Retracing GoI with Phil

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Memories of Phil

Enthusiasm, Laughter, Kindness

Paper with Phil

Abramsky, Samson, Esfandiar Haghverdi, and Philip Scott. "Geometry of interaction and linear combinatory algebras." Mathematical Structures in Computer Science 12, no. 5 (2002): 625-665.

Geometry of Interaction

Series of papers by Girard:

- Multiplicatives (1988, unpublished preprint)
- Towards a Geometry of Interaction (1989)
- Geometry of Interaction I: Interpretation of System F (1989)
- Geometry of Interaction II-V (1988,1995,2011)

Early work by Danos, Regnier and Malacaria

- Some results on the interpretation of λ-calculis in operator algebras (M & R, 1991)
- Local and asynchronous beta-reduction (D & R, 1993)

SA and Radha Jagadeesan (1992)

- New Foundations for the Geometry of Interaction
- Games and Full Completeness for Multiplicative Linear Logic

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  - dynamics of cut-elimination
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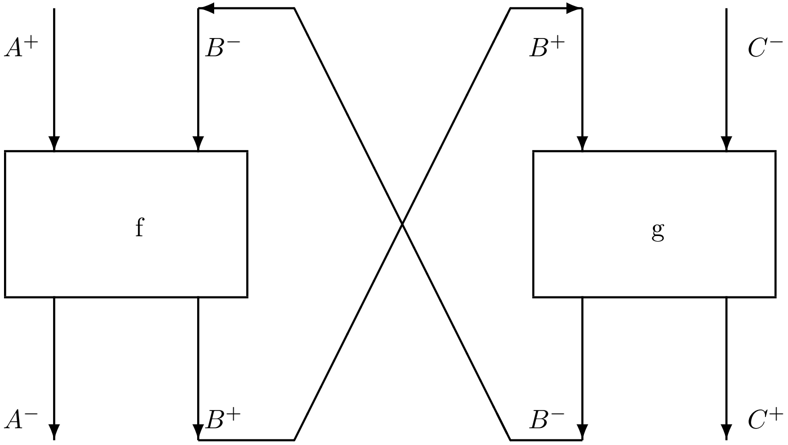
  - dynamics of cut-elimination
  - operator algebras: a red herring
- My perspective
  - Adapting Girard, novel way of interpolating between operational and denotational semantics
  - close connection with game semantics (being developed at the same time)
  - GoI = "game semantics without games"

GoI in one picture



Composition as symmetric feedback.

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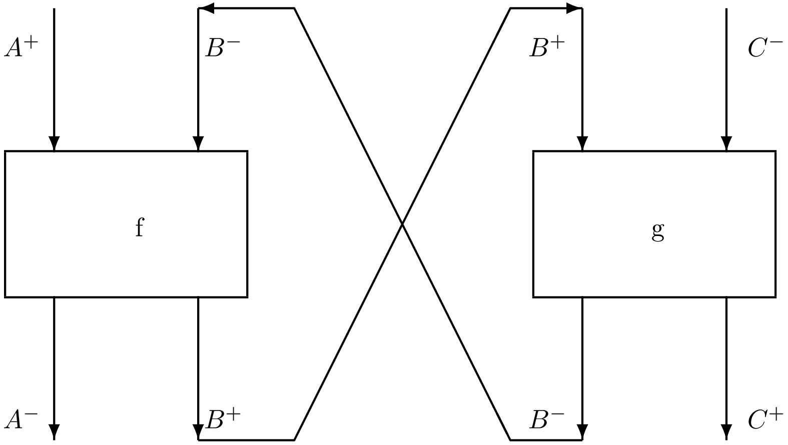
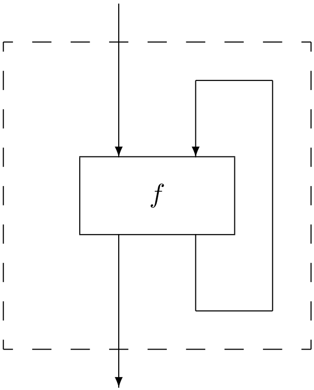


Composition as symmetric feedback.
Almost commutative! (up to symmetry).

Traced monoidal categories

$$\frac{f : A \otimes B \longrightarrow C \otimes B}{\mathsf{Tr}(f) : A \longrightarrow C}$$



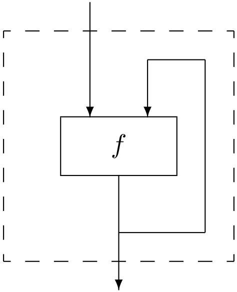
Linearity of Trace

Note that trace is linear, unlike the **iteration** concept of iterative theories (Elgot, Bloom, Esik) used to model flowcharts.

$$\frac{f : A \times B \longrightarrow B}{\mathsf{It}(f) : A \longrightarrow B}$$

$$f \circ \langle 1, \mathsf{It}(f) \rangle \quad = \quad \mathsf{It}(f)$$

There is no implicit **sharing** of wires as in iteration.



Mathematical essence of multiplicative GoI

- Feedback in a **linear** (monoidal) setting
- Building a free compact closed category from a traced category – "higher-order" vs. "first-order".
- The composition in this free compact closed category (i.e. symmetric feedback) subsumes the "Execution formula", hence the "dynamics of Cut-elimination".

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Avant la lettre:

- NFGoI built a traced monoidal category from domains and fixpoints (product based)
- In "Games and Full Completeness for MLL" it was observed that composition of history-free strategies was given by a suitable abstraction of the Execution formula (coproduct based)
- I showed these examples in detail to André Joyal during LiCS 1993 in Montreal.

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- “Particle style”: the tensor product is coproduct (disjoint union). **Rel**, **Pfn**, **PInj**, etc. Covers the original version of GoI introduced by Girard. Matrices over a Kleene algebra are another example. (The ‘magic formula’ for star is exactly the trace!)

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- "Acausal": Compact closed categories. $(\mathbf{Rel}, \times)$, $(\mathbf{FDVect}, \otimes)$. Physical interpretation: 'Physical Traces', S.A. and Bob Coecke.

Closing the gap

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However, to achieve a full understanding of GoI, more was needed:

- Extending the axiomatic framework to cover the whole of GoI, and computational universality
- GoI does not exactly give a model of Linear Logic (or λ-calculus) in the usual sense – many extensional equalities fail. What **is** it doing?

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I outlined an approach to answering these questions in lectures given in Siena and Edinburgh in 1997. This developed into a joint project with Phil and Esfan (who was doing his Ph.D. with Phil), and led to our paper.

Our answers

- We introduce **GoI situations** as an axiomatic framework for the full GoI construction, including exponentials.
- We propose **combinatory algebra** as the appropriate intensional setting to capture what GoI does.
- We introduce **linear combinatory algebras**, show that these give rise to standard, computationally universal combinatory algebras, and prove our **main result**:

Theorem

Every GoI situation gives rise to a linear combinatory algebra.

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Crucially, they capture the right level of intensionality for GoI, explaining why it suffices for computation.

The Curry combinators

Curry's original set of combinators was **B**, **C**, **K**, and **W**:

$$
\begin{aligned}
\mathbf{B} \cdot x \cdot y \cdot z &= x \cdot (y \cdot z) \\
\mathbf{C} \cdot x \cdot y \cdot z &= x \cdot z \cdot y \\
\mathbf{W} \cdot x \cdot y &= x \cdot y \cdot y
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They have the following principal types:

$$\begin{array}{lcll} \mathbf{I} & : & \alpha \to \alpha & \text{Axiom} \\ \mathbf{B} & : & (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma & \text{Cut} \\ \mathbf{C} & : & (\alpha \to \beta \to \gamma) \to \beta \to \alpha \to \gamma & \text{Exchange} \\ \mathbf{K} & : & \alpha \to \beta \to \alpha & \text{Weakening} \\ \mathbf{W} & : & (\alpha \to \alpha \to \beta) \to \alpha \to \beta & \text{Contraction} \end{array}$$

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These are structures $(A, \cdot, \mathbf{!})$, where in addition to the applicative structure $(A, \cdot)$ we have a unary operator $!$.

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There are constants $B, C, I, K, W, D, \delta, F$ satisfying:

1. $Bxyz = x(yz)$ Composition, Cut
2. $Cxyz = (xz)y$ Exchange
3. $Ix = x$ Identity
4. $Kx!y = x$ Weakening
5. $Wx!y = x!y!y$ Contraction
6. $D!x = x$ Dereliction
7. $\delta!x = !!x$ Comultiplication
8. $F!x!y = !(xy)$ Monoidal Functoriality

LCA and Linear Logic

The principal types:

1. $B : (\beta \multimap \gamma) \multimap (\alpha \multimap \beta) \multimap \alpha \multimap \gamma$
2. $C : (\alpha \multimap \beta \multimap \gamma) \multimap (\beta \multimap \alpha \multimap \gamma)$
3. $I : \alpha \multimap \alpha$
4. $K : \alpha \multimap \,!\beta \multimap \alpha$
5. $W : (!\alpha \multimap \,!\alpha \multimap \beta) \multimap \,!\alpha \multimap \beta$
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A combinatory version of the Girard translation results in:

Theorem

Given an LCA, we can construct a standard CA, where $x \bullet_s y := x \bullet \mathbf{!} y$.

GoI situations

A *GoI Situation* is a triple $(\mathbb{C}, T, U)$ where:

- $\mathbb{C}$ is a traced symmetric monoidal category
- $T : \mathbb{C} \longrightarrow \mathbb{C}$ is a traced symmetric monoidal functor with the following retractions (which are monoidal natural transformations):
  1. $e : TT \lhd T : e'$ (Comultiplication)
  2. $d : Id \lhd T : d'$ (Dereliction)
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We apply the GoI (or Int) construction to $\mathbf{C}$ to get a compact closed category $\mathcal{G}(\mathbf{C})$. We can define $\mathbf{!}$ on $\mathcal{G}(\mathbf{C})$ using the functor T.

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This matches the idea that in combinatory logic, in general λ-equations hold only for **closed terms**.

Completing the construction

We then show that weak linear categories give rise to linear combinatory algebras.
To get a type-free model, we assume a "reflexive object" U with retractions $U \otimes U \lhd U$, $I \lhd U$, $TU \lhd U$.

GoI Situation

$\Downarrow$ construct $\mathcal{G}(\mathbb{C})$

Weak Linear Category

$\Downarrow$ take $\mathcal{G}(\mathbb{C})(I, U)$

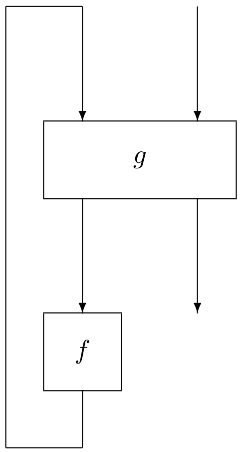
Linear Combinatory Algebra

$\Downarrow$ by standardization

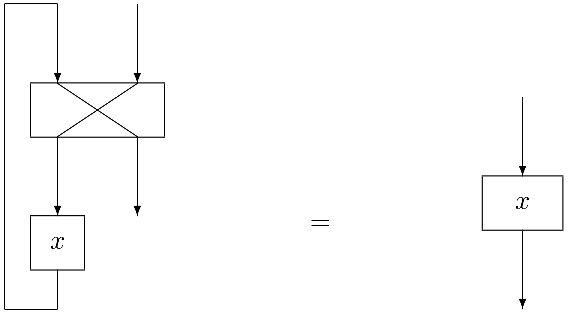
Combinatory Algebra

Application pictorially

This is just a special case of composition:

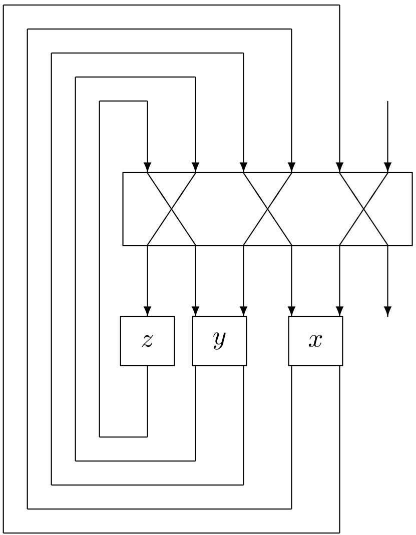


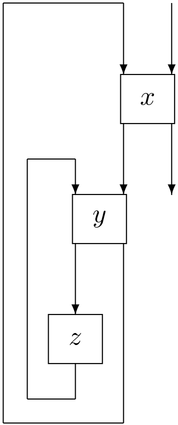
The I combinator



$$\mathbf{I} \bullet x = x$$

The B combinator





The W combinator

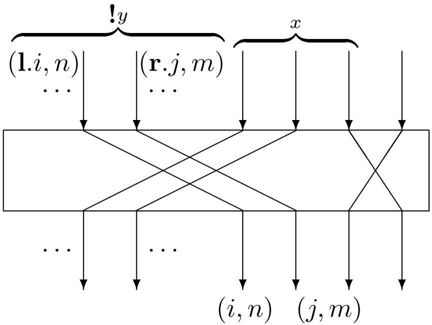
$$Wx\mathbf{!}y = x\mathbf{!}y\mathbf{!}y$$

Hilbert Hotelling:

$$\mathsf{p} : \mathbb{N} \times \mathsf{Pos} \xrightarrow{\cong} \mathsf{Pos}$$

Translation between dialects

W



Further Developments

- Phil and Esfan wrote a further series of papers on GoI. Topics include: typed GoI, unique decomposition categories, partial traces. Tutorial in *New Structures in Physics* (2010).
- Peter Hines has found many fascinating connections of the Hilbert hotel aspects, e.g. to the Thompson group, strictification of coherence, the Collatz conjecture, etc.
- Mark Lawson has studied algebraic aspects.
- Naohiko Hoshino, Ichiro Hasuo and Koko Muroya have studied extensions such as higher-order quantum GoI and memoryful GoI.
- Kuko Muroya and Dan Ghica developed a dynamic GoI.
- Carsten Furhmann and David Pym developed a categorical model of Classical logic using GoI ideas.
- I gave a very concrete account of GoI as the basis for a **structural approach to reversible computation**. This shows that one can build a combinatory algebra of **partial involutions** (graph matchings) in a very simple fashion.
- An interesting question arising from this work has been answered in recent work by Alberto Ciaffaglione, Pietro Di Gianantonio, Furio Honsell, Marina Lenisa, and Ivan Scagnetto.


Paintings