Partial Combinatory Algebras for Intensional Type Theory

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ACT 2024
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Motivation: homotopy BHK

The BHK interpretation specifies what counts as evidence for a given proposition.

The homotopy interpretation of ITT maintains that evidence for an identification (a term of identity type) is a path in some space.

This leads to the idea of having realizers carry higher-dimensional structure, cf. cubical computational type theory [Angiuli, Harper, Wilson, Favonia, Cavallo].

Partial combinatory algebras

Realizability models of type theory are traditionally built out of partial combinatory algebras (PCAs).

A PCA comprises a carrier set A and a partial binary operation

$$(-) \cdot (-) : A \times A \rightharpoonup A$$

There must exist "combinators" $\mathsf{k}, \mathsf{s} \in A$ satisfying:

$$\begin{aligned} &\mathsf{k}ab = a \\ &\mathsf{s}ab \downarrow \\ &\mathsf{s}abc \simeq ac(bc) \end{aligned}$$

Examples of PCAs

1. Untyped λ-terms modulo β:

$$t_1 \cdot t_2 \coloneqq t_1 t_2$$

$$\mathsf{k} \coloneqq \lambda xy.x \qquad \mathsf{s} \coloneqq \lambda fgx.fx(gx)$$

2. Categorical models of the λ-calculus, ie. cartesian closed categories $\mathcal{C}$ with a reflexive object $U \in \mathcal{C}$. The carrier set is $\mathcal{C}(1, U)$ and application is induced by

$$U \times U \xrightarrow{\mathsf{app} \times U} U^U \times U \xrightarrow{\mathsf{eval}} U$$

   These are total PCAs.

Examples of PCAs

3. Kleene's first algebra of natural numbers and partial recursive functions:

$$n \cdot m \coloneqq \varphi_n(m)$$

This is genuinely partial.

PCAs in cartesian restriction categories

PCAs can be defined internally to any cartesian restriction category (CRC) [Cockett, Hofstra]. Restriction categories axiomatize categories of partial maps.

Every (cartesian) category is trivially a (C)RC. The prototypical, non-trivial example is the category of sets and partial functions (with cartesian product).

The category of groupoids and partial functors (defined on a (full) subcategory) is a CRC (with cartesian product of groupoids).

PCAs in cartesian restriction categories

A PCA is an applicative structure

$$\bullet : A \times A \to A$$

such that the maps:

$$\begin{aligned} \mathsf{k} &= A \times A \xrightarrow{\pi_1} A \\ \mathsf{s} &= A \times A \times A \xrightarrow{\langle \pi_{13}, \pi_{23} \rangle} (A \times A) \times (A \times A) \xrightarrow{\bullet \times \bullet} A \times A \xrightarrow{\bullet} A \end{aligned}$$

are computable.

A map $f : A^n \to A$ is computable iff there is a total map ("code") $\ulcorner f \urcorner : 1 \to A$ such that

$$f = A^n \xrightarrow{\ulcorner f \urcorner \times A^n} A \times A^n \xrightarrow{\bullet^n} A$$

PCAs in groupoids 1: higher λ-calculi

Inspired by cubical type theory [Bezem, Cohen, Coquand, Huber, Mörtberg,...]

Judgements of the form: $\Psi \mid \Gamma \vdash t$, where Γ is a context of variables and Ψ is a dimension context.

$$\frac{i \mid \Gamma \vdash \alpha \qquad i \mid \Gamma \vdash \beta \qquad \cdot \mid \Gamma \vdash \beta[0/i] = \alpha[1/i]}{i \mid \Gamma \vdash \beta \circ \alpha} \text{ comp}$$

$$\frac{i \mid \Gamma \vdash \alpha}{i \mid \Gamma \vdash \alpha[1/i] \circ \alpha = \alpha} \text{ unit-left}$$

PCAs in groupoids 1: higher λ-calculi

Carrier groupoid:

- objects: terms of the form $\cdot \mid \Gamma \vdash t$
- morphisms $(\cdot \mid \Gamma \vdash t) \to (\cdot \mid \Gamma \vdash u)$: terms $i \mid \Gamma \vdash \alpha$ satisfying $\cdot \mid \Gamma \vdash \alpha[0/i] = t$ and $\cdot \mid \Gamma \vdash \alpha[1/i] = u$

The application functor is induced by application of the calculus.

The combinators are:

$$\cdot \mid \cdot \vdash \lambda xy.\, x \qquad \cdot \mid \cdot \vdash \lambda fgx.\, fx(gx)$$

We get a total PCA in the trivial CRC of groupoids and functors.

PCAs in groupoids 2: bicategorical models of λ-calculus

A bicategorical model of the λ-calculus is a cartesian closed bicategory $\mathfrak{C}$ with a pseudoreflexive object $U \in \mathfrak{C}$.

Examples:

- generalised species of structures [Fiore, Gambino, Hyland, Winskel]
- profunctorial Scott semantics [Galal]
- categorified relational (distributors-induced) model [Olimpieri]
- categorified graph model [Kerinec, Manzonetto, Olimpieri]

The carrier groupoid is the core of $\mathfrak{C}(1, U)$ and application is induced as in set case.

These are total PCAs but are weak in the sense that the combinator equations hold up to isomorphism.

PCAs in groupoids 3: categorified Kleene's first algebra

Let $\mathbf{FinBij}$ be the groupoid of finite sets and bijections. Fix an equivalence

$$|-| : \mathbf{FinBij} \xrightarrow{\sim} \mathbf{FinOrd}$$

between $\mathbf{FinBij}$ and its skeleton $\mathbf{FinOrd}$ of finite ordinals and bijections.

A partial recursive functor wrt $|-|$ is a partial functor $\mathbf{FinBij}^k \to \mathbf{FinBij}$ belonging to the smallest class $\mathcal{PR}$ containing: the constantly 0 functor, the projection functors and the $(-) + c$ functor for any finite set c, and closed under composition as well as...

PCAs in groupoids 3: categorified Kleene's first algebra

- primitive recursion: if $G : \mathbf{FinBij}^k \to \mathbf{FinBij}$ and $H : \mathbf{FinBij}^{k+2} \to \mathbf{FinBij}$ belong to $\mathcal{PR}$ then so does $F : \mathbf{FinBij}^{k+1} \to \mathbf{FinBij}$ defined by:

$$F(0, \vec{x}) \coloneqq G(\vec{x})$$
$$F(y, \vec{x}) \coloneqq H(|y| - 1, F(|y| - 1, \vec{x}), \vec{x})$$

- minimization: if $F : \mathbf{FinBij}^{k+1} \to \mathbf{FinBij}$ is in $\mathcal{PR}$ and is total then the partial functor $\mu F : \mathbf{FinBij}^k \to \mathbf{FinBij}$ defined as follows belongs to $\mathcal{PR}$.

$$\mu F(\vec{x}) \coloneqq n \iff \begin{cases} |F(i, \vec{x})| > 0 & (0 \leq i < n) \\ F(n, \vec{x}) = 0 \end{cases}$$

Partial and weak (code functions using finite sets, universal function and s-m-n theorems up to iso)?

Future work

- Definition of weak PCA in a cartesian restriction bicategory.
- Investigate realizability models over PCAs in groupoids (partial equivalence relations, assemblies, realizability toposes). What principles are satisfied/refuted?