CaTT contexts are finite computads

Thibaut Benjamin, Ioannis Markakis, Chiara Sarti

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ω-categories: why?

- Identity types in MLTT

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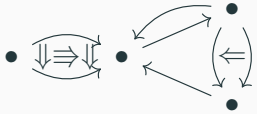
- Proof relevant rewriting
- Directed homotopy and concurrent programming

ω-categories: what?

- A globular set



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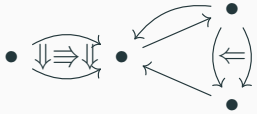

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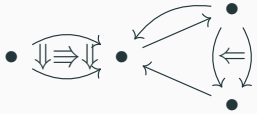

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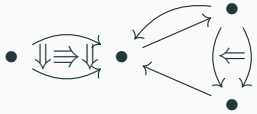

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- Satisfying various axioms...
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- Weakly!
  Up to a higher cell: witnessing an equivalence

ω-categories: how?

catt
Finster, Mimram
dependent type theory

computads
description by DFMRV [1]
inductively defined

[1] Dean, Finster, Markakis, Reutter, Vicary

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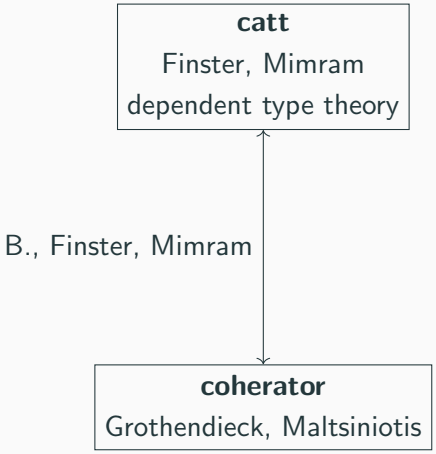

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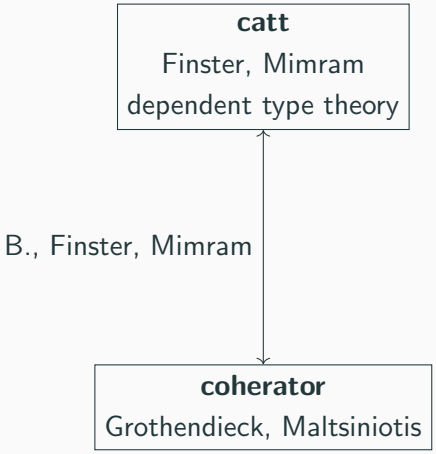

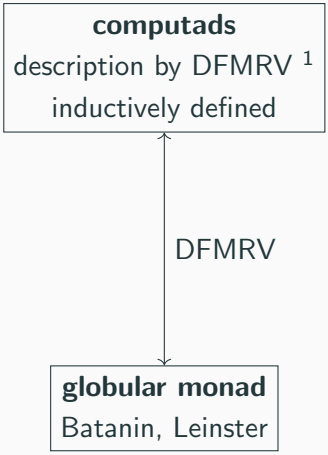

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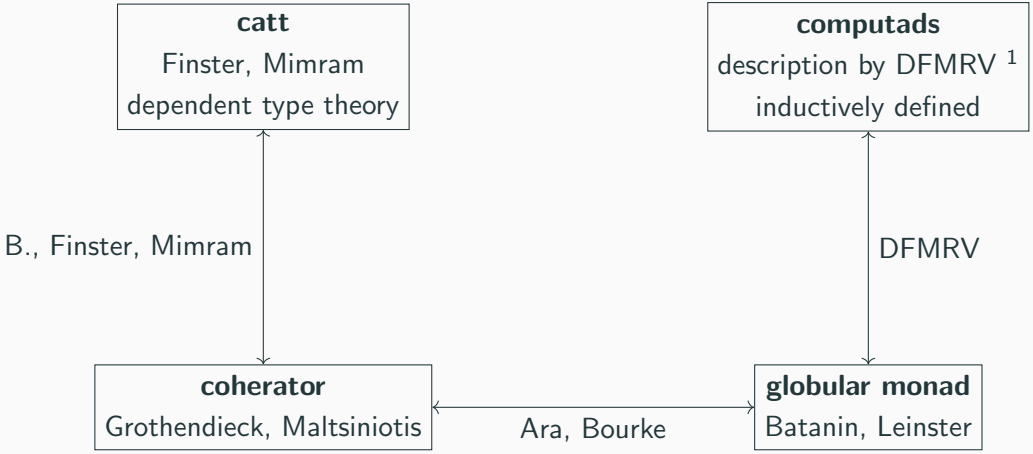

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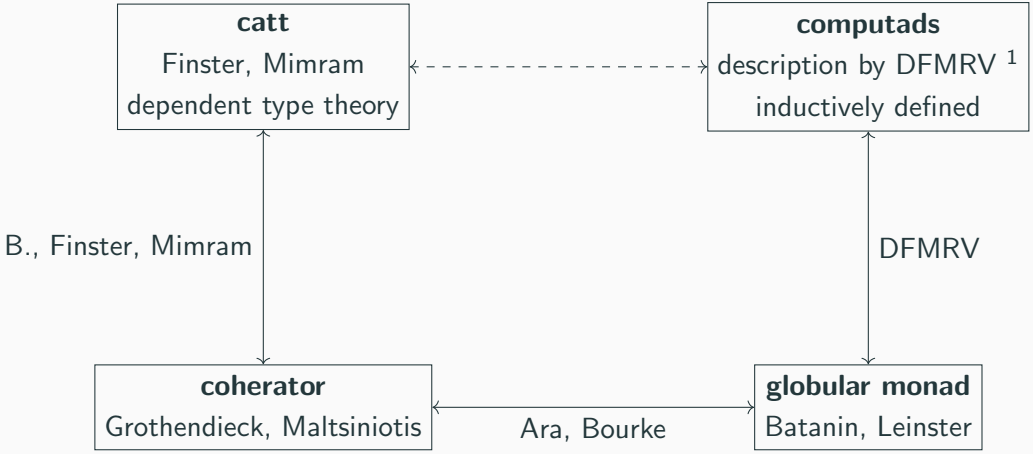

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Presentation of CaTT

Types and globular sets

CaTT has 2 type constructors:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash u : A \qquad \Gamma \vdash v : A}{\Gamma \vdash u \to_A v}$$

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- **Idea**: The arrow types represent abstract directed equality.
- **Structure**: Contexts using these types are globular sets

$$(x : \star)(y : \star)(f : x \to y)(g : x \to x)$$

Pasting schemes

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Pasting schemes : Example

$$\frac{}{(x : \star) \vdash_{\mathsf{ps}} x : \star}\ (\textsc{start})$$

x

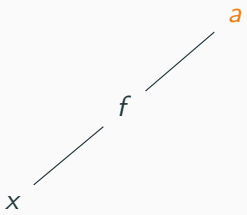
x

Pasting schemes : Example

$$\frac{}{(x : \star)\ (y : \star)(f : x \to y) \vdash_{\mathsf{ps}} f : x \to y}\ (\textsc{start})\ (\textsc{ext})$$

Pasting schemes : Example

$$\frac{}{(x : \star)\ (y : \star)(f : x \to y)\ (g : x \to y)(a : f \to g) \vdash_{\mathsf{ps}} a : f \to g}\ (\textsc{start})\ (\textsc{ext})\ (\textsc{ext})$$



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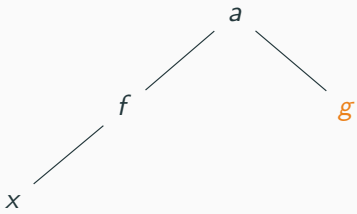

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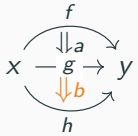

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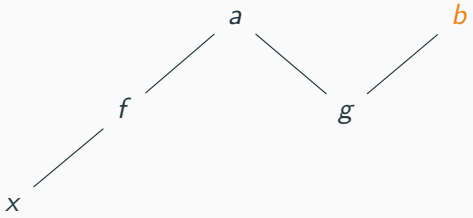

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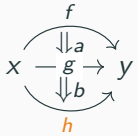

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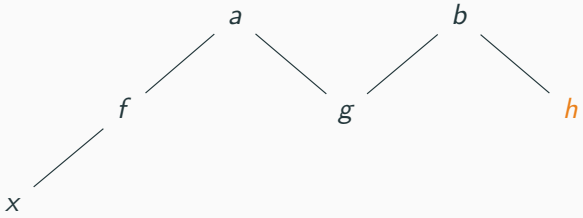

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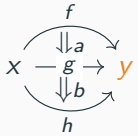

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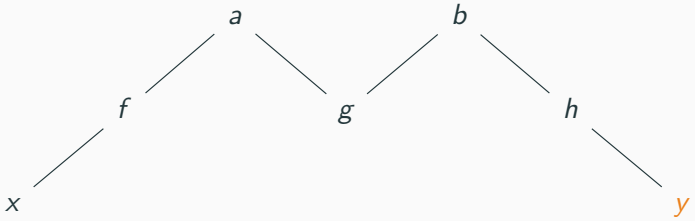

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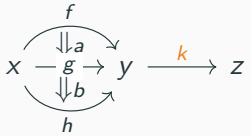

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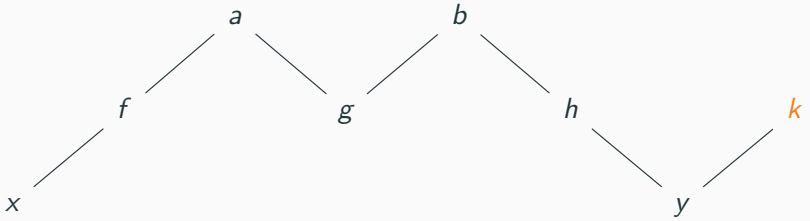

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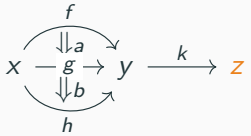

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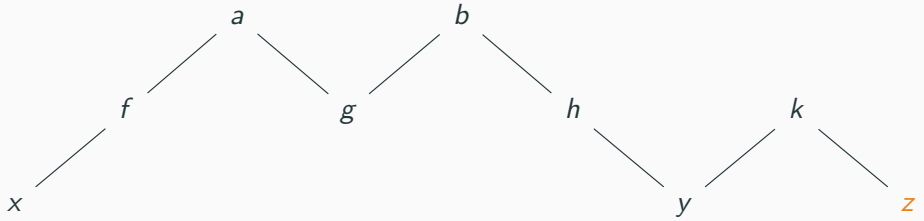

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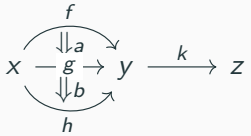

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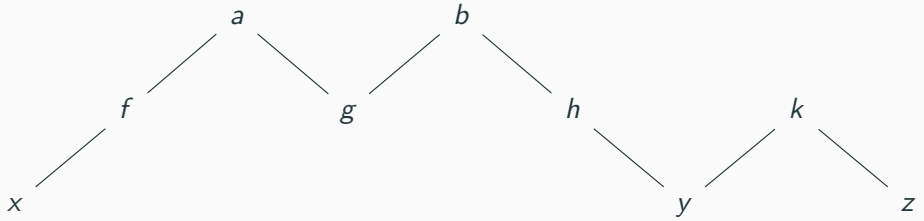

Term constructors: compositions and coherences

- **Idea**: Force every pasting scheme to have an essentially unique composite

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- **Existence**

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- **Essential uniqueness**

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- Includes proof automation and term synthesis.

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General Idea

Computads translate a the idea of freeness for higher categories.

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1-generators

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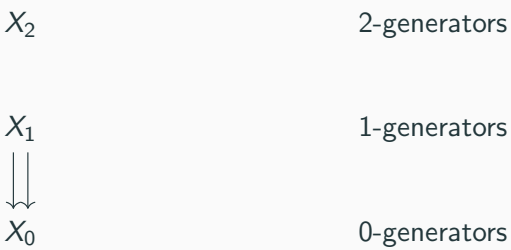

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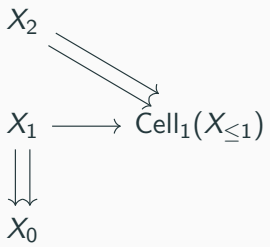

2-generators

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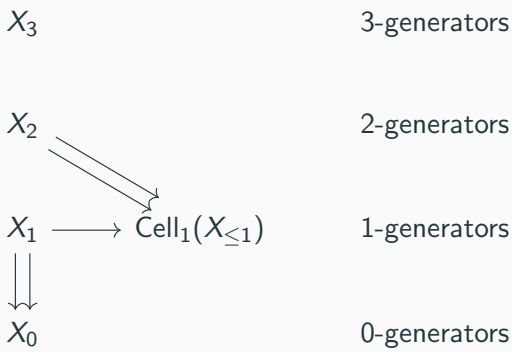

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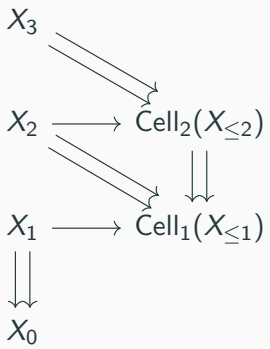

3-generators

2-generators

1-generators

0-generators

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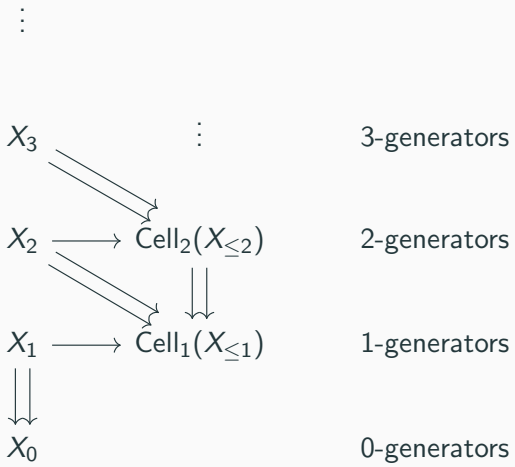

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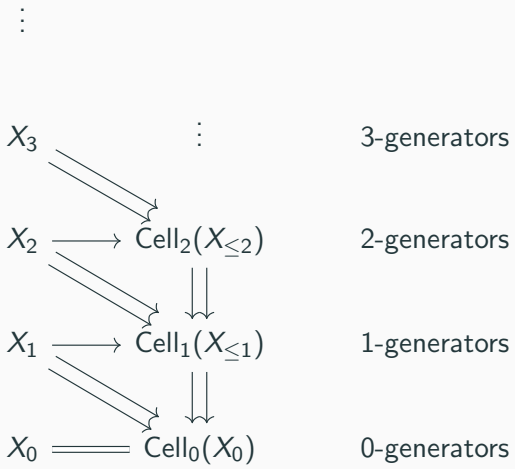
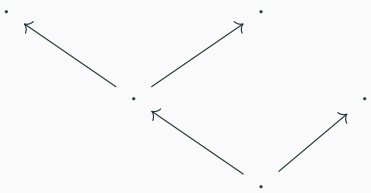
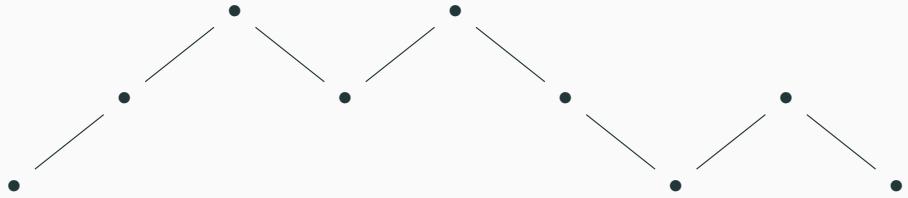


Batanin trees and pasting schemes

B :=





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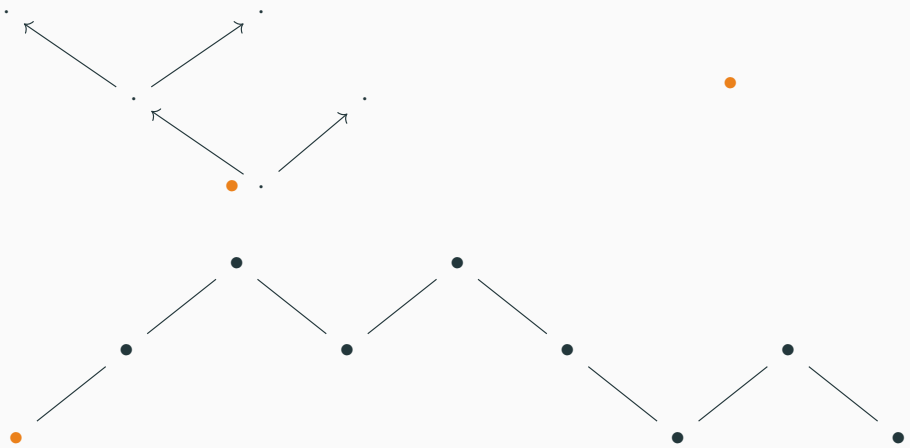

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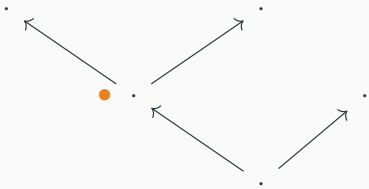

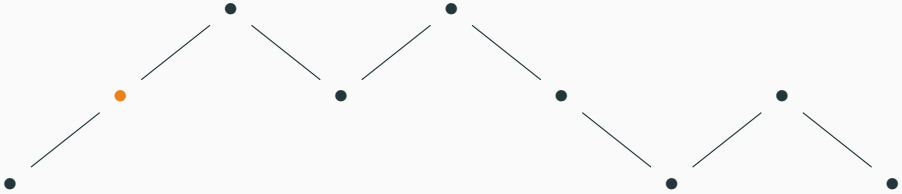

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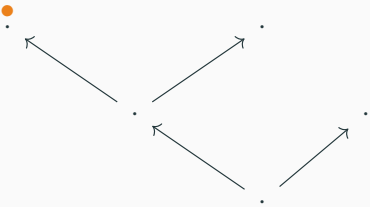

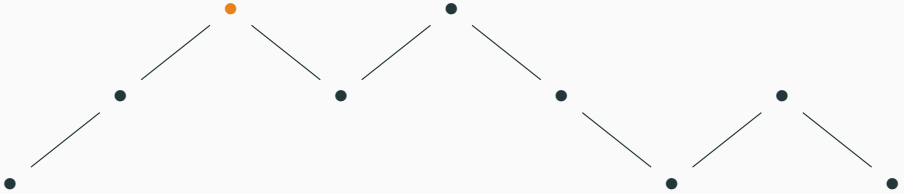

Batanin trees and pasting schemes

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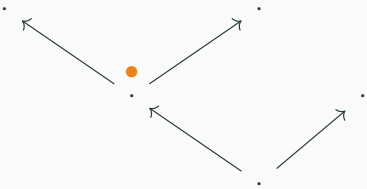

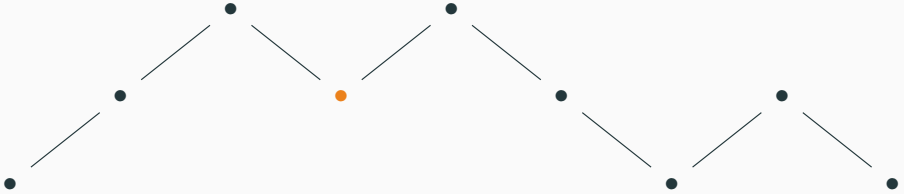

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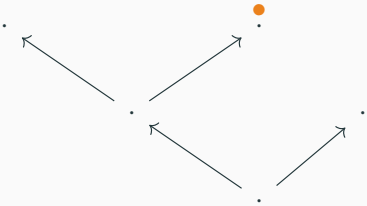

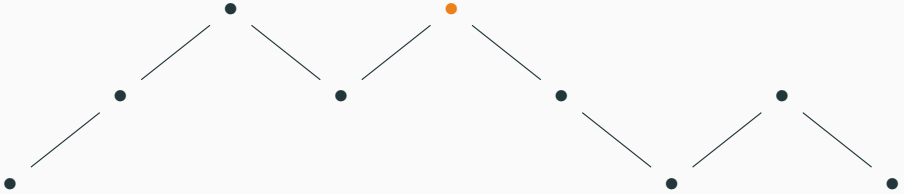

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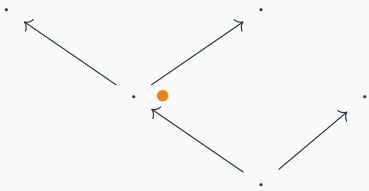

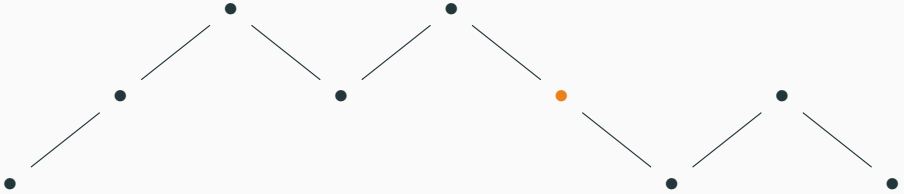

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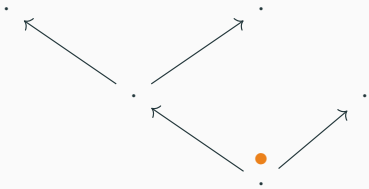

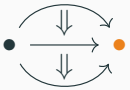

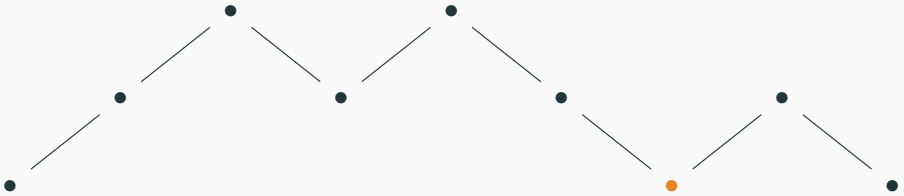

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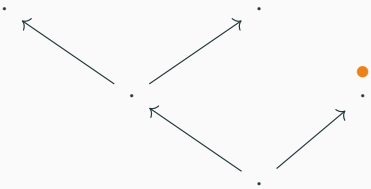

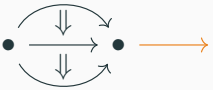

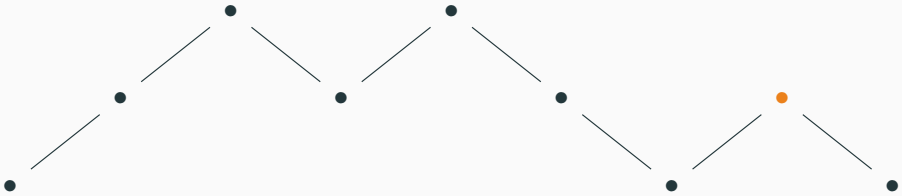

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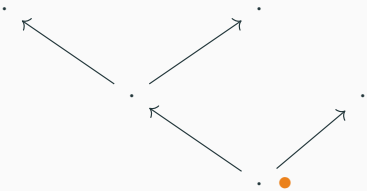

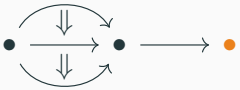

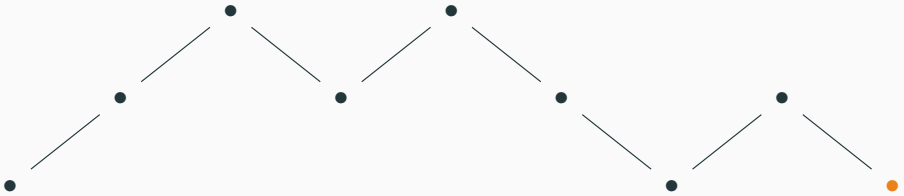

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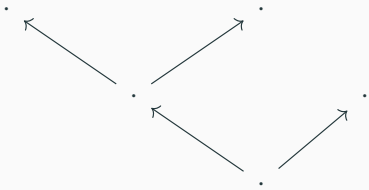
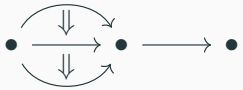
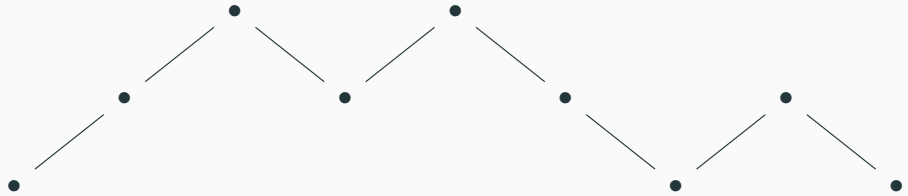


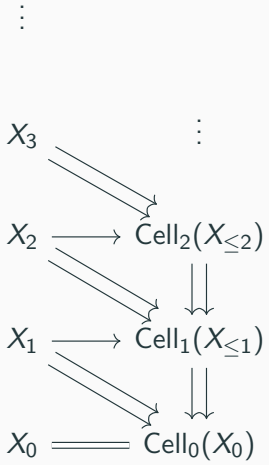
Pos(B) =





Description of $\mathsf{Cell}_n(X_{\leq n})$

The sets $\mathsf{Cell}_n(X_{\leq n})$ and the source and target maps are given by:



- $\mathrm{var}\, x$ where $x \in X_n$

$$s(\mathrm{var}\, x) = s(x) \qquad t(\mathrm{var}\, x) = t(x)$$

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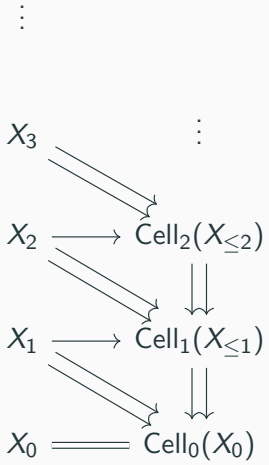

- $\mathrm{var}\, x$ where $x \in X_n$

$$s(\mathrm{var}\, x) = s(x) \qquad t(\mathrm{var}\, x) = t(x)$$

- $\mathrm{coh}(B, (u, v), \gamma)$, where B is a Batanin tree, $u, v \in \mathsf{Cell}_{n-1}(\mathrm{Pos}\, B)$ satisfying conditions similar to `CaTT`, and $\gamma : \mathsf{Cell}(\mathrm{Pos}(B)) \to X$

$$s(\mathrm{coh}(B, (u, v), \gamma)) = \gamma(u)$$
$$t(\mathrm{coh}(B, (u, v), \gamma)) = \gamma(v)$$

The category Comp of computads

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  - **objects:** computads
  - **morphisms:** $f : X \to Y$ = collection of maps $f_n : X_n \to \mathrm{Cell}_n(Y)$ respecting source and target

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  morphisms $\simeq$ substitutions

- CwF structure on $\mathrm{Comp}^{\mathrm{op}}$:
  - **types:** pair of parallel cells in $\mathrm{Cell}(X)$
  - **terms:** cells in $\mathrm{Cell}(X)$ with prescribed source/target
  - **extension:** adding new generator

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Comparison result

Summary and comparison

CaTT	description of computads
Finite	Infinite
Ordered	Unordered
Structural induction	Induction on dimension
Pasting schemes	Batanin trees
Named variables	Unnamed elements

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- Name of the elements: irrelevant with de Bruijn levels.
- Pasting schemes and Batanin trees equivalent.

Contribution

Theorem (B., Markakis, Sarti)
There exists a fully faithful morphism of CwF from $\mathcal{S}_{\texttt{CaTT}}$ *to* $\mathsf{Comp}^{\mathsf{op}}$. *Its essential image is made of finite computads.*

`CaTT` **contexts are finite computads**

Conclusion

- Weak ω-categories admit many different definitions, using various tools: monads, coherator, dependent type theory, inductive description of computads.

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- Network of connections, that we completed by showing that `CaTT` defines finite computads.
- Makes an explicit connection between algebras for the monad and models of the dependent type theory.