbb{E}, 0, +, \cdot' \rangle$ define the *effect (sub-)distributions* functor $D_{\mathbb{E}} : \mathbf{Set} \to \mathbf{Set}$

$$D_{\mathbb{E}} X = \left\{ \Delta \in \mathbb{E}^X \;\middle|\; \text{supp}(\Delta) \text{ is finite}, \sum_{\in \text{supp}()} \Delta(x) \leq 1_{\mathbb{E}} \right\} \qquad D_{\mathbb{E}} f(\Delta) = \lambda y. \sum_{x \in f^{-1}(y)} \Delta(x)$$

where $\text{supp}(\Delta)$ is the set $\{ x \in X \mid \Delta(x) \neq 0 \}$.

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$$D_{[0,1]}X \ni \{x \mapsto 1\}, \{x \mapsto 0.5, y \mapsto 0.5\}, \{z \mapsto 0.3\} \dots$$
$$D_{\mathcal{E}f_{\mathcal{H}}}X \ni \{x \mapsto \mathbb{I}_{\mathcal{H}}\}, \{x \mapsto |+\rangle\langle+|, y \mapsto |-\rangle\langle-|\}, \{z \mapsto 0.3 \cdot \mathbb{I}_{\mathcal{H}}\} \dots$$

ELTSs

Colagebras

F-Coalgebras: couples (X, c) with $X \xrightarrow{c} FX$.

Given $X \xrightarrow{c} FX$ and $Y \xrightarrow{d} FY$, a **coalegebra homomorphism** is a function $m : X \to Y$ such that $Fm \circ x = d \circ m$. F-coalgebras and their homomorphisms form the category $\mathbf{Coalg}_F$.

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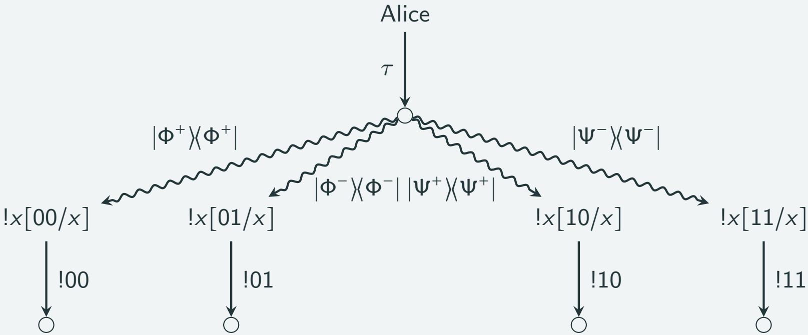
Example: Quantum Teleportaion

$$\text{Alice} := M_B(q_1, q_2 \triangleright x).c!x \qquad \text{Bob} := \sum_{i=00}^{11} c?i.Bob_i$$



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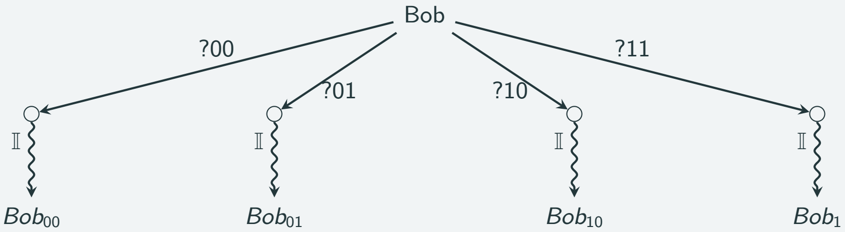

Behavioural Equivalences

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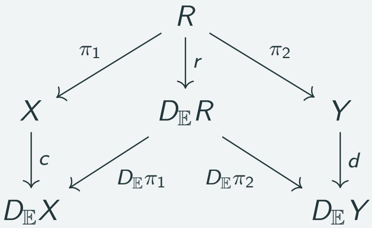
$R \subseteq X \times Y$ is an **Aczel-Mendler Bisimulation** if there exists a *span* of coalgebra homomorphisms. $\sim_{AM}$ is the largest AM-bisimulation.



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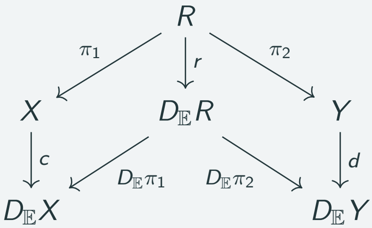
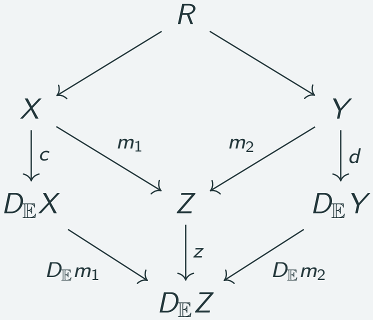


$R \subseteq X \times Y$ is a **Larsen-Skou Bisimulation** if it the **Set**−pullback of a *cospan* of coalgebra homomorpsisms. $\sim_{LS}$ is the largest LS bisimulation.



Weak Pullback Preservation

Theorem: $D_{\mathbb{E}}$ preserves weak pullbacks iff $\mathbb{E}$ is decomposable. In that case, $\mathbf{Coalg}_F$ has weak pullbacks.

$\mathbb{E}$ is *decomposable* if for all $a, b, c, d \in \mathbb{E}$ such that $a + b = c + d$ is defined, there exists $e_{11}, e_{12}, e_{21}, e_{22} \in \mathbb{E}$ such that:

$$\begin{array}{ccccc} a & & b & & \\ = & & = & & \\ e_{11} & + & e_{12} & = & c \\ + & & + & & \\ e_{21} & + & e_{22} & = & d \end{array}$$

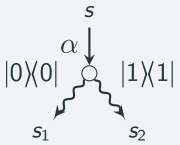
Theorem: Let $\mathbb{E}$ be an effect algebra. For any $D_{\mathbb{E}}$-coalgebras c and d:

$$x \sim_{AM} y \xrightarrow{\text{always}} x \sim_{LS} y \qquad\qquad x \sim_{AM} y \xLeftarrow{\mathbb{E}\text{ decomposable}} x \sim_{LS} y$$

The quantum case

An example inspired by [Ogawa 2014]:

c =



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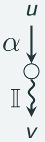

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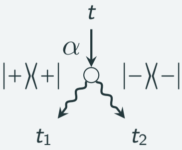

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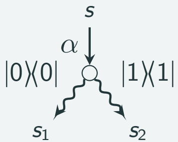

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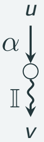

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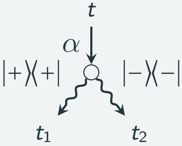

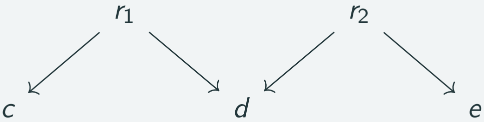

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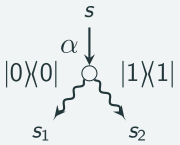

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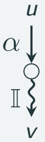

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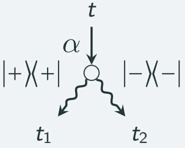

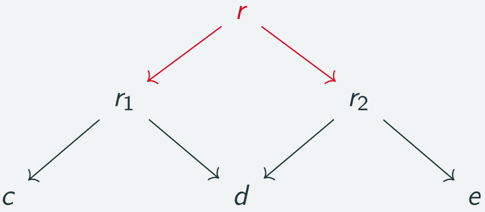

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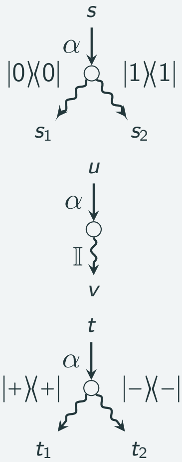

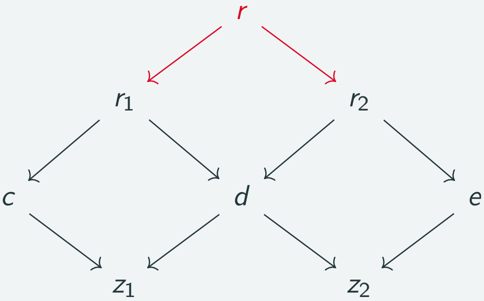

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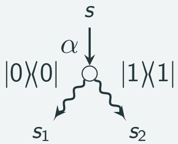
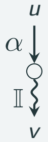
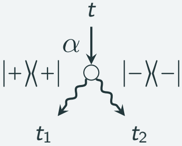
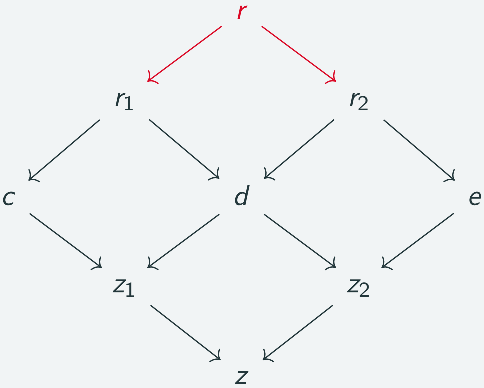
c = 

d = 

e = 



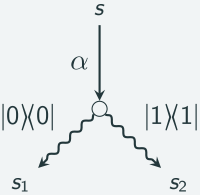
Why we choose Larsen-Skou Bisimilarity:

- It is an equivalence relation
- It corresponds to the coalgebraic notion of behavioural equivalence (i.e. identity in the final colagebra)
- **It is correct and complete with respect to the observable probabilistic behaviour**

Probabilistic Behaviour

We can **evaluate** a QLTS with a quantum input state $\rho \in DM_{\mathcal{H}}$, by updating its weights.

$C =$



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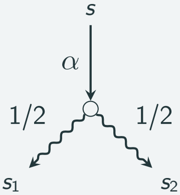

$C \downarrow_{|0\rangle\langle 0|} =$

Lifting Quantum Isomorphisms

- A quantum state $\rho \in DM_{\mathcal{H}}$ gives an *effect algebra homomorphism* $m_\rho : \mathcal{E}f_{\mathcal{H}} \to [0,1]$

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- A nat.transf. $\alpha_\rho : F \Rightarrow G$ gives a *functor of coalgebras* $\cdot \downarrow_\rho : \mathbf{Coalg}_F \to \mathbf{Coalg}_G$

$$x \downarrow_\rho = x \qquad c \downarrow_\rho = \alpha_\rho \circ c \qquad f \downarrow_\rho = f$$

Correctness and Completeness

Theorem: Thanks to functoriality, for any two QLTS c and d:

$$x \sim_{LS} y \qquad \Rightarrow \qquad \forall \rho \in DM_{\mathcal{H}} \quad x\downarrow_\rho \sim_{LS} y\downarrow_\rho$$

Moreover, if we restrict the weights to a finite sub-algebra $\mathcal{E}f'_{\mathcal{H}} \subsetneq \mathcal{E}f_{\mathcal{H}}$, we have an injective morphism from $\mathcal{E}f'_{\mathcal{H}}$ to $[0,1]$.

Theorem: For any $PD^{L}_{\mathcal{E}f'_{\mathcal{H}}}$-coalgebras c and d:

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What is the parallel composition of two $\mathbb{E}$LTSs? If $s \xrightarrow{!0} \mathfrak{D}$ ad $t \xrightarrow{?0} \mathfrak{T}$, what about $s \parallel t$? In the probabilistic case $s \parallel t \xrightarrow{\tau} \mathfrak{D} \cdot \mathfrak{T}$, the *joint probability distribution*.

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We need a natural transformation $\alpha : D_{\mathbb{E}}X \times D_{\mathbb{E}}Y \Rightarrow D_{\mathbb{E}}(X \times Y)$. We search for a **commutative monad**.

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If $\mathbb{E}$ is a **commutative monoid object** in **EA**

Then $D_{\mathbb{E}}$ is a **monad** on **Set**

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If $\mathbb{E}$ is a **commutative monoid object** in **EA**

There is an morphism $\nabla : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ that is commutative associative and unital

$$\nabla(e, f) = \nabla(f, e) \qquad \nabla(e, \nabla(f, g)) = \nabla(\nabla(e, f), g)$$
$$\nabla(e, 1_{\mathbb{E}}) = e \qquad \nabla(e, 0_{\mathbb{E}}) = 0_{\mathbb{E}}$$

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Effect Monoids and Monads

If $\mathbb{E}$ is a **commutative monoid object** in **EA**

There is an morphism $\nabla : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ that is commutative associative and unital

$$\nabla(e, f) = \nabla(f, e) \qquad \nabla(e, \nabla(f, g)) = \nabla(\nabla(e, f), g)$$

$$\nabla(e, 1_{\mathbb{E}}) = e \qquad \nabla(e, 0_{\mathbb{E}}) = 0_{\mathbb{E}}$$

Then $D_{\mathbb{E}}$ is a **monad** on **Set**

There are a unit $\eta : Id \Rightarrow T$ and a multiplication $\mu : D_{\mathbb{E}} D_{\mathbb{E}} \Rightarrow D_{\mathbb{E}}$ defined as

$$\eta(x) = 1_{\mathbb{E}} \bullet x \qquad \mu(\textstyle\sum_i e_i \bullet \Delta_i)x = \textstyle\sum_i \nabla(e_i, \Delta_i(x))$$

PCM of Quantum Systems

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- Let $Sys = \{S_i\}$ be a fixed set of quantum systems.
- $\mathcal{S} = \langle \mathcal{P}(Sys), \varnothing, \uplus \rangle$ forms a PCM where $\uplus$ is the partial disjoint union.
- Each collection of systems $C \in \mathcal{P}(Sys)$ has an associated a Hilbert space obtained by tensoring.

Graded Monoid of Quantum Effects

Quantum effects carry a commutative $\mathcal{S}$-graded effect monoid structure, i.e.:

- An effect algebra $\mathbb{E}_C = \mathcal{E}f_{\mathcal{H}_C}$ for any collection $C \in \mathcal{P}(Sys)$
- An operator $\nabla_{C,D} : \mathbb{E}_C \otimes \mathbb{E}_D \to \mathbb{E}_{C \uplus D}$ defined by $\nabla(E_1, E_2) = Sort_{C,D}(E_1 \otimes_k E_2)$,

Graded Effect Monoids and Monads

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Then $D_{\mathbb{E}}$ is a **commutative** M**-graded monad** on **Set**

There are endofunctors $T_n = D_{\mathbb{E}_n}$, a unit $\eta : Id \Rightarrow T_0$ and a multiplication $\mu : T_n T_m \Rightarrow T_{n+m}$ such that

$$\begin{array}{ccc} T_m T_n T_o & \xrightarrow{T_m \mu_{n,o}} & T_m T_{n+o} \\ {\scriptstyle \mu_{m,n} T_o}\downarrow & & \downarrow{\scriptstyle \mu_{m,n+o}} \\ T_{m+n} T_o & \xrightarrow[\mu_{m+n,o}]{} & T_{m+n+o} \end{array} \qquad \begin{array}{ccc} T_m & \xrightarrow{T_m \eta} & T_m T_0 \\ {\scriptstyle \eta T_m}\downarrow & \searrow\!\!= & \downarrow{\scriptstyle \mu_{m,0}} \\ T_0 T_m & \xrightarrow[\mu_{0,m}]{} & T_m \end{array}$$

There is a nat. transf. $\alpha : T_n X \times T_m Y \Rightarrow T_{n+m}(X \times Y)$ given by strength.

Synchronization

Consider the graded monad $Q_C = \mathcal{P}(D_{\mathbb{E}_C})^L$. Given two coalgebras $X \xrightarrow{c} Q_C X$ and $Y \xrightarrow{d} Q_D Y$ we define their CCS-style synchronization $c \parallel d : X \times Y \to Q_{C \uplus D}(X \times Y)$

$$\frac{s \xrightarrow{\mu} \mathfrak{D}}{s \parallel t \xrightarrow{\mu} \alpha(\mathfrak{D}, \{t \mapsto \mathbb{I}_D\})} \qquad \frac{t \xrightarrow{\mu} \mathfrak{T}}{s \parallel t \xrightarrow{\mu} \alpha(\{s \mapsto \mathbb{I}_C\}, \mathfrak{T})} \qquad \frac{s \xrightarrow{\mu} \mathfrak{D} \quad t \xrightarrow{\overline{\mu}} \mathfrak{T}}{s \parallel t \xrightarrow{\tau} \alpha(\mathfrak{D}, \mathfrak{T})}$$

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We have defined a functor $\cdot \parallel \cdot : \mathbf{Coalg}_{Q_C} \times \mathbf{Coalg}_{Q_D} \to \mathbf{Coalg}_{Q_{C \uplus D}}$. Since it preserves bisimilarity, we also have that

$$s \sim_{LS} s', \ t \sim_{LS} t' \qquad \Longrightarrow \qquad s \parallel t \sim_{LS} s' \parallel t'$$

Conclusion and Future Works

- We use $\mathbb{E}$LTSs to model probabilistic and quantum systems uniformly
- We define two bisimilarities, and prove that LS-bisimilarity is in general coarser
- In the quantum case, LS-bisimilarity is correct and complete with respect to the probabilistic behavioural equivalence
- When the effects chosen as weights form a graded monoid, we have a graded parallel composition of coalgebras

Future Work:

- Dealing with superoperator distributions instead of probability distributions
- Defining an adequate graded GSOS format and proving congruence for the standard CCS operators

Thank you

Question time